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The Einstein–Hilbert Type Action on Pseudo-Riemannian Almost-Product Manifolds

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We develop variation formulas for the quantities of extrinsic geometry of almost-product pseudo-Riemannian manifolds, and we consider the variations of metric preserving orthogonality of the distributions. These formulas are applied to study the Einstein–Hilbert type actions for the mixed scalar curvature and the extrinsic scalar curvature of a distribution. The Euler–Lagrange equations for these variations are derived in full generality and in several particular cases (foliations that are integrable plane fields, conformal submersions, etc.). The obtained Euler–Lagrange equations generalize the results for codimension-one foliations to the case of arbitrary codimension, and admit a number of solutions, e.g., twisted products and isoparametric foliations.

Key words: pseudo-Riemannian metric, almost-product manifold, foliation, second fundamental form, adapted variation, mixed scalar curvature, conformal submersion.

Mathematical Subject Classification 2010: 53C12, 53C44.

1. Introduction

Minimizing geometric quantities have been studied for a long time: recall, for example, isoperimetric inequalities and estimates of total curvature of submanifolds. In the context of foliations and distributions, Gluck and Ziller [6] studied the problem of minimizing functions as the volume defined for k-plane fields on a manifold. In all the cases mentioned above, they considered a fixed Riemannian metric and looked for geometric objects (submanifolds, foliations) minimizing geometric quantities defined usually as integrals of curvatures of different types.

The following approach to problems in geometry of codimension-one foliations is presented in [14, Chap. 2]: given a foliated manifold and a property Q of a submanifold, depending on the principal curvatures of the leaves, study Riemannian metrics, which minimize the integral of Q in the class of variations of metrics, such that the unit vector field orthogonal to the leaves is the same for all metrics of the variation family. Certainly, (like in some of the cases mentioned before) such Riemannian structures may not exist, but if they do, they usually have interesting geometric properties.

Our goal is to get new variation formulas for the quantities of extrinsic geometry of almost-product (e.g., foliated) pseudo-Riemannian manifolds for *adapted variations* of metrics and apply them to the study of the Einstein–Hilbert type actions. These functionals are defined like the classical Einstein–Hilbert action, the difference being the fact that the scalar curvature is replaced by the mixed scalar curvature $\mathrm{S}_{\rm mix}$ (i.e., an averaged mixed sectional curvature) or the extrinsic scalar curvature $\mathrm{S}_{\rm ex}$ of a non-degenerate distribution — the quantities studied by several geometers, see [3, 13, 17] and bibliographies therein.

Adapted variations that we apply generalize the approach of [14], to vary the metric in a way that preserves the almost-product structure of the manifold. We find the Euler–Lagrange equations and characterize the critical metrics in several classes of almost-product and foliated manifolds. The mixed Einstein–Hilbert action for a globally hyperbolic spacetime (M^4, g) was studied in [1], where the Euler–Lagrange equations (called the mixed gravitational field equations) were derived and their solutions for an empty space were examined. As we shall see shortly, the Euler–Lagrange equations for the Einstein–Hilbert type action involve several new tensors and a new type of Ricci curvature (introduced in [13], and studied in [1] for a codimension-one foliated globally hyperbolic space-time and in [2] for foliated closed Riemannian manifolds), whose properties are to be studied.

The paper develops methods of [14], where the variation formulas and functionals were studied for codimension-one foliations; our main result for this case (Euler–Lagrange equations in Section 3.3) coincides with an analogue of Einstein field equations from [1]. Our research poses open problems for further study, e.g., stability conditions of the action and the geometry of critical metrics with respect to all adapted variations of metric. Although adapted variations (of metric) preserve the orthogonal complement of a given distribution, note that, unlike $\mathrm{S}_{\rm mix}$, the extrinsic scalar curvature (see Section 2.4) does not depend explicitly on this complement. Therefore, in our subsequent work we shall also consider general variations more appropriate to this case.

The paper contains the introduction and two sections. Section 2 develops variation formulas for the quantities of extrinsic geometry for adapted variations of metrics on almost-product pseudo-Riemannian manifolds and applies them to study the total mixed scalar curvature and the total extrinsic scalar curvature of a distribution-analogues of the classical Einstein–Hilbert action. Its main result are the Euler–Lagrange equations for two types of adapted variations of metrics, the second of which preserves the volume of a relatively compact domain Ω (and yields an analogue of Einstein field equations). Section 3 is devoted to particular cases, e.g., foliated manifolds including flows, codimension-one foliations and conformal submersions with totally umbilical fibers. We give the examples (e.g., twisted products and isoparametric foliations) with necessary and sufficient conditions for critical metrics.

Throughout the paper everything (manifolds, distributions, etc.) is assumed to be smooth (i.e., C^∞-differentiable) and oriented. Following [3, 11], and in view of expected applications in theoretical physics, we consider pseudo-Riemannian metrics on open manifolds.

2. Einstein–Hilbert type action on almost-product manifolds

Let $\mathrm{Sym}^2(M)$ be the space of all symmetric $(0,2)$-tensors tangent to M. A pseudo-Riemannian metric of index q on M is an element $g\in\mathrm{Sym}^2(M)$ such that each g_x $(x\in M)$ is a *non-degenerate bilinear form of index* q on the tangent space T_xM. When $q=0$, i.e., g_x is positive definite, g is a Riemannian metric (resp., a Lorentz metric when $q=1$). At a point $x\in M$, a 2-dimensional linear subspace $X\wedge Y$ (called a plane section) of T_xM is *non-degenerate* if $W(X,Y):=g(X,X)\,g(Y,Y)-g(X,Y)\,g(X,Y)\neq 0$. For such section at x, the *sectional curvature* is the number $K(X,Y)=g(R(X,Y)X,Y)/W(X,Y)$. Here $R(X,Y)=\nabla_Y\nabla_X-\nabla_X\nabla_Y+\nabla_{[X,Y]}$ is the curvature tensor of the Levi-Civita connection ∇ of g.

The "musical" isomorphisms $\sharp$ and $\flat$ will be used for rank-1 and symmetric rank-2 tensors. For example, if $\omega\in T^1_0M$ is a 1-form and $X,Y\in\mathfrak{X}_M$, then $\omega(Y)=g(\omega^\sharp,Y)$ and $X^\flat(Y)=g(X,Y)$. For $(0,2)$-tensors A and B we have $\langle A,B\rangle=\mathrm{Tr}(A^\sharp B^\sharp)=\langle A^\sharp,B^\sharp\rangle$.

2.1. Preliminaries.

A subbundle $\widetilde{\mathcal{D}}\subset TM$ (called a distribution) is non-degenerate if $\widetilde{\mathcal{D}}_x$ is a non-degenerate subspace of (T_xM,g_x), i.e., g_x restricted to $\widetilde{\mathcal{D}}_x$ is non-degenerate bilinear form, for every $x\in M$; in this case, its complementary orthogonal distribution $\mathcal{D}$ (i.e., $\widetilde{\mathcal{D}}_x\cap\mathcal{D}_x=0$, $\widetilde{\mathcal{D}}_x\oplus\mathcal{D}_x=T_xM$ and $\widetilde{\mathcal{D}}_x\perp_g\mathcal{D}_x$ for any $x\in M$) is also non-degenerate. Thus, we are entitled to consider a connected manifold M^{n+p} with a pseudo-Riemannian metric g and a pair of complementary orthogonal non-degenerate distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ of ranks $\dim\widetilde{\mathcal{D}}_x=n$ and $\dim\mathcal{D}_x=p$ for every $x\in M$, called an *almost-product structure* on M, [7]. The following convention is adopted for the range of indices:

$$a,b,\ldots\in\{1,\ldots,n\},\quad i,j,\ldots\in\{1,\ldots,p\}.$$

The sectional curvature $K(X,Y)$ is called mixed if $X\in\widetilde{\mathcal{D}}$, $Y\in\mathcal{D}$. Let $\{E_a,\mathcal{E}_i\}$ be a local orthonormal frame adapted to $(\widetilde{\mathcal{D}},\mathcal{D})$, i.e., $\{E_a\}_{a=1\ldots n}$ belongs to $\widetilde{\mathcal{D}}$ and $\{\mathcal{E}_i\}_{i=1\ldots p}$ belongs to $\mathcal{D}$. Let $\epsilon_i=g(\mathcal{E}_i,\mathcal{E}_i)$, $\epsilon_a=g(E_a,E_a)$. The function on M,

$$\mathrm{S}_{\rm mix}=\sum_{a,i}K(E_a,\mathcal{E}_i)=\sum_{a,i}\epsilon_a\epsilon_i\,g\big(R(E_a,\mathcal{E}_i)E_a,\mathcal{E}_i\big),\tag{2.1}$$

is called the *mixed scalar curvature*, see [17]. If a distribution is spanned by a unit vector field N, i.e., $g(N,N)=\epsilon_N\in\{-1,1\}$, then $\mathrm{S}_{\rm mix}=\epsilon_N\,\mathrm{Ric}_N$, where Ric_N is the Ricci curvature in N-direction. For the surfaces foliated by curves, $\mathrm{S}_{\rm mix}$ is the Gaussian curvature.

Let $\mathfrak{X}_M$ (resp., $\mathfrak{X}_{\mathcal{D}}$ and $\mathfrak{X}_{\widetilde{\mathcal{D}}}$) be the module over $C^\infty(M)$ of all vector fields on M (resp., on $\mathcal{D}$ and $\widetilde{\mathcal{D}}$). For any $X\in\mathfrak{X}_M$, let $\widetilde{X}\equiv X^\top$ be the $\widetilde{\mathcal{D}}$-component (resp., $X^\perp$ the $\mathcal{D}$-component) of X with respect to the direct sum decomposition of TM. A tensor $B\in\mathrm{Sym}^2(M)$ is said to be *adapted* if $B(X^\top,Y^\perp)=0$ for any $X, Y \in \mathfrak{X}_M$. Let $\mathfrak{M} \equiv \mathfrak{M}(\widetilde{\mathcal{D}}, \mathcal{D})$ consist of all adapted symmetric tensors on $(M, \widetilde{\mathcal{D}}, \mathcal{D})$. Let $\mathrm{Riem}(M) \subset \mathrm{Sym}(M)$ be the subspace of pseudo-Riemannian metrics of given signature, and let $\mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D}) = \mathrm{Riem}(M) \cap \mathfrak{M}$.

We study pseudo-Riemannian structures on a manifold M, minimizing the functional

$$J_{\mathrm{mix},\Omega}(g) : g \mapsto \int_\Omega \mathrm{S}_{\mathrm{mix}}(g)\, \mathrm{d}\,\mathrm{vol}_g \tag{2.2}$$

for variations g_t ($g_0 = g$, $|t| < \varepsilon$) preserving orthogonality of $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, i.e., $g_t \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. In the paper, Ω from (2.2) is a relatively compact domain of M (and $\Omega = M$ when M is closed), containing supports of variations g_t. Let $\mathfrak{M}_{\widetilde{\mathcal{D}}}$ and $\mathfrak{M}_{\mathcal{D}}$ be, respectively, the spaces of symmetric $(0,2)$-tensors with the properties $B(X^\perp, Y) = 0$ and $B(X^\top, Y) = 0$ for any $X, Y \in \mathfrak{X}_M$. Then

$$\mathfrak{M} = \mathfrak{M}_{\widetilde{\mathcal{D}}} \oplus \mathfrak{M}_{\mathcal{D}}, \tag{2.3}$$

the decomposition is orthogonal with respect to the inner product induced on $\mathfrak{M}$ by a $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. For each $(0,2)$-tensor B tangent to M, we define its components $\widetilde{B}, B^\perp \in \Gamma(T^*M \otimes T^*M)$ by setting $\widetilde{B}(X, Y) = B(X^\top, Y^\top)$ and $B^\perp(X, Y) = B(X^\perp, Y^\perp)$. If $B \in \mathrm{Sym}^2(M)$, then $B \in \mathfrak{M} \Leftrightarrow B = B^\perp + \widetilde{B}$, see (2.3). In particular, $g = g^\perp + \widetilde{g}$ for any $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. Note that if $B \in \mathfrak{M}$, then $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ are $B^\sharp$-invariant.

Our purpose is to compute the directional derivatives

$$D_g J_{\mathrm{mix},\Omega} : T_g\, \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D}) \equiv \mathfrak{M} \to \mathbb{R} \tag{2.4}$$

for any $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ and study the geometry of almost-product or foliated manifolds $(M, \mathcal{D}, \widetilde{\mathcal{D}})$, where g is a critical point of $J_{\mathrm{mix},\Omega}$ with respect to all adapted variations supported in domain Ω. Certainly, we can restrict ourselves to the cases $D_g J_{\mathrm{mix},\Omega} : \mathfrak{M}_{\mathcal{D}} \to \mathbb{R}$ or $D_g J_{\mathrm{mix},\Omega} : \mathfrak{M}_{\widetilde{\mathcal{D}}} \to \mathbb{R}$, when g is critical either for $g^\perp$-variations, i.e., $D_g J_{\mathrm{mix},\Omega}(B) = 0$ for all $B \in \mathfrak{M}_{\mathcal{D}}$, or for $\widetilde{g}$-variations, i.e., $D_g J_{\mathrm{mix},\Omega}(B) = 0$ for all $B \in \mathfrak{M}_{\widetilde{\mathcal{D}}}$.

We define several tensors for one of distributions and introduce similar (e.g., $\widetilde{\mathcal{D}}$-valued) tensors for the second distribution using "$\sim$" notation. The symmetric $(0,2)$-tensor $r_{\mathcal{D}}$, given by

$$r_{\mathcal{D}}(X, Y) = \sum_a \epsilon_a\, g(R(E_a, X^\perp)E_a, Y^\perp), \qquad X, Y \in \mathfrak{X}_M, \tag{2.5}$$

is referred to as the *partial Ricci tensor* for $\mathcal{D}$. In particular, by (2.1),

$$\mathrm{Tr}_g\, r_{\mathcal{D}} = \mathrm{S}_{\mathrm{mix}}. \tag{2.6}$$

Note that the *partial Ricci curvature* $r_{\mathcal{D}}(X, X)$ in the direction of a unit vector $X \in \mathcal{D}$ is the "mean value" of sectional curvatures over all mixed planes containing X.

