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The Maximal "Kinematical" Invariance Group for an Arbitrary Potential Revised

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Dedicated to Professor V.A. Marchenko on his 95th birthday

Group classification of one particle Schrödinger equations with arbitrary potentials (C.P. Boyer, Helv. Phys. Acta **47** (1974), p. 450) is revised. The corrected completed list of non-equivalent potentials and the corresponding symmetries is presented together with exact identification of symmetry algebras and admissible equivalence transformations.

Key words: Schrödinger equation, Lie symmetries, equivalence transformations.

Mathematical Subject Classification 2010: 34L15, 34L20, 35R10.

1. Introduction

Symmetry is one of the key concepts of any consistent physical theory. For example, relativistic theories should possess Lorentz invariance, which is replaced by Galilei invariance for systems with velocities much less than the velocity of light.

A regular way for searching of continuous and some other symmetries was proposed long time ago by Sophus Lie. In particular, he created the grounds of the group classification of differential equations. Being applied to model equations of mathematical and theoretical physics, it presents an effective way for construction of theories with a priori requested symmetries.

A perfect example of the group classification of fundamental equations of physics is the completed description of continuous point symmetries possessed by Schrödinger equations with arbitrary potentials. It has been done more than forty years ago in papers [1, 12] and [3]. It was Niederer [12] who had found the maximal invariance group of the free Schrödinger equation. He was the first who demonstrated that this group is more extended than just the Galilei group discussed previously in [9], and includes also dilations and conformal transformations. Notice that in fact this result was predicted by Sophus Lie who described symmetries of the heat equation.

Symmetries of Schrödinger equations with nontrivial potentials were described in paper [1], but only for the case of one spatial variable. Then Boyer extended these results for more physically interesting planar and three dimensional systems.

The mentioned contributions had a big impact and occupy a place of honour in modern physics. They present a priori information about possible symmetries of one particle QM systems and so form group-theoretical grounds of quantum mechanics. The group classification is the necessary step in investigation of higher symmetries of Schrödinger equations requested for separation of variables [10] and description of superintegrable systems [22]. They also give rise for some new inspiring physical theories such as the Galilean conformal field theory [8].

Recently we extend the Boyer classification to the case of QM systems with position-dependent mass [16, 18, 19]. It was a good opportunity to revise the classical paper [3] bearing in mind that some results of this paper appear as particular cases of our analysis. As a result it was recognized, that the Boyer classification is incomplete and some systems with inequivalent symmetries are missing in [3]. In addition, the classification results of Boyer are presented in a rather inconvenient form. In the classification table given in [3] there is a list of potentials while the symmetry generators are absent, and the reader is supposed to look for them in different places of the paper text an make rather nontrivial speculation to identify them. Moreover, this identification is not well-defined, and some potentials include parameters which can be removed using equivalence transformations.

We believe that the physical community should be supplied by a conveniently presented and correct information on point continuous symmetries which can be admitted by the fundamental equation of quantum mechanics. It was the main reason to write the present work, where the Boyer results are verified and corrected. In addition, the methods of group analysis of differential equations are essentially developed in comparison with the seventieth of the previous century, and it was interesting to apply them to the well known and very important object of mathematical physics.

2. Determining equations

We will consider Schrödinger equations of the following generic form

$$\left(\mathrm{i}\frac{\partial}{\partial t} - H\right)\psi(t,\mathbf{x}) = 0, \tag{2.1}$$

where H is the Hamiltonian

$$H = -\frac{1}{2}\partial_a\partial_a + V(\mathbf{x}) \tag{2.2}$$

with

$$\partial_a = \frac{\partial}{\partial x_a}, \quad \mathbf{x} = (x_1, x_2, \dots, x_n)$$

and summation is imposed over the repeating indices a over the values $a = 1, 2, \dots, n$.

We will search for symmetries of equation (2.1) with respect to continuous groups of transformations of dependent and independent variables. We will not apply the generic Lie approach but, following [3], restrict ourselves to using its simplified version which is perfectly adopted to equation (2.1). Let us represent the infinitesimal operator of the searched transformation group in the form

$$Q = \xi^0 \partial_t + \xi^a \partial_a + \tilde{\eta} \equiv \xi^0 \partial_t + \frac{1}{2} \left(\xi^a \partial_a + \partial_a \xi^a \right) + \mathrm{i}\eta, \tag{2.3}$$

where $\tilde{\eta} = \frac{1}{2}\xi^a_a + \mathrm{i}\eta$, ξ^0, ξ^a and η are functions of independent variables and $\partial_t = \frac{\partial}{\partial t}$.

Generator (2.3) transforms solutions of equation (2.1) into solutions if it satisfies the following operator equation

$$QL - LQ = \alpha L, \tag{2.4}$$

where $L = \mathrm{i}\partial_t - H$ and α is one more unknown function of t and $\mathbf{x}$.