Let $T, h : \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}} \to \mathcal{D}$ be the integrability tensor and the second fundamental form of $\widetilde{\mathcal{D}}$,

$$T(X, Y) = \frac{1}{2}[X, Y]^\perp, \quad h(X, Y) = \frac{1}{2}(\nabla_X Y + \nabla_Y X)^\perp.$$

Using the local orthonormal frame $\{E_i,\mathcal{E}_a\}_{i\le p,a\le n}$, one can find the formulas

$$\langle h,h\rangle=\sum_{a,b}\epsilon_a\epsilon_b\,\big(h(E_a,E_b),h(E_a,E_b)\big),$$
$$\langle T,T\rangle=\sum_{a,b}\epsilon_a\epsilon_b\,g\big(T(E_a,E_b),T(E_a,E_b)\big).$$

The mean curvature vector of $\widetilde{\mathcal{D}}$ is $H=\operatorname{Tr}_g h=\sum_a\epsilon_a h(E_a,E_a)$. The distribution $\widetilde{\mathcal{D}}$ is called *totally umbilical*, *harmonic*, or *totally geodesic* if $h=\frac{1}{n}H\widetilde{g}$, $H=0$, or $h=0$, respectively.

The Weingarten operator A_Z of $\widetilde{\mathcal{D}}$ with respect to $Z\in\mathcal{D}$, and the operator $T^\sharp_Z$ are defined by

$$g\big(A_Z(X),Y\big)=g\big(h(X,Y),Z\big),\quad g\big(T^\sharp_Z(X),Y\big)=g\big(T(X,Y),Z\big).$$

For the local orthonormal frame $\{E_i,\,\mathcal{E}_a\}_{i\le p,\,a\le n}$ (adapted to the distributions) we use the following convention for various $(1,1)$-tensors: $\widetilde{T}^\sharp_a:=\widetilde{T}^\sharp_{E_a}$, $A_i:=A_{\mathcal{E}_i}$, etc.

The Divergence Theorem states that $\int_M(\operatorname{div}\xi)\,\mathrm{d}\operatorname{vol}_g=0$ when M is closed; this is also true when M is open and $\xi\in\mathfrak{X}_M$ is supported in a relatively compact domain $\Omega\subset M$. The $\mathcal{D}^\perp$-*divergence* of ξ is defined by $\operatorname{div}^\perp\xi=\sum_i\epsilon_i\,g(\nabla_{\mathcal{E}_i}\xi,\mathcal{E}_i)$. Thus, the divergence of ξ is

$$\operatorname{div}\xi=\operatorname{Tr}(\nabla\xi)=\operatorname{div}^\perp\xi+\widetilde{\operatorname{div}}\xi.$$

Recall that for a vector field $X\in\mathfrak{X}_{\mathcal{D}}$,

$$\operatorname{div}^\perp X=\operatorname{div}X+g(X,H).\tag{2.7}$$

Indeed, using $H=\sum_{a\le n}\epsilon_a h(E_a,E_a)$ and $g(X,E_a)=0$, one derives (2.7):

$$\operatorname{div}X-\operatorname{div}^\perp X=\sum_a\epsilon_a\,g(\nabla_{E_a}X,E_a)=-\sum_a\epsilon_a\,g\big(h(E_a,E_a),X\big)=-g(X,H).$$

For a $(1,2)$-tensor P define a $(0,2)$-tensor $\operatorname{div}^\perp P$ by

$$(\operatorname{div}^\perp P)(X,Y)=\sum_i\epsilon_i\,g\big((\nabla_{\mathcal{E}_i}P)(X,Y),\mathcal{E}_i\big).$$

Then the divergence of P is $(\operatorname{div}P)(X,Y)=\widetilde{\operatorname{div}}\,P+\operatorname{div}^\perp P$. For a $\mathcal{D}$-valued $(1,2)$-tensor P, similarly to (2.7), we have $\sum_a\epsilon_a\,g\big((\nabla_{E_a}P)(X,Y),E_a\big)=-g(P(X,Y),H)$ and

$$\operatorname{div}^\perp P=\operatorname{div}P+\langle P,H\rangle,\tag{2.8}$$

where $\langle P,H\rangle(X,Y):=g(P(X,Y),H)$ is a $(0,2)$-tensor. For example, $\operatorname{div}^\perp h=\operatorname{div}h+\langle h,H\rangle$.

For any function f on M, we introduce the following notation of the projections of its gradient onto distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$:

$$\nabla^\top f \equiv \widetilde{\nabla} f := (\nabla f)^\top, \quad \nabla^\perp f := (\nabla f)^\perp.$$

The $\widetilde{\mathcal{D}}$-Laplacian of a function f is given by the formula $\widetilde{\Delta}\, f = \widetilde{\operatorname{div}}\,(\widetilde{\nabla}\, f)$.

To study the problem we introduce several tensors.

Definition 2.1. The *$\mathcal{D}$-deformation* $\operatorname{Def}_{\mathcal{D}} Z$ of a vector field Z (e.g., $Z = H$) is the symmetric part of ∇Z restricted to $\mathcal{D}$,

$$2\operatorname{Def}_{\mathcal{D}} Z(X,Y) = g(\nabla_X Z, Y) + g(\nabla_Y Z, X), \quad X, Y \in \mathfrak{X}_{\mathcal{D}}.$$

The antisymmetric part of ∇Z restricted to $\mathcal{D}$ is regarded as a 2-form $d_{\mathcal{D}} Z$,

$$2 d_{\mathcal{D}} Z(X,Y) = g(\nabla_X Z, Y) - g(\nabla_Y Z, X), \quad X, Y \in \mathfrak{X}_{\mathcal{D}}.$$

Define self-adjoint $(1,1)$-tensors: $\mathcal{A} := \sum_i \epsilon_i A_i^2$ (called the *Casorati operator* of $\mathcal{D}$) and $\mathcal{T} := \sum_i \epsilon_i (T_i^\sharp)^2$. Define the symmetric $(0,2)$-tensor Ψ by the identity

$$\Psi(X,Y) = \operatorname{Tr}(A_Y A_X + T_Y^\sharp T_X^\sharp), \quad X, Y \in \mathfrak{X}_{\mathcal{D}}.$$

Proposition 2.2 (see [2])**.** *Let $g \in \operatorname{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. Then the following identities hold:*

$$\begin{aligned} r_{\mathcal{D}} &= \operatorname{div} \tilde{h} + \langle \tilde{h}, \widetilde{H} \rangle - \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat - \Psi + \operatorname{Def}_{\mathcal{D}} H, \\ d_{\mathcal{D}}\, H &= -\widetilde{\operatorname{div}}\, \widetilde{T} + \sum_a \epsilon_a \big(\widetilde{A}_a \widetilde{T}_a^\sharp + \widetilde{T}_a^\sharp \widetilde{A}_a \big)^\flat. \end{aligned} \tag{2.9}$$

The difference, called the *extrinsic curvature* of $\widetilde{\mathcal{D}}$,

$$R_{\rm ex}(X,Y,Z,W) = g\big(h(X^\top, Z^\top), h(Y^\top, W^\top)\big) - g\big(h(X^\top, W^\top), h(Z^\top, Y^\top)\big),$$

is useful in the study of extrinsic geometry of foliations, see [14]. The traces (along $\widetilde{\mathcal{D}}$),

$$\begin{aligned} \operatorname{Ric}_{\rm ex} &= \sum_a \epsilon_a\, R_{\rm ex}(\cdot, E_a, \cdot, E_a), \\ \operatorname{S}_{\rm ex} &= \sum_b \epsilon_b \operatorname{Ric}_{\rm ex}(E_b,\, E_b) = g(H,H) - \langle h, h \rangle, \end{aligned}$$

are the *extrinsic Ricci* and *scalar* curvatures of $\widetilde{\mathcal{D}}$.

Tracing $(2.9)_1$ over $\mathcal{D}$ and applying (2.6) and the equalities

$$\begin{aligned} \operatorname{Tr}_g \Psi &= \sum_i \epsilon_i \operatorname{Tr}_g (A_i^2 + (T_i^\sharp)^2) = \langle h, h \rangle - \langle T, T \rangle, \\ \operatorname{Tr} \mathcal{A} &= \langle h, h \rangle, \quad \operatorname{Tr} \mathcal{T} = -\langle T, T \rangle, \\ \operatorname{Tr}_g(\operatorname{div} h) &= \operatorname{div} H, \quad \operatorname{Tr}_g(\operatorname{Def}_{\mathcal{D}} H) = \operatorname{div} H + g(H,H) \end{aligned}$$

yield the formula (see also [17])

$$\operatorname{S}_{\rm mix} = \operatorname{S}_{\rm ex} + \widetilde{\operatorname{S}}_{\rm ex} + \langle T, T \rangle + \langle \widetilde{T}, \widetilde{T} \rangle + \operatorname{div}(H + \widetilde{H}), \tag{2.10}$$

which shows how $\operatorname{S}_{\rm mix}$ is built of the invariants of the extrinsic geometry of the distributions.

2.2. Variation formulas. Given an adapted pseudo-Riemannian metric g on $(M,\widetilde{\mathcal{D}},\mathcal{D})$, consider smooth 1-parameter variations of $g_0=g$,

$$\Big\{g_t\in \operatorname{Riem}(M,\widetilde{\mathcal{D}},\mathcal{D}):|t|<\varepsilon\Big\}. \tag{2.11}$$

Assume that the induced infinitesimal variations, presented by a symmetric $(0,2)$-tensor $B_t\equiv(\partial g_t/\partial t)\in\mathfrak{M}$, are supported in a relatively compact domain Ω in M. We adopt the notations

$$\partial_t\equiv\partial/\partial t,\quad B\equiv\partial_t g_t|_{t=0}\,. \tag{2.12}$$

Taking into account (2.3), it is sufficient to work with special curves $\{g_t\}_{|t|<\varepsilon}$ starting at $g\in\operatorname{Riem}(M,\widetilde{\mathcal{D}},\mathcal{D})$ called $g^\perp$-*variations*:

$$\Big\{g_t^\perp+\widetilde{g}:\ |t|<\varepsilon\Big\}, \tag{2.13}$$

as the associated infinitesimal variations B_t lie in $\mathfrak{M}_{\mathcal{D}}$. All the variational formulas for $B_t\in\mathfrak{M}_{\widetilde{\mathcal{D}}}$ have the same form as those obtained for variations (2.13), only with the roles of distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ interchanged.

For variations (2.11), (2.12) it is known how the Levi-Civita connection changes, see [16],

$$2g_t\big(\partial_t(\nabla^t_X Y),Z\big)=\big(\nabla^t_X B\big)(Y,Z)+\big(\nabla^t_Y B\big)(X,Z)-\big(\nabla^t_Z B\big)(X,Y),\quad X,Y,Z\in\mathfrak{X}_M. \tag{2.14}$$

Lemma 2.3. *Let a local $(\widetilde{\mathcal{D}},\mathcal{D})$-adapted frame $\{E_a,\mathcal{E}_i\}$ evolve by (2.11), (2.12) according to*

$$\partial_t E_a=-\frac12 B_t^\sharp(E_a),\qquad \partial_t\mathcal{E}_i=-\frac12 B_t^\sharp(\mathcal{E}_i).$$

Then, for all t, $\{E_a(t),\mathcal{E}_i(t)\}$ is a g_t-orthonormal frame adapted to $(\widetilde{\mathcal{D}},\mathcal{D})$.

Proof. For $\{E_a(t)\}$ (and similarly for $\{\mathcal{E}_i(t)\}$), we have

$$\begin{aligned}\partial_t\big(g_t(E_a,E_b)\big)&=g_t\big(\partial_t E_a(t),E_b(t)\big)+g_t\big(E_a(t),\partial_t E_b(t)\big)+\big(\partial_t g_t\big)(E_a(t),E_b(t))\\&=B_t\big(E_a(t),E_b(t)\big)-\frac12\,g_t(B_t^\sharp\big(E_a(t)),E_b(t)\big)\\&\quad-\frac12\,g_t\big(E_a(t),B_t^\sharp(E_b(t))\big)=0.\end{aligned}$$ □

Lemma 2.4 (see [2])**.** *For $g^\perp$-variations (2.11), (2.12) and $X,Y\in\mathfrak{X}_{\mathcal{D}}$, $Z\in\mathfrak{X}_{\widetilde{\mathcal{D}}}$, we have*

$$2g\big(\partial_t\widetilde{h}(X,Y),Z\big)=g\big((\widetilde{h}-\widetilde{T})(B^\sharp(X),Y)+(\widetilde{h}+\widetilde{T})\big(X,B^\sharp(Y)\big),Z\big)-\widetilde{\nabla}_Z B(X,Y), \tag{2.15}$$

$$2\partial_t\widetilde{H}=-\widetilde{\nabla}\big(\operatorname{Tr}B^\sharp\big),\quad \partial_t h=-B^\sharp\circ h,\quad \partial_t H=-B^\sharp(H). \tag{2.16}$$

Thus, $g^\perp$-variations preserve the properties of $\widetilde{\mathcal{D}}$: to be totally umbilical, totally geodesic and harmonic distribution.

Define symmetric $(0,2)$-tensors Φ_h and Φ_T (the last one vanishes when $n = 1$ and always when the suitable distribution is integrable), using the identities (with arbitrary $B \in \mathfrak{M}$)

$$\langle \Phi_h, B\rangle = B(H,\,H) - \sum_{a,b} \epsilon_a\epsilon_b\, B\big(h(E_a,E_b), h(E_a,E_b)\big),$$
$$\langle \Phi_T, B\rangle = -\sum_{a,b} \epsilon_a\epsilon_b\, B\big(T(E_a,E_b), T(E_a,E_b)\big).$$

Similarly, we define symmetric tensors $\Phi_{\widetilde{h}}$ and $\Phi_{\widetilde{T}}$. We have $\operatorname{Tr}_g \Phi_h = \mathrm{S}_{\rm ex}$ and $\operatorname{Tr}_g \Phi_T = -\langle T, T\rangle$. Define a $(1,1)$-tensor (with zero trace)

$$\mathcal{K} = \sum_i \epsilon_i \big[T_i^\sharp, A_i\big] = \sum_i \epsilon_i \big(T_i^\sharp A_i - A_i T_i^\sharp\big).$$

Remark 2.5. 1) Let g be definite on $\widetilde{\mathcal{D}}$. Then $\Phi_h = 0$ if and only if one of the following point-wise conditions holds:

(i) $h = 0$;

(ii) $H \neq 0$, $\mathrm{S}_{\rm ex} = 0$ and the image of h is spanned by H.

To show this, consider any vector $X \in \mathcal{D}$ such that $g(X,H) = 0$. Then

$$\begin{aligned}\big\langle \Phi_h, X^\flat \otimes X^\flat\big\rangle &= g(X,H)^2 - \sum_{a,b} \epsilon_a\epsilon_b\, g\big(X, h(E_a,E_b)\big)^2\\ &= -\sum_{a,b} \epsilon_a\epsilon_b\, g\big(X, h(E_a,E_b)\big)^2.\end{aligned}$$

Since all ϵ_a are of the same sign, the above sum is equal to zero if and only if every summand vanishes. Moreover, $\langle \Phi_h, H^\flat \otimes H^\flat\rangle = g(H,H)\,\mathrm{S}_{\rm ex}$ holds. Similarly, if $\Phi_T = 0$, then we have

$$\big\langle \Phi_T, X^\flat \otimes X^\flat\big\rangle = -\sum_{a,b} \epsilon_a\epsilon_b\, g\big(X, T(E_a,E_b)\big)^2 = 0 \quad (X \in \mathcal{D}).$$

Hence, if g is definite on $\widetilde{\mathcal{D}}$ ($\epsilon_a = \epsilon_b$), then the condition $\Phi_T = 0$ is equivalent to $T = 0$. Therefore, Φ_T can be viewed as a measure of non-integrability of $\mathcal{D}$.

2) If $\mathcal{D}$ is integrable, then $\widetilde{T}_a^\sharp = 0$ for all $a = 1,\dots,n$, hence $\widetilde{\mathcal{K}} := \sum_a \epsilon_a [\widetilde{T}_a^\sharp, \widetilde{A}_a] = 0$. Also, if $\mathcal{D}$ is totally umbilical, then every operator $\widetilde{A}_a$ is a multiple of identity and $\widetilde{\mathcal{K}}$ vanishes as well.

Lemma 2.6. *For $g^\perp$-variations, we have*

$$\partial_t \widetilde{\mathrm{S}}_{\rm ex} = \big\langle (\operatorname{div} \widetilde{H}) g^\perp - \operatorname{div} \widetilde{h} - \widetilde{\mathcal{K}}^\flat, B\big\rangle + \operatorname{div}\big(\langle \widetilde{h}, B\rangle - (\operatorname{Tr}_g B)\widetilde{H}\big), \tag{2.17}$$
$$\partial_t \mathrm{S}_{\rm ex} = -\langle \Phi_h, B\rangle, \tag{2.18}$$
$$\partial_t \langle \widetilde{T}, \widetilde{T}\rangle = \big\langle 2\widetilde{\mathcal{T}}^\flat, B\big\rangle, \quad \partial_t \langle T, T\rangle = -\langle \Phi_T, B\rangle. \tag{2.19}$$

Proof. Assume $\nabla_{E_a}\mathcal{E}_i \in \widetilde{\mathcal{D}}_x$ at a point $x \in M$. In the calculations below we use (2.14), Lemmas 2.3 and 2.4 and vanishing of B on $\widetilde{\mathcal{D}}$. First we obtain $(2.19)_1$:

$$
\begin{aligned}
\partial_t\langle\widetilde{T},\widetilde{T}\rangle &= 2\sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a g\big(\widetilde{T}(\mathcal{E}_i,\mathcal{E}_j),E_a\big)g\big(\widetilde{T}(\partial_t\mathcal{E}_i,\mathcal{E}_j)+\widetilde{T}(\mathcal{E}_i,\partial_t\mathcal{E}_j),E_a\big)\\
&= -\sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a g\left(\widetilde{T}(\mathcal{E}_i,\mathcal{E}_j),E_a\right)g\left(\widetilde{T}(B^\sharp(\mathcal{E}_i),\mathcal{E}_j)+\widetilde{T}(\mathcal{E}_i,B^\sharp(\mathcal{E}_j)),E_a\right)\\
&= -\sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a g(\widetilde{T}^\sharp_a(\mathcal{E}_i),\mathcal{E}_j)g\left((\widetilde{T}^\sharp_a B^\sharp + B^\sharp\widetilde{T}^\sharp_a)(\mathcal{E}_i),\mathcal{E}_j\right)\\
&= -\sum_{i,a}\epsilon_i\epsilon_a g\left((\widetilde{T}^\sharp_a B^\sharp + B^\sharp\widetilde{T}^\sharp_a)(\mathcal{E}_i),\widetilde{T}^\sharp_a(\mathcal{E}_i)\right)\\
&= \sum_{i,a}\epsilon_i\epsilon_a g\left(((\widetilde{T}^\sharp_a)^2 B^\sharp + \widetilde{T}^\sharp_a B^\sharp\widetilde{T}^\sharp_a)(\mathcal{E}_i),\mathcal{E}_i\right)\\
&= 2\operatorname{Tr}\Big(B^\sharp\sum_a\epsilon_a(\widetilde{T}^\sharp_a)^2\Big) = 2\operatorname{Tr}\big(\widetilde{\mathcal{T}}B^\sharp\big) = \langle 2\widetilde{\mathcal{T}}^\flat, B\rangle.
\end{aligned}
$$

Next, by (2.8) and (2.15), we obtain

$$
\begin{aligned}
\partial_t\langle\widetilde{h},\widetilde{h}\rangle &= 2\sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a\, g\big(\widetilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\big)g\big(\partial_t(\widetilde{h}(\mathcal{E}_i,\mathcal{E}_j)),E_a\big)\\
&= 2\sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a\, g\big(\widetilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\big)\\
&\qquad\times g\big((\partial_t\widetilde{h})(\mathcal{E}_i,\mathcal{E}_j)+\widetilde{h}(\partial_t\mathcal{E}_i,\mathcal{E}_j)+\widetilde{h}(\mathcal{E}_i,\partial_t\mathcal{E}_j),E_a\big)\\
&= \sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a\, g\big(\widetilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\big)\\
&\qquad\times\Big(g\big(\widetilde{T}(\mathcal{E}_i,B^\sharp(\mathcal{E}_j))-\widetilde{T}B^\sharp(\mathcal{E}_i),\mathcal{E}_j),E_a\big)-\nabla_{E_a}B(\mathcal{E}_i,\mathcal{E}_j)\Big)\\
&= \sum_{i,j,a}\epsilon_i\epsilon_j\epsilon_a\Big(g\big(\widetilde{A}_a(\mathcal{E}_i),\mathcal{E}_j\big)g\big([B^\sharp,\widetilde{T}^\sharp_a](\mathcal{E}_i),\mathcal{E}_j\big)\\
&\qquad-\nabla_{E_a}\Big(B(\mathcal{E}_i,\mathcal{E}_j)g\big(\widetilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\big)\Big)\\
&\qquad-\nabla_{E_a}g\big(\widetilde{h}(\mathcal{E}_i,\mathcal{E}_j),E_a\big)B(\mathcal{E}_i,\mathcal{E}_j)\Big)\\
&= \sum_{i,a}\epsilon_i\epsilon_a\Big(B\big(\widetilde{T}^\sharp_a(\mathcal{E}_i),\widetilde{A}_a(\mathcal{E}_i)\big)+B\big(\mathcal{E}_i,\widetilde{T}^\sharp_a\widetilde{A}_a(\mathcal{E}_i)\big)\Big)\\
&\qquad+\big\langle\widetilde{\operatorname{div}}\,\widetilde{h}-\langle\widetilde{h},\widetilde{H}\rangle,B\big\rangle-\operatorname{div}\big(\langle\widetilde{h},B\rangle\big)\\
&= \langle\operatorname{div}\widetilde{h}+\widetilde{\mathcal{K}}^\flat,B\rangle-\operatorname{div}\langle\widetilde{h},B\rangle.
\end{aligned}
$$

Here we used $\big(\widetilde{T}^\sharp_a\big)^* = -\widetilde{T}^\sharp_a$, $\big(\widetilde{A}_a\big)^* = \widetilde{A}_a$ and $\big(B^\sharp\big)^* = B^\sharp$, hence

$$
\begin{aligned}
\operatorname{Tr}\big(\widetilde{T}^\sharp_a\widetilde{A}_a B^\sharp\big) &= \operatorname{Tr}\big(B^\sharp(\widetilde{T}^\sharp_a\widetilde{A}_a)^*\big) = \operatorname{Tr}\big((\widetilde{T}^\sharp_a\widetilde{A}_a)^* B^\sharp\big)\\
&= \operatorname{Tr}\big(\widetilde{A}_a(\widetilde{T}^\sharp_a)^* B^\sharp\big) = -\operatorname{Tr}\big(\widetilde{A}_a\widetilde{T}^\sharp_a B^\sharp\big).
\end{aligned}
$$

Next, we get (2.17), applying $B(\widetilde{H},\widetilde{H})=0$ and

$$\partial_t g\big(\widetilde{H},\widetilde{H}\big)=2g(\partial_t\widetilde{H},\widetilde{H})=-g\big(\nabla(\operatorname{Tr}B^\sharp),\widetilde{H}\big).$$

Notice that $g\big(\nabla(\operatorname{Tr}B^\sharp),\widetilde{H}\big)=\operatorname{div}\big((\operatorname{Tr}B^\sharp)\widetilde{H}\big)-(\operatorname{div}\widetilde{H})\operatorname{Tr}B^\sharp$. We have

$$\begin{aligned}\partial_t g(H,H)&=B(H,H)+2g(\partial_t H,\,H)=B(H,H)-2g(B^\sharp(H),H)=-B(H,H),\\ \partial_t\langle h,h\rangle&=\partial_t\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b\,g\big(h(E_a,\,E_b),\mathcal{E}_i\big)^2\\ &=2\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b\,g\big(h(E_a,E_b),\mathcal{E}_i\big)\partial_t g\big(h(E_a,\,E_b),\mathcal{E}_i\big)\\ &=-\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b g\big(h(E_a,\,E_b),\mathcal{E}_i\big)g\big(h(E_a,E_b),B^\sharp(\mathcal{E}_i)\big)\\ &=-\sum_{a,b}\epsilon_a\epsilon_b\,B\big(h(E_a,E_b),h(E_a,E_b)\big).\end{aligned}$$

From the above, (2.18) follows. Finally, we have: $(2.19)_2$:

$$\begin{aligned}\partial_t\langle T,T\rangle&=\partial_t\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b\,g\big(T(E_a,E_b),\mathcal{E}_i\big)^2\\ &=2\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b\,g\big(T(E_a,E_b),\mathcal{E}_i\big)\partial_t\big(g\big(T(E_a,E_b),\mathcal{E}_i\big)\big)\\ &=2\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b\,g\big(T(E_a,E_b),\mathcal{E}_i\big)\Big(B\big(T(E_a,E_b),\mathcal{E}_i\big)+g\big(T(E_a,E_b),\partial_t\mathcal{E}_i\big)\Big)\\ &=\sum_{i,a,b}\epsilon_i\epsilon_a\epsilon_b\,g\big(T(E_a,E_b),\mathcal{E}_i\big)g\big(T(E_a,\,E_b),B^\sharp(\mathcal{E}_i)\big)\\ &=\sum_{a,b}\epsilon_a\epsilon_b\,B\big(T(E_a,E_b),T(E_a,E_b)\big).\end{aligned}$$

□

2.3. Euler–Lagrange equations. In [1], $J_{\mathrm{mix},\Omega}$ was considered as an analogue of the Einstein–Hilbert action $g\mapsto\int_\Omega S(g)\mathrm{d}\,\mathrm{vol}_g$, where $S(g)$ is the scalar curvature of metric g. However, to obtain the gravity part of the Einstein field equation as the Euler–Lagrange equation for the Einstein–Hilbert action, one does not consider all compactly supported variations of the metric – because for this kind of variation in $\dim M\neq 2$ only Ricci-flat metrics are critical. Instead, variations of the metric are required to keep the volume of the considered relatively compact domain Ω, and in consequence yield a more interesting, broader class of metrics (the Einstein metrics) as the critical points for the Einstein–Hilbert action.

In analogy to this approach, we consider variations that preserve the volume of Ω, yet still can be presented in the form (2.13). To obtain them, we conformally rescale arbitrary variations (2.13) on the distribution $\mathcal{D}$ by a function ϕ that — for our convenience — depends only on the parameter of variation t (i.e., it is constant on Ω for any given t). While these volume preserving variations are not any more compactly supported inside Ω (they do not vanish on the boundary of Ω for this particular choice of function ϕ), they accomplish our goal leading to "inhomogeneous" versions of the Euler–Lagrange equations obtained for variations (2.13).

In this section, we derive directional derivatives (2.4) and the Euler–Lagrange equations of the action $J_{\mathrm{mix},\Omega}$ on an open pseudo-Riemannian almost-product manifold for two types of $g^\perp$-variations (i.e., either preserving the volume or not).