Evaluating the commutator in the l.h.s. of (2.4) and equating coefficients for the linearly independent differentials we obtain the following system of equations for unknowns $\xi^0, \xi^a, \eta,$ V and α:

$$\dot{\xi}^0 = -a, \quad \xi^0_a = 0, \tag{2.5}$$

$$\xi^b_a + \xi^a_b - \frac{2}{n}\delta_{ab}\xi^i_i = 0, \tag{2.6}$$

$$\xi^i_i = -\frac{n}{2}\alpha, \tag{2.7}$$

$$\dot{\xi}^a + \eta_a = 0, \tag{2.8}$$

$$\xi^a V_a = \alpha V + \dot{\eta}. \tag{2.9}$$

In accordance with (2.5) both ξ^0 and α do not depend on $\mathbf{x}$. Equations (2.6) and (2.7) specify the x-dependence of coefficients ξ^a:

$$\xi^a = -\frac{\alpha}{2}x_a + \theta^{ab}x_b + \nu_a, \tag{2.10}$$

where α, $\theta^{ab} = -\theta^{ba}$ and ν^a are (in general time-dependent) parameters. Moreover, in accordance with (2.8), θ^{ab} are time independent, and so we will deal with five unknown functions α, η and ν^a depending on t. Then, integrating (2.8) we obtain the generic form of function η:

$$\eta = \frac{\dot{\alpha}}{4}x^2 - \dot{\nu}_a x_a + f(t), \tag{2.11}$$

and equation (2.9) is reduced to the following form:

$$\left(\frac{\alpha}{2}x_a - \nu_a - \theta^{ab}x_b\right) V_a + \alpha V + \frac{\ddot{\alpha}}{4}x^2 - \ddot{\nu}_a x_a - \dot{f} = 0. \tag{2.12}$$

Thus to classify point Lie symmetries of equation (2.1) it is necessary to find all non-equivalent solutions of equation (2.12). The evident equivalence transformations for (2.1), i.e., transformations which keep the generic form of this equation but can change the potential, are shifts, rotations and scalings of independent variables. Such transformations of the spatial variables form the Euclid group E(n) extended by simultaneous scaling of these variables. In addition, we can scale ψ and make the following transformations:

$$\psi \to \exp(\mathrm{i}tC)\psi, \quad V \to V + C, \tag{2.13}$$

where C is a constant.

It is not difficult to show that the mentioned transformations exhaust the equivalence group for equation (2.1) with arbitrary potential. However, for some particular potentials there are additional equivalence transformations which will be specified in the following.

3. Symmetries for equations with trivial, free fall, and isotropic oscillator potentials

Let us present the symmetries accepted by the Schrödinger equations with the trivial, isotropic oscillator and free fall potentials. They are well known, but we fix them for the readers convenience. In addition, the specific combinations of just these symmetries are accepted by the other systems classified below, and we will refer to them in the following classification tables.

Setting in (2.12) $V = 0$ we can easy solve the obtained equation and find the corresponding admissible symmetries (2.3). They are linear combinations of the following symmetry operators:

$$P_a = -\mathrm{i}\partial_a, \quad M_{ab} = x_a P_b - x_b P_a, \tag{3.1}$$

$$D = 2tP_0 - x_a P_a + \frac{\mathrm{i}n}{2}, \tag{3.2}$$

$$P_0 = \mathrm{i}\partial_t, \qquad G_a = tP_a - x_a, \tag{3.3}$$

$$A = tD - t^2 P_0 - \frac{x^2}{2}. \tag{3.4}$$

Operators (3.1)–(3.4) together with the unit operator I satisfy the following commutation relations:

$$[P_0, A] = \mathrm{i}D, \qquad [P_0, D] = 2\mathrm{i}P_0, \qquad [D, A] = 2\mathrm{i}A, \tag{3.5}$$

$$[P_a, D] = \mathrm{i}P_a, \qquad [G_a, D] = -\mathrm{i}G_a, \tag{3.6}$$

$$[P_0, G_a] = \mathrm{i}P_a, \qquad [P_a, G_b] = \mathrm{i}\delta_{ab} I, \tag{3.7}$$

$$[M_{ab}, M_{cd}] = \mathrm{i}(\delta_{ad}M_{bc} + \delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac}), \tag{3.8}$$

$$[P_a, M_{bc}] = \mathrm{i}(\delta_{ab}P_c - \delta_{ac}P_b), \qquad [G_a, M_{bc}] = \mathrm{i}(\delta_{ab}G_c - \delta_{ac}G_b) \tag{3.9}$$

(the remaining commutators are equal to zero) and form the Lie algebra schr$(1, n)$ whose dimension is $N = \frac{n^2+3n+8}{2}$. Commutation relations (3.7)–(3.9) specify the Lie algebra g$(1, n)$ of Galilei group, which is a subalgebra of schr$(1, n)$.