For arbitrary $f \in L^1(\Omega, \mathrm{d}\,\mathrm{vol}_g)$, denote by

$$f(\Omega, g) = \mathrm{Vol}^{-1}(\Omega, g)\int_\Omega f\,\mathrm{d}\,\mathrm{vol}_g$$

the mean value of f on Ω. Together with a family g_t of (2.13), consider on Ω the metrics

$$\bar g_t = \phi_t g_t^\perp + \widetilde g, \quad \phi_t := \big(\mathrm{Vol}(\Omega, g_t)/\,\mathrm{Vol}(\Omega, g)\big)^{-2/p}, \quad |t| < \varepsilon. \tag{2.20}$$

Recall, see [14], that the volume form evolves as

$$\partial_t\big(\mathrm{d}\,\mathrm{vol}_{g_t}\big) = \frac{1}{2}\big(\,\mathrm{Tr}_{g_t} B_t\big)\mathrm{d}\,\mathrm{vol}_{g_t}. \tag{2.21}$$

We will show that $\mathrm{Vol}(\Omega, \bar g_t) = \mathrm{Vol}(\Omega, g)$ for all t. As $\bar g_t$ are $\mathcal{D}$-conformal to g_t with constant scale ϕ_t, their volume forms are related as

$$\mathrm{d}\,\mathrm{vol}_{\bar g_t} = \phi_t^{p/2}\mathrm{d}\,\mathrm{vol}_{g_t}; \tag{2.22}$$

hence, $\mathrm{Vol}(\Omega, \bar g_t) = \int_\Omega \mathrm{d}\,\mathrm{vol}_{\bar g_t} = \mathrm{Vol}(\Omega, g)$. Let us differentiate (2.22) in order to obtain

$$\begin{aligned}\partial_t\big(\mathrm{d}\,\mathrm{vol}_{\bar g_t}\big) &= \big(\phi_t^{p/2}\big)'\mathrm{d}\,\mathrm{vol}_{g_t} + \phi_t^{p/2}\partial_t\big(\mathrm{d}\,\mathrm{vol}_{g_t}\big)\\ &= \frac{1}{2}\Big(\,\mathrm{Tr}\, B_t^\sharp - \big(\,\mathrm{Tr}_{g_t} B_t\big)(\Omega, g_t)\Big)\mathrm{d}\,\mathrm{vol}_{\bar g_t}.\end{aligned}$$

We have used (2.21) and the fact that $\phi_0 = 1$ and

$$\phi_t' = -\frac{\phi_t}{p}\big(\,\mathrm{Tr}_{g_t} B_t\big)(\Omega, g_t). \tag{2.23}$$

For $\bar g_t = \phi_t g_t + \widetilde g$, see (2.20), we have, see [1],

$$\begin{array}{lll} H_{\bar g} = \phi^{-1} H, & \langle T, T\rangle_{\bar g} = \phi\,\langle T, T\rangle_g, & \langle \widetilde T, \widetilde T\rangle_{\bar g} = \phi^{-2}\langle \widetilde T, \widetilde T\rangle_g,\\ \widetilde H_{\bar g} = \widetilde H, & \langle h_{\bar g}, h_{\bar g}\rangle_{\bar g} = \phi^{-1}\langle h, h\rangle_g, & \bar g(H_{\bar g}, H_{\bar g}) = \phi^{-1} g(H, H),\\ \widetilde h_{\bar g} = \phi\,\widetilde h, & \langle \widetilde h_{\bar g}, \widetilde h_{\bar g}\rangle_{\bar g} = \langle \widetilde h, \widetilde h\rangle_g, & \bar g(\widetilde H_{\bar g}, \widetilde H_{\bar g}) = g(\widetilde H, \widetilde H), \end{array} \tag{2.24}$$

where subscript $\bar g$ corresponds to geometric quantities calculated with respect to $\bar g$.

Next we give several technical lemmas.

Lemma 2.7. *For all $g^\perp$-variations* (2.11), (2.12) *and all $g^\perp$-variations preserving the volume of Ω, the evolution of* div *on a t-dependent vector field X is given by the formula*

$$\partial_t(\operatorname{div} X) = \operatorname{div}\big(\partial_t, X\big) + \frac{1}{2} X\big(\operatorname{Tr} B^\sharp\big). \tag{2.25}$$

Proof. First, consider the arbitrary $g^\perp$-variation g_t. Differentiating the formula $\operatorname{div} X \mathrm{d}\operatorname{vol}_g = \mathcal{L}_X(\mathrm{d}\operatorname{vol}_g)$, see [11], we obtain (2.25).

Extending the arbitrary vector $X \in T_xM$ and the basis $\{E_a, \mathcal{E}_i\}$ at T_xM to vector fields on a neighborhood of x, we obtain from the formula for the Levi-Civita connection the following:

$$\begin{aligned}
2\bar g\big(\overline\nabla_{E_a} X, E_a\big) &= 2\bar g\big([E_a, X], E_a\big) + X(\bar g(E_a, E_a))\\
&= 2g\big([E_a, X], E_a\big) + X\big(g(E_a, E_a)\big) = 2g\big(\nabla_{E_a} X, E_a\big),\\
2\bar g\big(\overline\nabla_{\phi^{-1/2}\mathcal{E}_i} X, \phi^{-1/2}\mathcal{E}_i\big) &= 2\phi^{-1}\bar g\big([\mathcal{E}_i, X], \mathcal{E}_i\big) + X\big(\phi^{-1}\bar g(\mathcal{E}_i, \mathcal{E}_i)\big)\\
&= 2g\big([\mathcal{E}_i, X], \mathcal{E}_i\big) + X\big(g(\mathcal{E}_i, \mathcal{E}_i)\big) = 2g\big(\nabla_{\mathcal{E}_i} X, \mathcal{E}_i\big).
\end{aligned}$$

It follows that for $g^\perp$-variations preserving the volume of Ω, the divergence $\operatorname{div}_{\bar g}$ with respect to metric $\bar g = \phi g^\perp + \widetilde g$ is given by

$$\begin{aligned}
\operatorname{div}_{\bar g} X &= \sum_a \epsilon_a\, \widetilde g\big(\overline\nabla_{E_a} X, E_a\big) + \sum_i \epsilon_i\, \phi g^\perp\big(\bar\nabla_{\phi^{-1/2}\mathcal{E}_i} X, \phi^{-1/2}\mathcal{E}_i\big)\\
&= \sum_a \epsilon_a\, g\big(\nabla_{E_a} X, E_a\big) + \sum_i \epsilon_i\, g\big(\nabla_{\mathcal{E}_i} X, \mathcal{E}_i\big) = \operatorname{div} X.
\end{aligned}$$

Hence, again we obtain (2.25). □

Lemma 2.8. *For any $g^\perp$-variation, g_t and $\bar g_t$ of* (2.20), *supporting in $\Omega \subset M$, we have*

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \operatorname{div}(H + \widetilde H)\mathrm{d}\operatorname{vol}_g = \begin{cases} 0 & \textit{for } g_t,\\ \frac{1}{2}\operatorname{div}\big(\frac{2-p}{p} H - \widetilde H\big)(\Omega, g)\int_\Omega (\operatorname{Tr}_g B)\mathrm{d}\operatorname{vol}_g & \textit{for } \bar g_t. \end{cases}$$

Proof. Using the equations for time derivatives of mean curvatures and the volume form, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \operatorname{div}\big(H + \widetilde H\big)\,\mathrm{d}\operatorname{vol}_g\\
&\qquad = \int_\Omega \partial_t\Big(\operatorname{div}\big(H + \widetilde H\big)\Big)\mathrm{d}\operatorname{vol}_g + \int_\Omega \operatorname{div}\big(H + \widetilde H\big)\partial_t\big(\mathrm{d}\operatorname{vol}_g\big)\\
&\qquad = -\int_\Omega \operatorname{div}\big(B^\sharp(H)\big)\,\mathrm{d}\operatorname{vol}_g + \int_\Omega \operatorname{div}\big(-\widetilde\nabla\big(\operatorname{Tr} B^\sharp\big)\big)\mathrm{d}\operatorname{vol}_g\\
&\qquad\quad + \frac{1}{2}\int_\Omega \Big(\operatorname{div}\big(\big(\operatorname{Tr} B^\sharp\big)(H + \widetilde H)\big) - \operatorname{div}\big(H + \widetilde H\big)\big(\operatorname{Tr} B^\sharp\big)\\
&\qquad\quad + \big(\operatorname{Tr} B^\sharp\big)\operatorname{div}\big(H + \widetilde H\big)\Big)\mathrm{d}\operatorname{vol}_g
\end{aligned}$$

$$= \int_\Omega \Big(-\operatorname{div}\big(\widetilde\nabla(\operatorname{Tr} B^\sharp)\big) - \operatorname{div}\big(B^\sharp(H)\big) + \frac{1}{2}\operatorname{div}\Big(\big(\operatorname{Tr} B^\sharp\big)\big(H+\widetilde H\big)\Big)\Big)\,\mathrm{d}\operatorname{vol}_g = 0,$$

since all the above terms are integrals of divergences of vector fields supported in Ω.

For $g^\perp$-variations preserving the volume of Ω, all the following derivatives with respect to t will be calculated at $t=0$. By Lemma 2.7 and using (2.23), we have

$$\partial_t\big(\operatorname{div} H_{\bar g}\big) = \partial_t(\operatorname{div} H) + \frac{1}{p}(\operatorname{div} H)\big(\operatorname{Tr}_g B\big)(\Omega, g),$$

while $\widetilde H_{\bar g} = \widetilde H$, see [1], and hence $\partial_t(\operatorname{div} \widetilde H_{\bar g}) = \partial_t\big(\operatorname{div} \widetilde H\big)$. We also have

$$\partial_t\big(\mathrm{d}\operatorname{vol}_{\bar g}\big) = \partial_t\big(\mathrm{d}\operatorname{vol}_g\big) - \frac{1}{2}\big(\operatorname{Tr}_g B\big)(\Omega, g)\,\mathrm{d}\operatorname{vol}_{\bar g}\,.$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \operatorname{div}(H_{\bar g} + \widetilde H_{\bar g})\,\mathrm{d}\operatorname{vol}_{\bar g} &= \int_\Omega \partial_t\big(\operatorname{div}(H+\widetilde H)\big)\,\mathrm{d}\operatorname{vol}_g \\ &\quad + \int_\Omega \operatorname{div}\big(H+\widetilde H\big)\partial_t\big(\mathrm{d}\operatorname{vol}_g\big) \\ &= \big(\operatorname{Tr}_g B\big)(\Omega, g)\int_\Omega \operatorname{div}\Big(\frac{2-p}{2p}H - \frac{1}{2}\widetilde H\Big)\,\mathrm{d}\operatorname{vol}_g\,. \qquad \square\end{aligned}$$

The next result compares adapted variations of the action (2.2) associated with metrics $\bar g_t$ and g_t.

Proposition 2.9. *The $g^\perp$-variations of a metric $g\in\operatorname{Riem}(M,\widetilde{\mathcal D},\mathcal D)$ for the action (2.2) associated with $\bar g_t$ and g_t are related by*

$$\frac{\mathrm{d}}{\mathrm{dt}}J_{\mathrm{mix},\Omega}(\bar g_t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{dt}}J_{\mathrm{mix},\Omega}(g_t)\Big|_{t=0} - \frac{1}{2}\mathrm{S}^*_{\mathrm{mix}}(\Omega, g)\int_\Omega \big(\operatorname{Tr}_g B\big)\,\mathrm{d}\operatorname{vol}_g, \tag{2.26}$$

where

$$\mathrm{S}^*_{\mathrm{mix}} = \mathrm{S}_{\mathrm{mix}} - \frac{2}{p}\big(\mathrm{S}_{\mathrm{ex}} + 2\langle\widetilde T,\widetilde T\rangle - \langle T,T\rangle + \operatorname{div} H\big). \tag{2.27}$$

Proof. Let us fix a $g^\perp$-variation g_t, see $(2.13)_1$. By (2.10) and Lemma 2.8, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}J_{\mathrm{mix},\Omega}(g_t) = \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega Q(g_t)\;\mathrm{d}\operatorname{vol}_{g_t},$$

where $Q(g) := \mathrm{S}_{\mathrm{mix}} - \operatorname{div}(H+\widetilde H)$ is represented using (2.10) as

$$Q(g) = \mathrm{S}_{\mathrm{ex}}(g) + \widetilde{\mathrm{S}}_{\mathrm{ex}}(g) + \langle\widetilde T,\widetilde T\rangle_g + \langle T,T\rangle_g. \tag{2.28}$$

Hence, and by (2.24),

$$Q(\bar g_t) = Q(g_t) + \big(\phi_t^{-1}-1\big)\mathrm{S}_{\mathrm{ex}}(g_t) + \big(\phi_t^{-2}-1\big)\langle\widetilde T,\widetilde T\rangle_{g_t} + \big(\phi_t - 1\big)\langle T,T\rangle_{g_t}.$$

Differentiating the above at $t = 0$ and using $\phi_0 = 1$, we get

$$\partial_t Q(\bar g_t)|_{t=0} = \partial_t Q(g_t)|_{t=0} - \phi_0'\big(\mathrm{S}_{\rm ex}(g) + 2\langle\widetilde T,\widetilde T\rangle_g - \langle T,T\rangle_g\big),$$

where $\phi_0' = -\frac{1}{p}\,(\operatorname{Tr}_g B)(\Omega,g)$, see (2.23). Using Lemma 2.8, we obtain

$$\frac{\rm d}{\rm dt} J_{{\rm mix},\Omega}(g_t)\Big|_{t=0} = \int_\Omega \Big\{\,\partial_t Q(g_t)|_{t=0} + \frac12 Q(g)\operatorname{Tr}_g B\Big\}\,{\rm d\,vol}_g,$$

$$\frac{\rm d}{\rm dt} J_{{\rm mix},\Omega}(\bar g_t)\Big|_{t=0} = \int_\Omega \Big\{\,\partial_t Q(\bar g_t)|_{t=0} + \frac12 Q(g)\big(\operatorname{Tr}_g B + p\phi_0'\big)\Big\}\,{\rm d\,vol}_g$$
$$+ \frac{\rm d}{\rm dt}\int_\Omega {\rm div}\,\big(H_{\bar g_t} + \widetilde H_{\bar g_t}\big)\,{\rm d\,vol}_{\bar g_t\,|\,t=0}. \tag{2.29}$$

Hence,

$$\begin{aligned}\frac{\rm d}{\rm dt} J_{{\rm mix},\Omega}(\bar g_t)\Big|_{t=0} &= \int_\Omega \Big\{\,\partial_t Q(\bar g_t)|_{t=0} + \frac12 Q(g)\big(\operatorname{Tr}_g B - (\operatorname{Tr}_g B)(\Omega,g)\big)\Big\}\,{\rm d\,vol}_g\\ &\quad + \frac{\rm d}{\rm dt}\int_\Omega {\rm div}(H_{\bar g_t} + \widetilde H_{\bar g_t})\,{\rm d\,vol}_{\bar g_t}\Big|_{t=0}\\ &= \int_\Omega \partial_t Q(g_t)|_{t=0}\ {\rm d\,vol}_g + \frac12\int_\Omega Q(g)(\operatorname{Tr}_g B)\,{\rm d\,vol}_g\\ &\quad + \frac12\bigg(\Big(\frac2p\Big(\mathrm{S}_{\rm ex} + 2\langle\widetilde T,\widetilde T\rangle_g - \langle T,T\rangle_g + {\rm div}\,H\Big)\Big)(\Omega,g)\\ &\quad - \Big(Q(g) - {\rm div}(H+\widetilde H)\Big)(\Omega,g)\bigg)\int_\Omega(\operatorname{Tr}_g B)\,{\rm d\,vol}_g\,.\end{aligned}$$

Using definition of $Q(g)$ and (2.27) we get (2.26). □

Remark 2.10. It should be stressed that as in [2], we work with two types of variations of metric, (2.13) and (2.20); the second of which preserves the volume of Ω. The formulas containing $\mathrm{S}^*_{\rm mix}$ correspond to (2.20). To obtain similar formulas, corresponding to 1-parameter variations of the form (2.13), one should merely delete the mean value terms $\mathrm{S}^*_{\rm mix}(\Omega,g)$ in the previous identities. Considering a closed manifold M instead of Ω, we obtain

$$\mathrm{S}^*_{\rm mix} = \mathrm{S}_{\rm mix} - \frac2p\big(\mathrm{S}_{\rm ex} + 2\langle\widetilde T,\widetilde T\rangle - \langle T,T\rangle\big)\quad \text{(for } g^\perp\text{-variations)}.$$

The next theorem gives the Euler–Lagrange equations of the variational principle $\delta J_{{\rm mix},\Omega}(g) = 0$ on a relatively compact domain Ω of a manifold M with an almost-product structure. These have a form $P = \lambda\tilde g$ (on $\widetilde{\mathcal D}$) and $P = \lambda\, g^\perp$ (on $\mathcal D$) for certain tensors P and functions λ on M.

Theorem 2.11 (Euler–Lagrange equations)**.** *A metric* $g \in \mathrm{Riem}(M,\widetilde{\mathcal D},\mathcal D)$ *is critical for the action* (2.2) *with respect to* $g^\perp$*-variations* (2.11), (2.12) *if and only if*

$$r_{\mathcal D} - \langle\tilde h,\widetilde H\rangle + \widetilde{\mathcal A}^\flat - \widetilde{\mathcal T}^\flat + \Phi_h + \Phi_T + \Psi - {\rm Def}_{\mathcal D} H + \widetilde{\mathcal K}^\flat$$
$$= \frac12\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega,g) + {\rm div}(\widetilde H - H)\big)g^\perp. \tag{2.30}$$

Proof. Applying Lemma 2.6 to (2.28), using (2.8) and removing integrals of divergences of vector fields compactly supported in Ω, we get

$$\int_\Omega \partial_t Q|_{t=0}\,\mathrm{d}\,\mathrm{vol}_g = \int_\Omega \big\langle\,\mathrm{div}(\widetilde{H}\,g^\perp - \widetilde{h}) + 2\widetilde{\mathcal{T}}^\flat - \Phi_h - \Phi_T - \widetilde{\mathcal{K}}^\flat, B\big\rangle\,\mathrm{d}\,\mathrm{vol}_g,$$

where $B = \{\partial_t g_t\}|_{t=0} \in \mathfrak{M}_{\mathcal{D}}$. Notice that $\mathrm{Tr}_g\,B = \langle B, g^\perp\rangle$. Then, by (2.29), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\,J_{\mathrm{mix},\Omega}(g_t)|_{t=0} = \int_\Omega \Big\langle\,\mathrm{div}\,\big(\widetilde{H} g^\perp - \widetilde{h}\big) + 2\widetilde{\mathcal{T}}^\flat - \Phi_h - \Phi_T - \widetilde{\mathcal{K}}^\flat + \frac{1}{2}\big(\mathrm{S}_{\mathrm{mix}} - \mathrm{div}(H + \widetilde{H})\big)g^\perp, B\Big\rangle\,\mathrm{d}\,\mathrm{vol}_g\,. \qquad (2.31)$$

By (2.31) and Proposition 2.9, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}} J_{\mathrm{mix},\Omega}(\bar{g}_t)\Big|_{t=0} = \int_\Omega \Big\langle\,\mathrm{div}(\widetilde{H} g^\perp - \widetilde{h}) + 2\widetilde{\mathcal{T}}^\flat - \Phi_h - \Phi_T - \widetilde{\mathcal{K}}^\flat + \frac{1}{2}\big(\mathrm{S}_{\mathrm{mix}} - \mathrm{S}^*_{\mathrm{mix}}(\Omega, g) - \mathrm{div}(\widetilde{H} + H)\big)g^\perp, B\Big\rangle\,\mathrm{d}\,\mathrm{vol}_g\,. \quad (2.32)$$

If the metric g is critical for the action $J_{\mathrm{mix},\Omega}$ with respect to $g^\perp$-variations, then the integral in (2.32) is zero for arbitrary symmetric tensor $B \in \mathfrak{M}$ vanishing on $\widetilde{\mathcal{D}}$. That yields

$$\mathrm{div}\,\widetilde{h} - 2\widetilde{\mathcal{T}}^\flat + \Phi_h + \Phi_T + \widetilde{\mathcal{K}}^\flat = \frac{1}{2}\,\big(\mathrm{S}_{\mathrm{mix}} - \mathrm{S}^*_{\mathrm{mix}}(\Omega, g) + \mathrm{div}(\widetilde{H} - H)\big)g^\perp. \quad (2.33)$$

Using the partial Ricci tensor, see Proposition 2.2, and replacing $\mathrm{div}\widetilde{h}$ in (2.33) according to $(2.9)_1$, we rewrite (2.33) as (2.30). □

Remark 2.12. Differentiating (2.20) with respect to t, we see that variations $\partial_t\bar{g}$ preserving the volume of Ω do not vanish on the boundary of Ω, and hence they are not a subclass of adapted variations (2.11). Comparing (2.33) for general adapted variations and those given by (2.20), one can see that metrics critical for variations (2.11) supported inside a domain Ω remain critical for variations (2.20) preserving the volume of this set if and only if the following equality holds:

$$(p-2)\int_\Omega \mathrm{div}(\widetilde{H} - H)\mathrm{d}\,\mathrm{vol}_g = 0.$$

The above equality is satisfied in particular when we consider as Ω the entire, closed manifold M, see [2]. Also note that for general adapted variations (2.11) the Euler–Lagrange equations are supposed to hold everywhere on M, since the set Ω containing the support of a variation is assumed to be arbitrary. On the other hand, metrics critical with respect to variations preserving the volume of a fixed domain Ω satisfy the Euler–Lagrange equation only at the points of Ω (and since Ω explicitly appears in that equation, it cannot be assumed free).

Example 2.13 (Hopf fibrations). Let both distributions be totally geodesic. Then (2.30) reads

$$r_{\mathcal{D}} - \widetilde{\mathcal{T}}^{\flat} + \Phi_T + \Psi = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega, g)\big)g^{\perp},$$

where $\Psi(X,Y) = \mathrm{Tr}_g(T^\sharp_Y T^\sharp_X)$ $(X, Y \in \mathfrak{X}_{\mathcal{D}})$. Also $\mathrm{S}_{\rm mix} = \langle T, T\rangle + \langle \widetilde{T}, \widetilde{T}\rangle$ and, see (2.27),

$$\mathrm{S}^*_{\rm mix} = \frac{1}{p}\big((p-4)\langle \widetilde{T}, \widetilde{T}\rangle + (p+2)\langle T, T\rangle\big).$$

Note that this is the case of Hopf fibrations, when $\widetilde{\mathcal{D}}$ is a non-integrable, totally geodesic distribution with integrable orthogonal complement.

2.4. Total extrinsic scalar curvature. The variational formulas from Section 2.2 can be applied also to other functionals depending on extrinsic geometry of distributions. In particular, we can consider integrals of extrinsic scalar curvatures $\widetilde{\mathrm{S}}_{\rm ex}$ and $\mathrm{S}_{\rm ex}$. Since the variational formulas for both these quantities are similar, we shall examine only $\widetilde{\mathrm{S}}_{\rm ex}$. We consider adapted variations of the functional

$$J_{\widetilde{\rm ex},\Omega}(g) : g \to \int_\Omega \widetilde{\mathrm{S}}_{\rm ex}(g)\,\mathrm{d}\,\mathrm{vol}_g\,. \tag{2.34}$$

Note that for $p = 1$ we have $\widetilde{\mathrm{S}}_{\rm ex} = 0$ for any metric.

Proposition 2.14 (Euler–Lagrange equations)**.** *Let* $\dim \mathcal{D} > 1$. *A pseudo-Riemannian metric* $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$ *is critical for the action* (2.34) *with respect to all adapted variations if and only if*

$$\operatorname{div}\tilde{h} + \widetilde{\mathcal{K}}^{\flat} = -\frac{1}{2(p-1)}(\widetilde{\mathrm{S}}_{\rm ex} - \widetilde{\mathrm{S}}^*_{\rm ex}(\Omega, g))g^{\perp} \qquad (\textit{for } g^{\perp}\textit{-variations}), \tag{2.35}$$

$$\Phi_{\tilde{h}} = \frac{1}{n}\widetilde{\mathrm{S}}_{\rm ex}\tilde{g}, \quad \textit{and if } \ n \ne 2 \textit{ then } \widetilde{\mathrm{S}}_{\rm ex} = \widetilde{\mathrm{S}}^*_{\rm ex}(\Omega, g) \quad (\textit{for } \tilde{g}\textit{-variations}), \tag{2.36}$$

where $\widetilde{\mathrm{S}}^*_{\rm ex} = \widetilde{\mathrm{S}}_{\rm ex}$ *for variations* $\bar{g}_t$ *(preserving* $\mathrm{Vol}(\Omega, g)$*), and* $\widetilde{\mathrm{S}}^*_{\rm ex} = 0$ *for variations* g_t.

Proof. The formula for $g^{\perp}$-variation of $\widetilde{\mathrm{S}}_{\rm ex}$ was given in (2.17), and we can write the $\tilde{g}$-variation of $\widetilde{\mathrm{S}}_{\rm ex}$ from (2.18) as

$$\partial_t \widetilde{\mathrm{S}}_{\rm ex} = -\langle \Phi_{\tilde{h}}, B\rangle, \tag{2.37}$$

interchanging the roles of $\mathcal{D}$ and $\widetilde{\mathcal{D}}$. Using (2.17), (2.21), (2.37), and removing divergences of compactly supported vector fields, we obtain for $g^{\perp}$-variations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} J_{\widetilde{\rm ex},\Omega}(g_t)\Big|_{t=0} &= \int_\Omega (\partial_t \widetilde{\mathrm{S}}_{\rm ex})\,\mathrm{d}\,\mathrm{vol}_g + \int_\Omega \widetilde{\mathrm{S}}_{\rm ex}\,\partial_t(\mathrm{d}\,\mathrm{vol}_g)\\ &= \int_\Omega \Big\langle \big(\operatorname{div}\widetilde{H}\big)g^{\perp} - \operatorname{div}\tilde{h} - \widetilde{\mathcal{K}}^{\flat}, B\Big\rangle\,\mathrm{d}\,\mathrm{vol}_g + \frac{1}{2}\int_\Omega \widetilde{\mathrm{S}}_{\rm ex}\big(\mathrm{Tr}_g\,B\big)\mathrm{d}\,\mathrm{vol}_g\end{aligned}$$

$$= \int_\Omega \Big\langle \big(\operatorname{div} \widetilde H\big) g^\perp - \operatorname{div} \widetilde h - \widetilde{\mathcal K}^\flat + \frac12 \widetilde{\rm S}_{\rm ex} g^\perp, B \Big\rangle \,{\rm d}\,{\rm vol}_g,$$

and for $\widetilde g$-variations:

$$\frac{\rm d}{{\rm d}t} J_{\widetilde{\rm ex},\Omega}(g_t)\Big|_{t=0} = \int_\Omega \Big\langle -\Phi_{\widetilde h} + \frac12 \widetilde{\rm S}_{\rm ex} \widetilde g, B \Big\rangle \,{\rm d}\,{\rm vol}_g\,.$$

In the case of variations preserving the volume of Ω, using the notation of Section 2.3 and methods employed in the proof of Proposition 2.9, we get for $g^\perp$-variations:

$$\frac{\rm d}{{\rm d}t} J_{\widetilde{\rm ex},\Omega}(\bar g_t)\Big|_{t=0} = \frac{\rm d}{{\rm d}t} J_{\widetilde{\rm ex},\Omega}(g_t)\Big|_{t=0} - \frac12 \widetilde{\rm S}_{\rm ex}(\Omega, g) \int_\Omega \big(\operatorname{Tr}_g B\big)\,{\rm d}\,{\rm vol}_g,$$

and for $\widetilde g$-variations:

$$\frac{\rm d}{{\rm d}t} J_{\widetilde{\rm ex},\Omega}(\bar g_t)\Big|_{t=0} = \frac{\rm d}{{\rm d}t} J_{\widetilde{\rm ex},\Omega}(g_t)\Big|_{t=0} - \frac{n-2}{2n} \widetilde{\rm S}_{\rm ex}(\Omega, g) \int_\Omega \big(\operatorname{Tr}_g B\big)\,{\rm d}\,{\rm vol}_g\,.$$

Therefore, we obtain the following Euler–Lagrange equations for the action (2.34) (terms $\widetilde{\rm S}^*_{\rm ex}$ appear only in the case of $\bar g_t$-variations preserving the volume of Ω):

$$\operatorname{div} \widetilde h + \widetilde{\mathcal K}^\flat = \frac12 \Big(2 \operatorname{div} \widetilde H + \widetilde{\rm S}_{\rm ex} - \widetilde{\rm S}^*_{\rm ex}(\Omega, g)\Big)\, g^\perp \qquad (\text{for } g^\perp\text{-variations}), \tag{2.38}$$

$$\Phi_{\widetilde h} = \frac12 \Big(\widetilde{\rm S}_{\rm ex} - \frac{n-2}{n} \widetilde{\rm S}^*_{\rm ex}(\Omega, g)\Big)\, \widetilde g \qquad (\text{for } \widetilde g\text{-variations}). \tag{2.39}$$

Taking traces of (2.38) and (2.39) yields

$$\operatorname{div} \widetilde H = \frac{p}{2(1-p)} \Big(\widetilde{\rm S}_{\rm ex} - \widetilde{\rm S}^{\,*}_{\rm ex}(\Omega, g)\Big), \quad (n-2)\Big(\frac12 \widetilde{\rm S}_{\rm ex} - \widetilde{\rm S}^*_{\rm ex}(\Omega, g)\Big) = 0. \tag{2.40}$$

Using (2.40) in (2.38) and (2.39) completes the proof. □

3. Particular cases: foliations

In this section, we assume that a pseudo-Riemannian manifold (M^{n+p}, g) is endowed with an n-dimensional foliation $\mathcal F$. Since $\widetilde{\mathcal D} = T\mathcal F$, we write $r_{\mathcal F} = r_{\widetilde{\mathcal D}}$ and obtain dual to (2.9) equations

$$r_{\mathcal F} = \operatorname{div} h + \langle h, H\rangle - \mathcal A^\flat - \widetilde\Psi + {\rm Def}_{\mathcal F} \widetilde H, \qquad d_{\mathcal F} \widetilde H = 0. \tag{3.1}$$

Note that $\Psi(X, Y) = \operatorname{Tr}_g(A_Y A_X)$. Definition (2.27) takes the form

$${\rm S}^*_{\rm mix} = {\rm S}_{\rm mix} - \begin{cases} \frac2p \big({\rm S}_{\rm ex} + 2\langle \widetilde T, \widetilde T\rangle + \operatorname{div} H\big) & (\text{for } g^\perp \text{variations}), \\ \frac2n \big(\widetilde{\rm S}_{\rm ex} - \langle \widetilde T, \widetilde T\rangle + \operatorname{div} \widetilde H\big) & (\text{for } \widetilde g\text{-variations}). \end{cases} \tag{3.2}$$

From Theorem 2.11, we obtain the following.