Let us present additional identities satisfied by operators (3.1)–(3.4) (see, e.g., [6]):

$$P_a G_b - P_b G_a = M_{ab}, \tag{3.10}$$

$$P_aG_a + G_aP_a = 2D + 2t(P^2 - 2P_0), \tag{3.11}$$
$$G_aG_a = 2A + t^2(P^2 - 2P_0). \tag{3.12}$$

On the set of solutions of equation (2.2) the term in brackets is equal to $-2V$. Since in our case $V = 0$ relations (3.10)–(3.12) express generators M_{ab}, D and A via bilinear combination of Galilei group generators P_a and G_a. In other words, the invariance with respect to rotation, dilatation and conformal transformations appears to be a consequence of the symmetry with respect to the displacement and Galilei transformations.

Operators (3.1)–(3.4) generate the N-parametric symmetry group which is much more extended than the equivalence group for equation (2.1) with arbitrary potential. Moreover, there are three additional equivalence transformations

$$\begin{gathered}\mathbf{x} \to \tilde{\mathbf{x}} = \frac{\mathbf{x}}{\sqrt{1+t^2}}, \qquad t \to \tilde{t} = \frac{1}{\omega}\mathrm{arctan}(t),\\ \psi(t,\mathbf{x}) \to \tilde{\psi}(\tilde{t},\tilde{\mathbf{x}}) = (1+t^2)^{\frac{n}{4}} \mathrm{e}^{\frac{-\mathrm{i}\omega t\mathbf{x}^2}{2(1+t^2)}}\psi(t,\mathbf{x}),\end{gathered} \tag{3.13}$$

$$\begin{gathered}\mathbf{x} \to \tilde{\mathbf{x}} = \frac{\mathbf{x}}{\sqrt{1-t^2}}, \qquad t \to \tilde{t} = \frac{1}{\omega}\mathrm{arctanh}(t),\\ \psi(t,\mathbf{x}) \to \tilde{\psi}(\tilde{t},\tilde{\mathbf{x}}) = (1-t^2)^{\frac{n}{4}} \mathrm{e}^{\frac{\mathrm{i}\omega t\mathbf{x}^2}{2(1-t^2)}}\psi(t,\mathbf{x})\end{gathered} \tag{3.14}$$

and

$$\begin{gathered}x_a \to x'_a = x_a - \frac{1}{2}\kappa_a t^2, \qquad t \to t' = t,\\ \psi(t,\mathbf{x}) \to \psi'(t',\mathbf{x}') = \exp\left(-it\kappa_a x_a + \frac{\mathrm{i}}{3}\kappa^2 t^3\right)\psi(t,\mathbf{x})\end{gathered} \tag{3.15}$$

which keep the generic form of equation (2.1) but change the trivial potential $V = 0$ to

$$V = \frac{1}{2}\omega^2 x^2, \tag{3.16}$$
$$V = -\frac{1}{2}\omega^2 x^2, \tag{3.17}$$

and

$$V = \kappa_a x_a \tag{3.18}$$

correspondingly.

The transformations which reduce the isotropic harmonic and repulsive oscillators to the free particle Schrödinger equation were discovered by Niederer [13]. Formulae (3.13) and (3.14) present transformations for wave functions dependent on n spatial variables while in [13] we can find them only for $n = 1$. Let us note that in fact the origin of this transformation is much more extended: it can be applied for any equation (2.1) with potential V being a homogeneous function of degree -2.

Mapping (3.15) connects the systems with trivial and free fall potentials [14]. However, its origin can be extended to all potentials linearly dependent on a reduced number m of spatial variables with $m < n$.

Symmetries of equation (2.2) with quadratic potential (3.16) can be obtained from (3.1)–(3.4) applying transformations (3.13). They include P_0, M_{ab} and generators presented in the following formulae:

$$A_1^+ = \frac{1}{\omega}\sin(2\omega t)P_0 - \cos(2\omega t)\left(x_aP_a - \frac{\mathrm{i}n}{2}\right) - \omega\sin(2\omega t)x^2, \tag{3.19}$$

$$A_2^+ = \frac{1}{\omega}\cos(2\omega t)P_0 + \sin(2\omega t)\left(x_aP_a - \frac{\mathrm{i}n}{2}\right) - \omega\cos(2\omega t)x^2, \tag{3.20}$$

$$B_a^+(\omega) = \sin(\omega t)P_a - \omega x_a\cos(\omega t), \quad \hat{B}_a^+(\omega) = \cos(\omega t)P_a + \omega x_a\sin(\omega t). \tag{3.21}$$

For the case of the repulsive oscillator potential (3.17) we have the following symmetries:

$$\langle P_0, M_{ab}, A_1^-, A_2^-,\ B_a^-, \hat{B}_a^-\rangle \tag{3.22}$$

whose explicit form can be obtained from (3.19)–(3.21) changing $\omega \to \mathrm{i}\omega$.