Corollary 3.1 (Euler–Lagrange equations)**.** *Let $\mathcal{F}$ be a foliation with a transversal distribution $\mathcal{D}$ on M. Then a metric $g \in \mathrm{Riem}(M, T\mathcal{F}, \mathcal{D})$ is critical for the action* (2.2) *with respect to all adapted variations,* (2.11), (2.12)*, if and only if*

$$\begin{aligned} & r_{\mathcal{D}} - \langle \widetilde{h}, \widetilde{H}\rangle + \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat + \Phi_h + \Psi - \mathrm{Def}_{\mathcal{D}}\, H + \widetilde{\mathcal{K}}^\flat \\ & \quad = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega, g) + \mathrm{div}(\widetilde{H} - H)\big) g^\perp \qquad (\textit{for } g^\perp\textit{-variations}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} & r_{\mathcal{F}} - \langle h, H\rangle + \mathcal{A}^\flat + \Phi_{\widetilde{h}} + \Phi_{\widetilde{T}} + \widetilde{\Psi} - \mathrm{Def}_{\mathcal{F}} \widetilde{H} \\ & \quad = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega, g) + \mathrm{div}(H - \widetilde{H})\big) \widetilde{g} \qquad (\textit{for } \widetilde{g}\textit{-variations}) . \end{aligned} \tag{3.4}$$

The system (3.3), (3.4) (called the "mixed gravitational field equations") admits a certain number of solutions (e.g., twisted products and isoparametric foliations, see below), and we propose that they should have applications in theoretical physics, see discussion in [1].

A pseudo-Riemannian manifold may admit many different geometrically interesting types of foliations: totally geodesic ($h = 0$) and Riemannian ($\widetilde{h} = 0$) foliations are the most common examples; totally umbilical ($h = \frac{1}{n} H \widetilde{g}$) and conformal ($\widetilde{h} = \frac{1}{p} \widetilde{H} g^\perp$) foliations are also popular. The simple examples of geodesic foliations are parallel circles or winding lines on a flat torus.

Example 3.2. Let $\mathcal{F}$ be a totally umbilical foliation (i.e., $h = \frac{1}{n} H \widetilde{g}$ and $T = 0$). Then

$$\Phi_h = \frac{n-1}{n} H^\flat \otimes H^\flat,\ \mathcal{A}^\flat = \frac{1}{n^2} g(H, H) \widetilde{g},\ \Psi = \frac{1}{n} H^\flat \otimes H^\flat,\ \mathrm{S}_{\rm ex} = \frac{n-1}{n} g(H, H).$$

Hence, the fundamental equation $(2.9)_1$ and the Euler–Lagrange equation (3.3) read as

$$r_{\mathcal{D}} - \mathrm{div}\, \widetilde{h} - \langle \widetilde{h}, \widetilde{H}\rangle + \widetilde{\mathcal{A}}^\flat + \widetilde{\mathcal{T}}^\flat + \frac{1}{n} H^\flat \otimes H^\flat - \mathrm{Def}_{\mathcal{D}} H = 0, \tag{3.5}$$

$$\begin{aligned} & r_{\mathcal{D}} - \langle \widetilde{h}, \widetilde{H}\rangle + \widetilde{\mathcal{A}}^\flat - \widetilde{\mathcal{T}}^\flat + H^\flat \otimes H^\flat - \mathrm{Def}_{\mathcal{D}}\, H + \widetilde{\mathcal{K}}^\flat \\ & \quad = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega, g) + \mathrm{div}(\widetilde{H} - H)\big) g^\perp \quad (\text{for } g^\perp\text{-variations}). \end{aligned} \tag{3.6}$$

3.1. Critical adapted metrics. Using the natural representation of $\mathrm{O}(p) \times \mathrm{O}(n)$ on the tangent bundle TM, A.M. Naveira [10] described thirty-six different classes of Riemannian almost-product manifolds; some of them are foliations, e.g., harmonic, totally umbilical and totally geodesic. Following this line of research, several geometers completed the geometric interpretation and gave examples for each class. We will characterize critical metrics for the action (2.2) in some distinguished classes of foliations.

Theorem 3.3. *Let complementary orthogonal distributions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ determine totally umbilical foliations with $n, p > 1$ of a pseudo-Riemannian manifold*

(M,g). *Then* g *is critical for the action* (2.2) *with respect to* $g^\perp$*-variations if and only if the leaves of* $\widetilde{\mathcal D}$ *are totally geodesic and*

$$r_{\mathcal D}=\frac{1}{p}\,{\rm S}_{\rm mix}\,g^\perp\quad with\quad \begin{cases}({\rm div}\,\widetilde H)(\Omega,g)=0 & if\ p\ne 2,\\ {\rm S}_{\rm mix}={\rm const} & if\ p=2.\end{cases}\tag{3.7}$$

Proof. We have the identity, see (3.5) with $\widetilde T=0$ and $\tilde h=\frac{1}{p}\widetilde H g^\perp$,

$$r_{\mathcal D}+\frac{1}{n}H^\flat\otimes H^\flat-{\rm Def}_{\mathcal D}H=\frac{1}{p}\left(\frac{p-1}{p}g(\widetilde H,\widetilde H)+{\rm div}\,\widetilde H\right)g^\perp.\tag{3.8}$$

Hence, or by (2.10),

$$ {\rm S}_{\rm mix}=\frac{n-1}{n}g(H,H)+\frac{p-1}{p}g(\widetilde H,\widetilde H)+{\rm div}(H+\widetilde H).$$

Let the metric g be critical with respect to $g^\perp$-variations. By (3.6), we have

$$\begin{aligned}&r_{\mathcal D}+H^\flat\otimes H^\flat-{\rm Def}_{\mathcal D}H\\ &\quad=\frac{1}{2}\left({\rm S}_{\rm mix}-{\rm S}^*_{\rm mix}(\Omega,g)+\frac{2(p-1)}{p^2}g(\widetilde H,\widetilde H)+{\rm div}(\widetilde H-H)\right)g^\perp.\end{aligned}\tag{3.9}$$

The difference of (3.9) and (3.8) is

$$\begin{aligned}&\frac{n-1}{n}H^\flat\otimes H^\flat\\ &\quad=\frac{1}{2}\left(\frac{n-1}{n}g(H,H)+\frac{p-1}{p}g(\widetilde H,\widetilde H)-{\rm S}^*_{\rm mix}(\Omega,g)+\frac{2(p-1)}{p}\,{\rm div}\,\widetilde H\right)g^\perp.\end{aligned}$$

As the symmetric $(0,2)$-tensor $H^\flat\otimes H^\flat$ has rank ≤ 1 and $g^\perp$ has rank $p>1$, we obtain $H=0$; hence, the leaves of $\widetilde{\mathcal D}$ are totally geodesic. By (3.8), the tensor $r_{\mathcal D}$ is conformal on $\mathcal D$ (i.e., $\mathcal D$-conformal). We also get ${\rm S}^*_{\rm mix}={\rm S}_{\rm mix}$ and

$$ {\rm S}_{\rm mix}+\frac{p-2}{p}\,{\rm div}\,\widetilde H={\rm S}^*_{\rm mix}(\Omega,g).$$

Thus, $\int_\Omega({\rm div}\,\widetilde H)\,{\rm d\,vol}=0$ for $p\ne 2$. The proof of opposite statement is similar. □

Example 3.4. Let $M=M_1\times M_2$ be the product of pseudo-Riemannian manifolds (M_i,g_i) $(i\in\{1,2\})$, and let $\pi_i:M\to M_i$ and $d\pi_i:TM\to TM_i$ be canonical projections. Given twisting functions $f_i\in C^\infty(M)$, a *double-twisted product* $M_1\times_{(f_1,f_2)}M_2$ is M with the metric $g=e^{f_1}\pi_1^*g_1+e^{f_2}\pi_2^*g_2$. If $f_1={\rm const}$, then we have a twisted product (a warped product if, in addition, $f_2=F\circ\pi_1$ for some $F\in C^\infty(M_1)$). The leaves $M_1\times\{y\}$ (tangent to $\widetilde{\mathcal D}$) and the fibers $\{x\}\times M_2$ (tangent to $\mathcal D$) are totally umbilical in (M,g) and this property characterizes double-twisted products (cf. [12]). For any double-twisted product, we have $T=0$ and

$$A_Y=-Y(f_1)\widetilde{\rm id},\quad h=-(\nabla^\perp f_1)\tilde g,\quad H=-n\nabla^\perp f_1,$$

(and similarly for $\widetilde{T}, \widetilde{A}_X, \widetilde{h}, \widetilde{H}$) where $X \in \widetilde{\mathcal{D}}$ and $Y \in \mathcal{D}$ are unit vectors. In this case, see (2.7),

$$\operatorname{div} \widetilde{H} = -p\widetilde{\Delta} f_2 - p^2 g(\widetilde{\nabla} f_2, \widetilde{\nabla} f_2), \quad \operatorname{div} H = -n\Delta^\perp f_1 - n^2 g(\nabla^\perp f_1, \nabla^\perp f_1).$$

By (2.10),

$$\mathrm{S}_{\rm mix} = \operatorname{div}(H + \widetilde{H}) + \frac{n-1}{n} g(H, H) + \frac{p-1}{p} g(\widetilde{H}, \widetilde{H}).$$

Let g be critical for the action (2.2) with respect to $g^\perp$-variations. By Theorem 3.3, the leaves (of $\widetilde{\mathcal{D}}$) are totally geodesic, and (3.7) hold. Note that $\widetilde{\Delta} e^{pf_2} = e^{pf_2}[p\widetilde{\Delta} f_2 + p^2 g(\widetilde{\nabla} f_2, \widetilde{\nabla} f_2)]$.

Summarizing, we conclude that a pseudo-Riemannian double-twisted product metric g is critical for (2.2) with respect to $g^\perp$-variations if and only if the following conditions hold:

(i) $r_{\mathcal{D}}$ is $\mathcal{D}$-conformal;

(ii) if $p \neq 2$, then $\widetilde{\Delta} e^{p\,f_2} = 0$; hence, $e^{p\,f_2}$ is $\widetilde{\mathcal{D}}$-harmonic when $g_{|\widetilde{\mathcal{D}}}$ is definite (recall that nonconstant positive harmonic functions exist on a complete manifold with nonnegative curvature outside a compact set [9]; by S.T. Yau's theorem (1975), there are no nonconstant positive harmonic functions on a complete manifold with nonnegative Ricci curvature);

(iii) f_1 does not depend on M_2; hence, it is the twisted product of $(M_1\,, e^{f_1} g_1)$ and $(M_2\,, g_2)$.

The following theorem continues Example 2.13: one of distributions becomes integrable.

Theorem 3.5. *Let a distribution $\mathcal{D}$ be nowhere integrable and $\widetilde{\mathcal{D}}$ tangent to a totally geodesic Riemannian foliation on a pseudo-Riemannian manifold (M, g). Then g is critical for the action (2.2) with respect to $g^\perp$-variations if and only if*

$$r_{\mathcal{D}} = (\mathrm{S}_{\rm mix}/p) g^\perp, \quad \textit{where } \mathrm{S}_{\rm mix} = \mathrm{const} \quad \textit{when } p \neq 4. \tag{3.10}$$

Proof. By conditions, $p > 1$, $h = 0 = \widetilde{h}$ and $T = 0$. Thus, $(2.9)_1$ reads as

$$r_{\mathcal{D}} = -\widetilde{\mathcal{T}}^\flat. \tag{3.11}$$

By tracing (3.11), we find $\mathrm{S}_{\rm mix} = \langle \widetilde{T}, \widetilde{T} \rangle$. From (3.3), we obtain

$$r_{\mathcal{D}} - \widetilde{\mathcal{T}}^\flat = \frac{1}{2}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega, g)\big) g^\perp \quad (\text{for } g^\perp\text{-variations}), \tag{3.12}$$

where $\mathrm{S}^*_{\rm mix} = \frac{p-4}{p}\langle \widetilde{T}, \widetilde{T} \rangle$, see (3.2). Adding (3.11) and (3.12), we obtain

$$r_{\mathcal{D}} = \frac{1}{4}\big(\mathrm{S}_{\rm mix} - \mathrm{S}^*_{\rm mix}(\Omega, g)\big) g^\perp. \tag{3.13}$$

Tracing (3.13), we get $(p-4)\mathrm{S}_{\rm mix} = p\mathrm{S}^*_{\rm mix}(\Omega, g)$, hence, $\mathrm{S}_{\rm mix} = \mathrm{const}$ when $p \neq 4$. This and (3.13) complete the proof. □

Theorem 3.6. *Let $\mathcal{F}$ be a totally geodesic foliation of a pseudo-Riemannian manifold (M,g) with integrable normal bundle $\mathcal{D}$. Then g is critical for the action (2.2) with respect to all adapted variations if and only if*

(i) $\operatorname{div}\Big(\widetilde h - \dfrac{1}{p}\widetilde H\, g^\perp\Big) = 0,$

(ii) $\Phi_{\widetilde h} = \dfrac{1}{n}\widetilde{\rm S}_{\rm ex}\widetilde g$ *and* ${\rm S}_{\rm ex} = {\rm const}$ *when* $n \ne 2$.