Analogously, staring with realization (3.1)–(3.4) and making transformations (3.15) it is not difficult to find symmetries for equation (2.2) with the free fall potential. We will not write the related cumbersome expression which can be easily obtained making the changes

$$P_0 \to P_0 + \kappa_aG_a + \frac{1}{2}\kappa^2t^2, \quad P_a \to P_a + \kappa_at, \quad x_a \to x_a + \frac{1}{2}\kappa_at^2 \tag{3.23}$$

in all generators (3.1)–(3.4).

4. Classification results for arbitrary potentials

Let us consider equation (2.12) for arbitrary potential V. Its terms are products of functions of different independent variables which makes it possible to make the effective separation of these variables and reduce the problem to solution of systems of ordinary differential equations for time-dependent functions α, ν^a and f. Then the corresponding potentials are easily calculated integrating equations (2.12) with found functions of t.

There are different ways to realize this program. We use the algebraic approach whose main idea is to exploit the basic property of symmetry operators, i.e., the fact that they should form a basis of a Lie algebra. This algebra by definition includes operator P_0 and the unit operator.

In this section we present all non-equivalent $3d$ and $2d$ potentials which correspond to more extended symmetries. They are collected in the following Tables 1 and 2 while calculation details can be found in Appendix.

Table 1: Non-equivalent potentials and symmetries for $3d$ Schrödinger equation

No	Potential V	Symmetries	Invariance algebras
1	$G(\tilde{r}, x_3) + \kappa\varphi$	$L_3 + \kappa t$	$\mathsf{n}_{3,1}$ if $\kappa \neq 0$, $\mathsf{n}_{1,1}$ if $\kappa = 0$
2	$G(\tilde{r}, x_3 - \varphi) + \kappa\varphi$	$L_3 + P_3 + \kappa t$	$\mathsf{n}_{3,1}$ if $\kappa \neq 0$, $3\mathsf{n}_{1,1}$ if $\kappa = 0$
3	$\frac{1}{r^2}G(\frac{r}{\tilde{r}}, r^\kappa e^{-\varphi})$	$D + \kappa L_3$	$\mathsf{s}_{2,1} \oplus \mathsf{n}_{1,1}$
$4^\star$	$G(x_1, x_2)$	G_3, P_3	$\mathsf{n}_{4,1}$
5^*	$\frac{1}{r^2}G(\varphi, \frac{\tilde{r}}{r})$	A, D	$\mathrm{sl}(2,R) \oplus \mathsf{n}_{1,1}$
6^*	$\frac{1}{r^2}G(\frac{\tilde{r}}{r})$	A, D, L_3	$\mathrm{sl}(2,R) \oplus 2\mathsf{n}_{1,1}$
$7^\star$	$G(\tilde{r}) + \kappa\varphi$	$L_3 + \kappa t, G_3, P_3$	$\mathsf{s}_{5,14}$ if $\kappa \neq 0$, $\mathsf{n}_{4,1} \oplus \mathsf{n}_{1,1}$ if $\kappa = 0$
8	$\frac{1}{\tilde{r}^2}G(\tilde{r}^\kappa e^{-\varphi})$	$D + \kappa L_3, G_3, P_3$	$\mathsf{s}_{5,38}$
$9^{\star\star}$	$\frac{1}{\tilde{r}^2}G(\varphi)$	$A,\ D,\ G_3,\ P_3$	$\mathrm{sl}(2,R) \Subset \mathsf{n}_{3,1}$
$10^\star$	$G(x_1)$	$G_3,\ P_3,\ P_2,\ G_2,\ L_1$	$\mathrm{g}(1,2)$
11	$G(r)$	$L_1,\ L_2,\ L_3$	$\mathrm{so}(3) \oplus 2\mathsf{n}_{1,1}$
12^*	$\frac{\kappa}{r^2}$	$A,\ D,\ L_1,\ L_2,\ L_3$	$\mathrm{sl}(2,R) \oplus \mathrm{so}(3) \oplus \mathsf{n}_{1,1}$
$13^{\star\star}$	$\frac{\kappa}{\tilde{r}^2}$	$A,\ D,\ G_3,\ P_3,\ L_3$	$\mathrm{sl}(2,R) \Subset \mathsf{n}_{3,1} \oplus \mathsf{n}_{1,1}$
$14^{\star\star}$	$\frac{\kappa}{x_1^2}$	$A,\ D,\ G_2,\ G_3,\ P_2,\ P_3,\ L_1$	$\mathrm{schr}(1,2)$
15	$\varepsilon\frac{\omega^2 x_3^2}{2} + G(x_1, x_2)$	$B_3^\varepsilon(\omega),\ \hat{B}_3^\varepsilon(\omega)$	$\mathsf{s}_{4,9}$ if $\varepsilon = 1$, $\mathsf{s}_{4,8}$ if $\varepsilon = -1$
16	$\varepsilon\frac{\omega^2 x_3^2}{2} + G(\tilde{r}) + \mu\varphi$	$B_3^\varepsilon(\omega),\ \hat{B}_3^\varepsilon(\omega),\ L_3 + \mu t$	$\mathsf{s}_{5,16}$ if $\varepsilon = 1,\ \mu \neq 0$ $\mathsf{s}_{5,15}$ if $\varepsilon = -1, \mu \neq 0$, $\mathsf{s}_{4,9} \oplus \mathsf{n}_{1,1}$ if $\varepsilon = 1,\ \mu = 0$, $\mathsf{s}_{4,8} \oplus \mathsf{n}_{1,1}$ if $\varepsilon = -1, \mu = 0$
$17^\star$	$\varepsilon\frac{\omega^2 x_2^2}{2} + G(x_1)$	$B_2^\varepsilon(\omega),\ \hat{B}_2^\varepsilon(\omega),\ P_3,\ G_3$	$\mathsf{s}_{6,160}$ if $\varepsilon = -1$, $\mathsf{s}_{6,161}$ if $\varepsilon = 1$
18	$\varepsilon_1\frac{\omega_1^2 x_1^2}{2} + \varepsilon_2\frac{\omega_2^2 x_2^2}{2}$ $+G(x_3)$	$B_k^{\varepsilon_k}(\omega_k),\ \hat{B}_k^{\varepsilon_k}(\omega_k),\ k = 1, 2$	$\mathsf{s}_{6,162}$ if $\varepsilon_1 = \varepsilon_2 = -1$, $\mathsf{s}_{6,164}$ if $\varepsilon_1\varepsilon_2 = -1$, $\mathsf{s}_{6,166}$ if $\varepsilon_1 = \varepsilon_2 = 1$
19	$\varepsilon\frac{\omega^2\tilde{r}^2}{2} + G(x_3)$	$B_k^\varepsilon(\omega),\ \hat{B}_k^\varepsilon(\omega),\ L_3$	$\mathsf{s}_{6,162}$ if $\varepsilon = -1$, $\mathsf{s}_{6,166}$ if $\varepsilon = 1$
20	$\varepsilon_1\frac{\omega_1^2 x_1^2}{2} + \varepsilon_2\frac{\omega_2^2 x_2^2}{2}$ $+\varepsilon_3\frac{\omega_3^2 x_3^2}{2}$	$B_a^{\varepsilon_a}(\omega_a), \hat{B}_a^{\varepsilon_a}(\omega_a), a = 1, 2, 3$	$\mathsf{s}_{8,1}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$

$21^\star$	$\varepsilon_1\frac{\omega_1^2x_1^2}{2}+\varepsilon_2\frac{\omega_2^2x_2^2}{2}$	$B_k^{\varepsilon_k}(\omega_k),\ \hat{B}_k^{\varepsilon_k}(\omega_k),\ P_3,\ G_3$	$\mathsf{s}_{8,2}(\varepsilon_1,\varepsilon_2)$
22	$\varepsilon\frac{\omega^2\tilde{r}^2}{2}+\varepsilon_3\frac{\omega_3^2x_3^2}{2}$	$L_3,\ B_a^{\varepsilon_a}(\omega_a),\ \hat{B}_a^{\varepsilon_a}(\omega_a)$	$\mathsf{s}_{9,1}(\varepsilon,\varepsilon_3)$
$23^\star$	$\varepsilon\frac{\omega^2\tilde{r}^2}{2}$	$G_3,\ P_3,\ L_3,\ B_k^{\varepsilon}(\omega),\ \hat{B}_k^{\varepsilon}(\omega)$	$\mathsf{s}_{9,2}(\varepsilon)$
$24^\star$	$\varepsilon\frac{\omega^2x_3^2}{2}$	$B_3^{\varepsilon}(\omega),\ \hat{B}_3^{\varepsilon}(\omega),\ P_k,\ G_k,\ L_3$	$\mathsf{s}_{9,3}(\varepsilon)$

Here $G(\cdot)$ are arbitrary function of variables given in the brackets, μ, κ and ω_a are arbitrary real parameters, ε_1, ε_2, and ε_3 can take values ±1 independently, subindexes a and k take all values 1, 2, 3 and 1, 2 correspondingly. In addition, we denote $r=\sqrt{x_1^2+x_2^2+x_3^2}$, $\tilde{r}=\sqrt{x_1^2+x_2^2}$ and $\varphi=\arctan(x_2/x_1)$.

All presented systems by construction admit symmetries P_0 and I, the latter is the unit operator. The additional symmetries are presented in Columns 3, where P_a, $L_a=\frac{1}{2}\varepsilon_{abc}M^{bc}$, D, A, $B_a^{\varepsilon}(\omega_a)$, and $\hat{B}_a^{\varepsilon}(\omega_a)$ are generators (3.1)–(3.4) and (3.19)–(3.22). The related symmetry algebras are fixed in the fourth columns, where $\mathsf{n}_{a,b}$ and $\mathsf{s}_{a,b}$ are nilpotent and solvable Lie algebras of dimension a. To identify these algebras for $a\le 6$ we use the notations presented in [21]. The symbol $2\mathsf{n}_{1,1}$ (or $3\mathsf{n}_{1,1}$) denotes the direct sum of two (or three) one-dimension algebras. In addition, $\mathrm{g}(1,2)$ and $\mathrm{shcr}(1,2)$ are Lie algebras of Galilei and Schrödinger groups in (1+2) dimensional space.