Proof. Using (2.33) and its dual with $\widetilde T = 0$, rewrite Euler–Lagrange equations (3.3), (3.4) as

$$\operatorname{div}(\widetilde h - \widetilde H g^\perp) + \Phi_h = \frac{1}{2}\big({\rm S}_{\rm ex} + \widetilde{\rm S}_{\rm ex} - {\rm S}^*_{\rm mix}(\Omega, g)\big)g^\perp \quad (\text{for } g^\perp\text{-variations}), \tag{3.14}$$

$$\operatorname{div}(h - H\widetilde g) + \Phi_{\widetilde h} = \frac{1}{2}\big({\rm S}_{\rm ex} + \widetilde{\rm S}_{\rm ex} - {\rm S}^*_{\rm mix}(\Omega, g)\big)\widetilde g \quad (\text{for } \widetilde g\text{-variations}). \tag{3.15}$$

We need to show the following (for totally geodesic foliations with integrable normal bundle):

(I) g is critical for the action $J_{{\rm mix},\Omega}$ with respect to $g^\perp$-variations if and only if (i) holds;

(II) g is critical for the action $J_{{\rm mix},\Omega}$ with respect to $\widetilde g$-variations if and only if (ii) holds.

(I) To show necessity of (i), observe that for totally geodesic foliations $h = 0$; hence, (3.14) reads

$$\operatorname{div}(\widetilde h - \widetilde H g^\perp) = \frac{1}{2}\big(\widetilde{\rm S}_{\rm ex} - {\rm S}^*_{\rm mix}(\Omega, g)\big)g^\perp, \tag{3.16}$$

where ${\rm S}^*_{\rm mix} = {\rm S}_{\rm mix}$. Tracing of (3.16) yields

$$(1-p)\operatorname{div}\widetilde H = \frac{p}{2}\big(\widetilde{\rm S}_{\rm ex} - {\rm S}^*_{\rm mix}(\Omega, g)\big). \tag{3.17}$$

Therefore, from (3.16) and (3.17) we obtain (i): $\operatorname{div}(\widetilde h - (\widetilde H/p)\, g^\perp) = 0$.

(II) To show necessity of (ii), from (3.15) with $h = 0$, we obtain for $T\mathcal{F}$-variations,

$$\Phi_{\widetilde h} = \frac{1}{2}\big(\widetilde{\rm S}_{\rm ex} - {\rm S}^*_{\rm mix}(\Omega, g)\big)\widetilde g, \tag{3.18}$$

where ${\rm S}^*_{\rm mix} = {\rm S}_{\rm mix} - \frac{2}{n}(\widetilde{\rm S}_{\rm ex} + \operatorname{div}\widetilde H)$. Tracing of (3.18) yields

$$\widetilde{\rm S}_{\rm ex} = \frac{n}{2}\big(\widetilde{\rm S}_{\rm ex} - {\rm S}^*_{\rm mix}(\Omega, g)\big). \tag{3.19}$$

From (3.18) and (3.19), we obtain $\Phi_{\widetilde h} = \frac{1}{n}\widetilde{\rm S}_{\rm ex}\widetilde g$. It also follows from (3.19) that $\frac{n-2}{n}\widetilde{\rm S}_{\rm ex} = \frac{n}{2}{\rm S}^*_{\rm mix}(\Omega, g)$ for $n \ne 2$. The proof of opposite statements is similar. □

In light of Theorems 3.5 and 3.6, it might be interesting to study totally geodesic foliations

(a) with totally geodesic normal bundle and for which (3.10) holds;

(b) with integrable normal bundle and for which conditions (i) and (ii) hold.

3.2. Flows ($n = 1$). Let the distribution $\widetilde{\mathcal{D}}$ be spanned by a nonsingular vector field N, then N defines a flow (a one-dimensional foliation). An example is provided by a circle action $S^1 \times M \to M$ without fixed points. Assume that $|g(N, N)| = 1$ and let $\epsilon_N = g(N, N)$. Thus, $\mathrm{S}_{\rm mix} = \epsilon_N \operatorname{Ric}_N$, and the partial Ricci tensor takes a particularly simple form:

$$r_{\widetilde{\mathcal{D}}} = \epsilon_N \operatorname{Ric}_N \widetilde{g}, \quad r_{\mathcal{D}} = \epsilon_N (R_N)^\flat,$$

where $R_N = R(N, \cdot)N$ and $\operatorname{Ric}_N = \sum_i \varepsilon_i g(R_N(\mathcal{E}_i), \mathcal{E}_i)$. The action (2.2) reduces itself to

$$J_{\mathrm{mix},\Omega}(g) = \epsilon_N \int_\Omega \operatorname{Ric}_N \mathrm{d}\operatorname{vol}_g . \tag{3.20}$$

We have $\widetilde{h} = \widetilde{h}_{\rm sc} N$, where $\widetilde{h}_{\rm sc} = \epsilon_N \langle \widetilde{h}, N\rangle$ is the scalar second fundamental form of $\mathcal{D}$. Define the functions $\widetilde{\tau}_i = \operatorname{Tr} \widetilde{A}_N^i$ ($i \geq 0$), where $\widetilde{A}_N$ is the (adjoint to $\widetilde{h}_{\rm sc}$) Weingarten operator of $\mathcal{D}$. It is easy to check that $\widetilde{\mathrm{S}}_{\rm ex} = \widetilde{\tau}_1^2 - \widetilde{\tau}_2$ and

$$\begin{aligned} \operatorname{div} N &= \sum_i \epsilon_i g(\nabla_{\mathcal{E}_i} N, \mathcal{E}_i) = -g(N, \sum_i \epsilon_i \nabla_{\mathcal{E}_i}\mathcal{E}_i) = -g(N, \widetilde{H}) = -\widetilde{\tau}_1, \\ \operatorname{div}(\widetilde{\tau}_1 N) &= N(\widetilde{\tau}_1) + \widetilde{\tau}_1 \operatorname{div} N = N(\widetilde{\tau}_1) - \widetilde{\tau}_1^2. \end{aligned}$$

The curvature of the flow lines is $H = \epsilon_N \nabla_N N$. It is easy to see that (3.2) takes the form

$$\mathrm{S}^*_{\rm mix} = \epsilon_N \operatorname{Ric}_N - 2 \begin{cases} \frac{2}{p}\langle \widetilde{T}, \widetilde{T}\rangle + \frac{1}{p}\operatorname{div} H & (\text{for } g^\perp\text{-variations}), \\ \epsilon_N(N(\widetilde{\tau}_1) - \widetilde{\tau}_2) - \langle \widetilde{T}, \widetilde{T}\rangle & (\text{for } \widetilde{g}\text{-variations}). \end{cases}$$

From Theorem 2.11 (or Corollary 3.1) we obtain the following.

Corollary 3.7 (Euler–Lagrange equations)**.** *Let a distribution $\widetilde{\mathcal{D}}$ be spanned by a unit vector field N with respect to the metric $g \in \operatorname{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$. Then g is critical for the action (3.20) with respect to all adapted variations if and only if*

$$\begin{aligned} &\epsilon_N \big(R_N + \widetilde{A}_N^2 - (\widetilde{T}_N^\sharp)^2 + [\widetilde{T}_N^\sharp, \widetilde{A}_N]\big)^\flat - \widetilde{\tau}_1 \widetilde{h}_{\rm sc} + H^\flat \otimes H^\flat - \operatorname{Def}_{\mathcal{D}} H \\ &\quad = \frac{1}{2}\big(\epsilon_N \operatorname{Ric}_N - \mathrm{S}^*_{\rm mix}(\Omega, g) + \operatorname{div}(\epsilon_N \widetilde{\tau}_1 N - H)\big)\, g^\perp \ (\textit{for } g^\perp\textit{-variations}), \end{aligned} \tag{3.21}$$

$$\epsilon_N \operatorname{Ric}_N + \mathrm{S}^*_{\rm mix}(\Omega, g) - 4\langle \widetilde{T}, \widetilde{T}\rangle - \operatorname{div}(\epsilon_N \widetilde{\tau}_1 N + H) = 0 \ (\textit{for } \widetilde{g}\textit{-variations}). \tag{3.22}$$

Proof. An easy computation shows that

$$\begin{gathered} \widetilde{\mathcal{A}} = \epsilon_N \widetilde{A}_N^2, \quad \langle \widetilde{h}_{\rm sc} N, \widetilde{H}\rangle = \widetilde{\tau}_1 \widetilde{h}_{\rm sc}, \quad \Psi = H^\flat \otimes H^\flat, \quad \widetilde{\Psi} = (\epsilon_N \widetilde{\tau}_2 - \langle \widetilde{T}, \widetilde{T}\rangle)\widetilde{g}, \\ \mathcal{A} = g(H, H)\widetilde{\mathrm{id}}, \quad \mathcal{T} = 0, \quad \langle h,\, H\rangle = g(H, H)\widetilde{g}, \\ H = \epsilon_N \nabla_N N, \quad h = H\widetilde{g}, \quad \langle h, h\rangle = g(H, H), \\ \widetilde{H} = \epsilon_N \widetilde{\tau}_1 N, \quad \widetilde{\tau}_1 = \epsilon_N \operatorname{Tr}_g \widetilde{h}_{\rm sc}, \quad \langle \widetilde{h},\ \widetilde{h}\rangle = \epsilon_N \widetilde{\tau}_2, \quad \operatorname{Def}_{\widetilde{\mathcal{D}}} \widetilde{H} = \epsilon_N N(\widetilde{\tau}_1)\widetilde{g}. \end{gathered} \tag{3.23}$$

Notice that $(H^\flat \otimes H^\flat)(X, Y) = g(H, X) g(H, Y)$. Substituting (3.2) and

$$\Phi_h = 0 = \mathrm{S}_{\rm ex}, \quad \widetilde{\mathrm{S}}_{\rm ex} = \epsilon_N(\widetilde{\tau}_1^2 - \widetilde{\tau}_2), \quad \widetilde{\mathcal{T}} = \epsilon_N \widetilde{T}_N^{\sharp 2}$$

into (2.30) yields (3.21). Substituting (3.2) and

$$h = H\,\widetilde{g}, \quad \Phi_{\widetilde{h}} = \epsilon_N(\widetilde{\tau}_1^2 - \widetilde{\tau}_2)\widetilde{g}, \quad \Phi_{\widetilde{T}} = -\langle\widetilde{T},\widetilde{T}\rangle\widetilde{g}$$

into equation dual to (2.30) yields (3.22). □

By (2.8), we have $\operatorname{div}\widetilde{h} = N(\widetilde{h}_{\rm sc}) - \widetilde{\tau}_1\widetilde{h}_{\rm sc}$ and $\operatorname{div} h = (\operatorname{div} H)\widetilde{g}$. Then, see $(2.9)_1$ and (2.10),

$$\begin{aligned}\epsilon_N\big(R_N + \widetilde{A}_N^2 + (\widetilde{T}_N^\sharp)^2\big)^\flat &= N(\widetilde{h}_{\rm sc}) - H^\flat\otimes H^\flat + \operatorname{Def}_{\mathcal D} H,\\ \epsilon_N \operatorname{Ric}_N &= \operatorname{div}(\nabla_N N) + \epsilon_N(N(\widetilde{\tau}_1) - \widetilde{\tau}_2) + \langle\widetilde{T},\widetilde{T}\rangle.\end{aligned} \tag{3.24}$$

Remark that $(3.24)_2$ is simply the trace of $(3.24)_1$.

A flow of a unit vector N is *geodesic* if the orbits are geodesics ($h = 0$), and is *Riemannian* if the metric is bundle-like ($\widetilde{h} = 0$). A nonsingular Killing vector clearly defines a Riemannian flow; moreover, a Killing vector of unit length generates a geodesic Riemannian flow. A manifold with such N-flow is called *Sasakian* if the sectional curvature of every section containing N equals one, in other words, its curvature satisfies the following condition:

$$R(X,N)Y = g(N,Y)X - g(X,Y)N.$$

Corollary 3.8 (of Theorem 3.5)**.** *Let a unit vector field N generates a geodesic Riemannian flow on a pseudo-Riemannian manifold (M^{p+1}, g). Then g is critical for the action (3.20) with respect to $g^\perp$-variations if and only if*

$$R_N = (1/p)\operatorname{Ric}_N \operatorname{id}^\perp, \quad \textit{where } \operatorname{Ric}_N = \operatorname{const} \textit{ when } p \neq 4. \tag{3.25}$$

Moreover, if p is odd, then $\operatorname{Ric}_N = 0$ and M splits, and if $\operatorname{Ric}_N \neq 0$, then p is even.