Table 2: Non-equivalent potentials and symmetries for $2d$ Schrödinger equation

No	Potential V	Symmetries	Invariance algebras
1	$G(\tilde{r})+\kappa\varphi$	$L_3+\kappa t$	$\mathsf{n}_{3,1}$ if $\kappa\neq0$, $3\mathsf{n}_{1,1}$ if $\kappa=0$
2	$\frac{1}{\tilde{r}^2}G(r^{\kappa}e^{-\varphi})$	$D+\kappa L_3$	$\mathsf{s}_{2,1}\oplus\mathsf{n}_{1,1}$
3^*	$\frac{1}{\tilde{r}^2}G(\varphi)$	$A,\ D$	$\mathrm{sl}(2,R)\oplus\mathsf{n}_{1,1}$
$4^\star$	$G(x_1)$	$P_2,\ G_2,$	$\mathsf{n}_{4,1}$
5^*	$\frac{\kappa}{\tilde{r}^2}+\kappa\varphi$	$A,\ D,\ L_3+\kappa t$	$\mathrm{sl}(2,R)\oplus\mathsf{n}_{3,1}$ if $\kappa\neq0$, $\mathrm{sl}(2,R)\oplus3\mathsf{n}_{1,1}$ if $\kappa=0$
$6^{\star\star}$	$\frac{\kappa}{x_1^2}$	$A,\ D,\ G_2,\ P_2$	$\mathrm{sl}(2,R)\oplus\mathsf{n}_{4,1}$
7	$\varepsilon\frac{\omega^2x_1^2}{2}+G(x_2)$	$B_1^{\varepsilon}(\omega),\ \hat{B}_1^{\varepsilon}(\omega)$	$\mathsf{s}_{4,9}$ if $\varepsilon=1$, $\mathsf{s}_{4,8}$ if $\varepsilon=-1$
8	$\varepsilon\frac{\omega^2x_1^2}{2}$	$B_1^{\varepsilon}(\omega),\ \hat{B}_1^{\varepsilon}(\omega),\ P_2,\ G_2$	$\mathsf{s}_{6,160}$ if $\varepsilon=-1$, $\mathsf{s}_{6,161}$ if $\varepsilon=1$
9	$\varepsilon_1\frac{\omega_1^2x_1^2}{2}+\varepsilon_2\frac{\omega_2^2x_2^2}{2}$	$B_1^{\varepsilon_1}(\omega_1),\ B_2^{\varepsilon_2}(\omega_2),\ \hat{B}_1^{\varepsilon_1}(\omega_1),$ $\hat{B}_2^{\varepsilon_2}(\omega_2)$	$\mathsf{s}_{6,162}$ if $\varepsilon=\varepsilon'=-1$, $\mathsf{s}_{6,164}$ if $\varepsilon\varepsilon'=-1$, $\mathsf{s}_{6,166}$ if $\varepsilon=\varepsilon'=1$

In the tables we specify also the admissible equivalence transformations additional to ones belonging to the extended Euclid group. Namely, the star near the item number means that the corresponding Schrödinger equation admits additional equivalence transformation (3.15) for independent variables x_a provided $\frac{\partial V}{\partial x_a} = 0$. The asterisk marks the items which specify equations admitting transformation (3.13) and (3.14).

The algebras of symmetries presented in Items 20–24 of Table 1 are solvable and have dimension $d \geq 8$. We denote them formally as $\mathsf{s}_{d,a}(\cdot)$ without referring to any data base, since the classification of algebras of such dimensions is still far from the completeness. Let us present commutation relations which specify these algebras:

$$
\begin{aligned}
&[P_0, B_a^\varepsilon] = \mathrm{i}\omega \hat{B}_a^\varepsilon, \quad &&[P_0, \hat{B}_a^\varepsilon] = \mathrm{i}\varepsilon\omega B_a^\varepsilon, \quad &&[P_0, G_3] = \mathrm{i}P_3, \quad &&[B_a^\varepsilon, \hat{B}_b^\varepsilon] = \mathrm{i}\delta_{ab} I,\\
&[B_2^\varepsilon, L_3] = \mathrm{i}B_1^\varepsilon, \quad &&[B_1^\varepsilon, L_3] = -\mathrm{i}B_2^\varepsilon, \quad &&[\hat{B}_2^\varepsilon, L_3] = \mathrm{i}\hat{B}_1^\varepsilon, \quad &&[\hat{B}_1^\varepsilon, L_3] = -\mathrm{i}\hat{B}_2^\varepsilon,
\end{aligned}
$$

where only nontrivial commutators are presented.

Thus we classify all non-equivalent Lie symmetries admitted by $3d$ and $2d$ Schrödinger equations (2.2). Some of them are new, see discussion in the following section.

5. Discussion

Our revision of continuous point symmetries of the main equation of quantum mechanics is seemed to be successful. Using the algebraic approach to the group classification of partial differential equations, we recover the classical results presented in [3], but also find four systems, missing in the Boyer classification. These systems are represented in Items 1, 2, 7 and 16 of Table 1 and Items 1, 5 of Table 2.

The potential $V = \kappa \arctan(x_2/x_1)$ missing in the Boyer classification satisfies the Laplace equation and so belongs to the class of harmonic potential fields which find many interesting applications including such exotic ones as the robot navigation. Mathematically, the presented new potentials and the corresponding symmetries can be used for the classification of superintegrable quantum mechanical systems admitting higher order integrals of motion.

We present a correct list of inequivalent point continuous symmetries and the corresponding potentials which can be admitted by one-particle Schrödinger equation. This list does not include the infinite symmetry group of transformations $\psi \to \psi + \tilde{\psi}$ where $\tilde{\psi}$ is an arbitrary solution of equation (2.2). In accordance with the superposition rule, such evident symmetries are valid for all linear equations.

In the classification tables the systems admitting additional equivalence transformations are clearly indicated. The invariance algebras are specified using notations proposed in [21]. We believe that this information is important and useful. In particular, the reader interested in the Casimir operators of the symmetry algebras can easy find them in book [21].

Notice that the low-dimension algebras of dimension $d \leq 5$ and some class of the algebras of dimension 6 had been classified by Mubarakzianov [11], see also more contemporary and accessible papers [2, 4, 20] were his results are slightly corrected.

A natural question arises: why the new systems presented here were not recognized by Boyer? (I am indebted to Prof. W. Miller, Jr for setting this question.) For readers familiar with paper [3] it is possible to indicate two miss points there. First, in equation (2.2g) giving the generic form of rotation generators the admissible term $C_{ij}t$ with constant antisymmetric tensor C_{ij} is missing. This term cannot be reduced to zero if parameters g_i present in this formula are trivial. Secondly, Boyer did not use a list of non-equivalent subalgebras of algebra $\tilde{\mathrm{e}}(3)$, and as a result the symmetry presented in Item 2 of Table 1 was overlooked.

In any case, Charles Boyer was the first who made the group classification of $3d$ Schrödinger equations with arbitrary potentials. Moreover, he deduced the determining equations (2.12) for arbitrary number of spatial variables. Up to minor misprints, the list of non-equivalent symmetries presented by him is correct but incomplete, and I appreciate the chance to make a small addition to these well known results. For group classification of nonlinear Schrödinger equations and their conditional symmetries see papers [7, 15, 17].

We restrict our analysis to equation (2.2) with two and three spatial variables. Its extension to equations with more variables is not too difficult but more cumbersome problem, see the last paragraph of the Appendix.

Appendix. Some details of calculations

We will not reproduce detailed calculations requested to solve the determining equations but present some points of the algebraic approach which was used to do it.

Considering various differential consequences of equation (2.12) it is possible to show that up to equivalence transformations (3.13)–(3.15) the generic symmetry (2.3) with coefficients (2.10), (2.11) is nothing but a linear combination of symmetries (3.1)–(3.4) and (3.19)–(3.22) and the unit operator multiplied by yet indefinite function $f(t)$. Thus to find all non-equivalent solutions of equation (2.12) we have to go over such (non-equivalent) combinations, which are restricted by the following condition: the corresponding functions α, ν^a, θ^{ab} and f should be proportional to the same function $\Phi(t)$. In accordance with (3.1)–(3.4) and (3.19)–(3.22) this function can be scalar, linear in t, trigonometric or hyperbolic.

The scalar function $\Phi(t)$ corresponds to linear combinations of generators P_a, $L_a = \frac{1}{2}\varepsilon_{abc}$ and D presented by equations (3.1), (3.2). These generators form a basis of extended Euclidean algebra $\tilde{\mathrm{e}}(3)$, whose non-equivalent subalgebras has been classified in [5]. In particular, this algebra has four non-equivalent one-dimensional subalgebras spanned on the following generators:

$$L_3 = M_{12}, \quad L_3 + P_3, \quad D + \mu L_3, \quad P_3. \tag{A.1}$$

The corresponding non-zero coefficients in equation (2.12) are $\theta^{12} = 1$ for L_3, $\theta^{12} = \nu^3 = 1$ for $L_3 + P_3$, $a = 1$, $\theta^{12} = \mu$ for $D + \mu L_3$ and $\nu^3 = 1$ for P_3. In addition, arbitrary function f should be linear in t, i.e., $\eta = f = \kappa t$, and just this function can be added to all generators (A.1). In particular, let $Q = L_3 + \kappa t$ then equation (2.12) is reduced to the following form:

$$L_3 V = -\mathrm{i}\kappa \qquad \text{or} \qquad \frac{\partial V}{\partial \varphi} = \kappa, \quad \varphi = \arctan\left(\frac{x_2}{x_1}\right),$$

and so $V = \kappa\varphi$. Just this case is missing in Boyer classification.

Solving such defined class of equations (2.12) we obtain potentials presented in Items 1–5 of Table 1. In Item 4 we set $\kappa = 0$ since this parameter can be reduced to zero using mapping inverse to (3.15). The additional symmetry G_3 is presented there since it generates the same equation for potential as P_3. In Item 5 we also have the additional symmetry A which requests the same potential as symmetry D.

In general, it is possible to fix the following pairs and triplets of "friendly symmetries"

$$\langle P_a, G_a\rangle, \quad \langle A, D\rangle, \quad \langle B_a^\varepsilon, \hat{B}_a^\varepsilon\rangle, \quad \langle (P_1, P_2), L_3\rangle, \quad \langle (B_1^\varepsilon, B_2^\varepsilon), L_3\rangle. \tag{A.2}$$

If equation (2.2) admit a symmetry from one of the presented pairs, it automatically admit also the other symmetry. Symmetries presented in brackets induce the third symmetry from the triplet.

The next step is to exploit the non-equivalent two-dimensional subalgebras of the extended Euclid algebra, which can be spanned on the following basis elements:

$$\langle L_3 + \kappa t, P_3\rangle, \quad \langle D + \kappa L_3, P_3\rangle, \quad \langle P_2, P_3\rangle, \quad \langle D, L_3\rangle. \tag{A.3}$$

Since any sets (A.3) includes at least one basis element from list (A.1), we have to solve equation (2.12) generated by the second element for potentials presented in Items 1–5 of Table 1. As a result, we obtain potentials included in Items 6–10. The corresponding symmetry algebras are extended at the cost of "friendly symmetries".

Analogously, considering non-equivalent three dimensional subalgebras

$$\langle D, P_3, L_3\rangle, \quad \langle D, P_1, P_2\rangle, \quad \langle L_1, L_2, L_3\rangle, \quad \langle L_3, P_1, P_2\rangle \tag{A.4}$$

$$\langle P_1, P_2, P_3\rangle, \quad \langle L_3 + P_3, P_1, P_2\rangle, \quad \langle D + \mu L_3, P_1, P_2\rangle, \quad \mu > 0, \tag{A.5}$$

we obtain potentials presented in Items 11–14 and recover once more the cases given in Items 7, 8, 10. Notice that now we have also a simple algebra so(3) realized by L_1, L_2 and L_3. In addition, algebras (A.5) are valid only for scalar potentials.

Thus we have described all symmetry algebras including generators with time-independent coefficients ξ^a. In fact some of them include the coefficients linear in t, but such generators appear automatically as "friendly symmetries". And if we now will consider specially realizations linear in t, the list of potentials presented in Items 1–14 will not be extended.

To obtain the remaining part of the table we have to search for symmetries including hyperbolic functions. This job is reduced to simple enumeration of possibilities with one, two, or three pairs of operators B_a^ε, $\hat{B}_a^\varepsilon$ with the same or different frequency parameters $\omega = \omega_a$.

Let us note that our analysis can be directly extended to the case of equation (2.2) with more large number n of spatial variables. For example, for $n = 4$ we have to start with the following one-dimensional subalgebras of the extended Euclidean algebra $\tilde{e}(4)$:

$$\begin{gathered} P_4,\ M_{12},\ M_{12}+M_{34},\ M_{12}+\alpha M_{34}\ (0<\alpha<1),\ M_{12}+\nu P_4\ (\nu>0), \\ D+\lambda M_{12},\ D+M_{12}+M_{34},\ M_{12}+\alpha M_{34}+\beta D\ (0<\alpha\leq 1,\ \beta>1) \end{gathered} \tag{A.6}$$

whose number is more extended than in $3d$ case (compare with equation (A.1)) but not dramatically large. On the other hand, the determining equation (2.12) is defined for arbitrary n and it can be solved for any of operators (5). Then we can use two-, three-, $\cdots$ dimensional subalgebras of algebra $\tilde{e}(4)$ which can be found in [5] and solve the related equation (2.12) in analogy with the above.

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Ревізія максимальної "кінематичної" групи інваріантності для довільного потенціалу

A.G. Nikitin

Переглянуто групову класифікацію одночастинкового рівняння Шредінгера з довільним потенціалом (C.P. Boyer, Helv. Phys. Acta **47** (1974), p. 450). Представлено виправлений перелік нееквівалентних потенціалів та відповідних симетрій разом з точною ідентифікацією алгебр симетрій та допустимих перетворень еквівалентності.

Ключові слова: рівняння Шредінгера, симетрія Лі, перетворення еквівалентності.