Proof. By Theorem 3.5, we have (3.25), and (3.11) reads $R_N = -(\widetilde{T}_N^\sharp)^2$. Tracing this we obtain $\epsilon_N \operatorname{Ric}_N = \langle\widetilde{T},\widetilde{T}\rangle$. In our case, (2.27) reads

$$\mathrm{S}^*_{\rm mix} = \begin{cases}\frac{p-4}{p}\langle\widetilde{T},\widetilde{T}\rangle & (\text{for } g^\perp\text{-variations}),\\ 3\langle\widetilde{T},\widetilde{T}\rangle & (\text{for } \widetilde{g}\text{-variations}).\end{cases}$$

For a geodesic Riemannian N-flow, (3.21), (3.22) reduce to

$$\begin{aligned}\epsilon_N\big(R_N - (\widetilde{T}_N^\sharp)^2\big)^\flat &= \frac{1}{2}\big(\epsilon_N \operatorname{Ric}_N - \mathrm{S}^*_{\rm mix}(\Omega,g)\big)\,g^\perp \qquad &(\text{for } g^\perp\text{-variations}),\\ \epsilon_N \operatorname{Ric}_N &= -\mathrm{S}^*_{\rm mix}(\Omega,g) + 4\,\langle\widetilde{T},\widetilde{T}\rangle &(\text{for}\,\widetilde{g}\text{-variations}).\end{aligned}$$

For p being odd, the skew-symmetric operator $\widetilde{T}_N^\sharp$ has zero eigenvalue; hence, $R_N = 0 = \widetilde{T}$; and by de Rham's Decomposition Theorem, (M,g) splits. □

Finally, observe that we can examine codimension-one foliations and distributions with critical metrics for other actions with respect to adapted variations, for example, (2.34). Since the case of $p = 1$ is trivial for this action, we consider $n = 1$ instead. The next result provides applications to foliations whose leaves have constant second mean curvature, see [14, Section 1.1.1].

Proposition 3.9. *Let (M, g) be a pseudo-Riemannian manifold, and a distribution $\widetilde{\mathcal{D}}$ be spanned by a complete in Ω unit vector field N. If g is critical for the action (2.34) with respect to all adapted variations, then $\widetilde{\mathrm{S}}_{\mathrm{ex}}(\Omega, g) \le 0$ and*

$$\widetilde{\tau}_1 = 0, \quad \widetilde{\tau}_2 = \mathrm{const}. \tag{3.26}$$

Proof. From (2.35), we obtain

$$\nabla_N \widetilde{h}_{\mathrm{sc}} - \widetilde{\tau}_1 \widetilde{h}_{\mathrm{sc}} + \epsilon_N \big[\widetilde{T}^\sharp_N, \widetilde{A}_N\big]^\flat = 0.$$

By tracing the above, we get $N(\widetilde{\tau}_1) = \widetilde{\tau}_1^2$, and in view of completeness in Ω of the flow of N, the only solution is $\widetilde{\tau}_1 = 0$, namely $(3.26)_1$. From (2.36) with $n = 1$ and $\Phi_{\widetilde{h}} = \epsilon_N(\widetilde{\tau}_1^2 - \widetilde{\tau}_2)\widetilde{g}$, we obtain

$$\widetilde{\tau}_1^2 - \widetilde{\tau}_2 = \epsilon_N \widetilde{\mathrm{S}}_{\mathrm{ex}}, \quad \widetilde{\mathrm{S}}_{\mathrm{ex}} = \widetilde{\mathrm{S}}_{\mathrm{ex}}(\Omega, g),$$

which together with $\widetilde{\mathrm{S}}_{\mathrm{ex}} = \widetilde{\mathrm{S}}^*_{\mathrm{ex}}$ and $\widetilde{\tau}_1 = 0$ yields $\widetilde{\tau}_2 = -\epsilon_N \widetilde{\mathrm{S}}^*_{\mathrm{ex}}(\Omega, g)$. Hence critical metrics of (2.34) with respect to all adapted variations are those with constant $\widetilde{\tau}_2$. □

For $n = 1$, the critical metrics of the action (2.34) with respect to all adapted variations also satisfy the differential equation $\nabla_N \widetilde{h}_{\mathrm{sc}} + \epsilon_N \big[\widetilde{T}^\sharp_N, \widetilde{A}_N\big]^\flat = 0$, which in the case of integrable $\mathcal{D}$, together with (3.26), yield the system of equations studied in Section 3.3, see (3.50) with interchanged $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, and $\widetilde{\tau}_2(\Omega, g)$ in place of $-\,\mathrm{Ric}_N(\Omega, g)$.

3.3. Codimension-one foliations.

The structure theory and dynamics of codimension-one foliations on manifolds are fairly well understood. The simplest examples of codimension-one foliations are the level surfaces of a function $u : M \to \mathbb{R}$ with no critical points. Geometric properties of such foliations correspond to analytic properties of their defining functions. As a particular example one can consider isoparametric functions (e.g., see [1] for globally hyperbolic spacetime). In this section, we analyze adapted critical metrics of the action (2.2) for codimension-one foliations and give their full classification on 3-dimensional manifolds admitting a certain global coordinate system.

Let $\mathcal{F}$ be a codimension-one foliation with a normal $N \in \mathfrak{X}_M$ of a pseudo-Riemannian manifold (M^{n+1}, g). Assume that $|g(N, N)| = 1$ and let $\epsilon_N = g(N, N)$. We have, see (2.5),

$$r_{\mathcal{D}} = \epsilon_N \,\mathrm{Ric}_N\, g^\perp, \quad r_{\mathcal{F}} = \epsilon_N (R_N)^\flat,$$

where $R_N = R(N,\cdot)N$ and $\mathrm{Ric}_N = \sum_a \varepsilon_a g(R_N(E_a), E_a)$. Then again, as in Section 3.2, the action (2.2) reduces itself to (3.20). Let $h_{\rm sc}$ be the scalar second fundamental form, and A_N the Weingarten operator of $\mathcal{F}$. We have $T = 0 = \widetilde{T}$ and

$$h_{\rm sc}(X,Y) = \epsilon_N g(\nabla_X Y, N), \quad A_N(X) = -\nabla_X N, \quad (X, Y \in T\mathcal{F}).$$

Define the functions $\tau_k = \operatorname{Tr} A_N^k$ $(k \ge 0)$, see [14], which can be expressed using the elementary symmetric functions σ_k's,

$$\det(\mathrm{id} + t\,A_N) = 1 + \sum_{1\le k\le n} \sigma_k\, t^k,$$

called the *k-th mean curvatures* of $\mathcal{F}$. For example, $\tau_1 = \epsilon_N \operatorname{Tr} h_{\rm sc}$ is the "usual" mean curvature of $\mathcal{F}$, and

$$\tau_1 = \sigma_1 = \operatorname{Tr} A_N = -\operatorname{div} N, \quad \tau_2 = \sigma_1^2 - 2\sigma_2 = \operatorname{Tr} A_N^2.$$

Evidently, the functions τ_{n+i} $(i > 0)$ are not independent: they can be expressed as polynomials of $(\tau_1, \dots, \tau_n)$ using the *Newton formulas*

$$\begin{aligned} &\tau_j - \tau_{j-1}\sigma_1 + \ldots + (-1)^{j-1}\tau_1\sigma_{j-1} + (-1)^j j\sigma_j = 0 && (1 \le j \le n),\\ &\tau_j - \tau_{j-1}\sigma_1 + \ldots + (-1)^n \tau_{j-n}\sigma_n = 0 && (j > n). \end{aligned}$$

Notice that $\mathcal{A} = \epsilon_N A_N^2$ and $\widetilde{\mathcal{A}} = g(\widetilde{H}, \widetilde{H})N$, where $\widetilde{H} = \epsilon_N \nabla_N N$ is the curvature vector of N-curves. By $(3.1)_1$ and $\widetilde{\Psi} = \widetilde{H}^\flat \otimes \widetilde{H}^\flat$, we obtain

$$\epsilon_N (R_N + A_N^2)^\flat = \nabla_N h_{\rm sc} - \widetilde{H}^\flat \otimes \widetilde{H}^\flat + \epsilon_N \mathrm{Def}_{\mathcal{F}} \widetilde{H}. \tag{3.27}$$

Then we find, taking trace of (3.27), that (see also [17] in terms of σ's)

$$\mathrm{Ric}_N = N(\tau_1) - \tau_2 + \operatorname{div} \widetilde{H}. \tag{3.28}$$

It is easy to see that (3.2) takes the form

$$\mathrm{S}^*_{\rm mix} = \epsilon_N \,\mathrm{Ric}_N - 2\epsilon_N \begin{cases} N(\tau_1) - \tau_2 & (\text{for } g^\perp\text{-variations}),\\ \frac{1}{n} \operatorname{div} \widetilde{H} & (\text{for } \widetilde{g}\text{-variations}). \end{cases} \tag{3.29}$$

From Theorem 2.11 (or Corollary 3.1) we obtain the following.

Proposition 3.10 (Euler–Lagrange equations). *Let $\mathcal{F}$ be a codimension-one foliation of a pseudo-Riemannian manifold (M^{n+1}, g), and a normal distribution $\mathcal{D}$ be spanned by a unit vector field N. Then g is critical for the action (3.20) with respect to all adapted variations if and only if*

$$\mathrm{Ric}_N + \epsilon_N \mathrm{S}^*_{\rm mix}(\Omega, g) - \big(N(\tau_1) - \tau_1^2\big) - \operatorname{div} \widetilde{H} = 0 \quad (\textit{for } g^\perp\textit{-variations}), \tag{3.30}$$

$$\epsilon_N \big(R_N + A_N^2\big)^\flat - \tau_1 h_{\rm sc} + \widetilde{H}^\flat \otimes \widetilde{H}^\flat - \epsilon_N \mathrm{Def}_{\mathcal{F}}\big(\widetilde{H}\big)$$

$$= \frac{1}{2}\Big(\epsilon_N \operatorname{Ric}_N - \mathrm{S}^*_{\rm mix}(\Omega, g) + \epsilon_N \operatorname{div}\big(\tau_1 N - \widetilde{H}\big)\Big)\,\widetilde{g} \quad (\textit{for } \widetilde{g}\textit{-variations}). \tag{3.31}$$

One may rewrite (3.30), (3.31) *using* (3.27), (3.28), *as*

$$\tau_1^2 - \tau_2 = -\epsilon_N \mathrm{S}^*_{\rm mix}(\Omega, g) \qquad (\textit{for } g^\perp\textit{-variations}), \tag{3.32}$$

$$\begin{aligned}\nabla_N h_{\rm sc} - \tau_1 h_{\rm sc} = \frac{1}{2}\big(2\epsilon_N\big(N(\tau_1) - \tau_1^2\big)\\ + \epsilon_N\big(\tau_1^2 - \tau_2\big) - \mathrm{S}^*_{\rm mix}(\Omega, g)\big)\,\widetilde{g} \quad (\textit{for } \widetilde{g}\textit{-variations}).\end{aligned} \tag{3.33}$$

Remark 3.11. Note that adapted variations provide the same Euler–Lagrange equations as in [1], where the action (3.20) was examined in a foliated globally hyperbolic space-time, and the Euler–Lagrange equations (called the mixed gravitational field equations) were derived using variation formulas for the curvature tensor. There, $\mathcal{D}$ was spanned by a unit (for initial metric g), time-like vector field N with integrable orthogonal distribution $\widetilde{\mathcal{D}}$. Equations (3.32) and (3.33) are formulated in terms of a newly introduced tensor $\operatorname{Ric}_{\mathcal{D}}(g)$, whose trace is denoted by $\operatorname{Scal}_{\mathcal{D}}(g)$. For unit vectors $X, Y \in \widetilde{\mathcal{D}}$, we have in coordinate free form:

$$\begin{aligned}
\operatorname{Ric}_{\mathcal{D}}(g)(X, Y) &= \big(\nabla_N h_{\rm sc} - \tau_1 h_{\rm sc}\big)(X, Y),\\
\operatorname{Ric}_{\mathcal{D}}(g)(X, N) &= \operatorname{div}(A_N(X)),\\
\operatorname{Ric}_{\mathcal{D}}(g)(N, X) &= -\operatorname{div}(A_N(X)),\\
\operatorname{Ric}_{\mathcal{D}}(g)(N, N) &= -\operatorname{div} H.
\end{aligned}$$

The Euler–Lagrange equations in [1] for the action (3.20) take the following form:

$$\operatorname{Ric}_{\mathcal{D}}(g) - \frac{1}{2}\operatorname{Scal}_{\mathcal{D}}(g) g + \operatorname{Ric}_N\big(N^\flat \otimes N^\flat + \frac{1}{2} g\big) = 0. \tag{3.34}$$

Since one should actually use only the symmetric part of $\operatorname{Ric}_{\mathcal{D}}(g)$ in (3.34), its both sides vanish when evaluated on the pair (X, N), where $X \in \widetilde{\mathcal{D}}$. Also, (3.34) reduces to (3.30) when evaluated on the pair (N, N) (with $\mathrm{S}^*_{\rm mix}(\Omega, g) = 0$, because in [1] the volume preserving variations is not considered), while evaluating (3.34) on $X, Y \in \widetilde{\mathcal{D}}$ yields (3.31).

Lemma 3.12. *Let $\mathcal{F}$ be a codimension-one foliation of a pseudo-Riemannian space (M^{n+1}, g), and let g be critical for* (3.20) *with respect to all adapted variations and let the unit normal field N of $\mathcal{F}$ be complete in a domain Ω of M. Then the function* $\operatorname{div}(\nabla_N N)$ *is non-positive somewhere in Ω, and* (3.32), (3.33) *read*

$$\tau_1^2 - \tau_2 = \operatorname{Ric}_N(\Omega, g) - 2\widehat{C}, \qquad \nabla_N h_{\rm sc} - \tau_1 h_{\rm sc} = \frac{\epsilon_N}{n}\widehat{C}\widetilde{g}, \tag{3.35}$$

where $\widehat{C} = \mathrm{const} \le 0$ and τ_1 is a global solution of the following ODE (along N-lines):

$$N(\tau_1) - \tau_1^2 = \widehat{C}. \tag{3.36}$$

Proof. Recall that $\widetilde{H} = \nabla_N N$. Put

$$x = \big(N(\tau_1) - \tau_1^2\big)(\Omega, g),\ y = \big(\tau_1^2 - \tau_2\big)(\Omega, g),\ z = \big(\operatorname{div} \widetilde{H}\big)(\Omega, g),\ J = \operatorname{Ric}_N(\Omega, g).$$

Integrating (3.32) and using (3.29), we obtain $2x + y = J$. Integrating the trace of (3.33) and using (3.32) and (3.29), we obtain $2(n-1)x + ny + 2z = nJ$. The rank 2 linear system

$$\{2x + y = J, \quad 2(n-1)x + ny + 2z = nJ\} \quad (\text{with variables } \ x, y, z)$$

admits a 1-parameter family of solutions $x = z = \widehat{C}$, $y = J - 2\widehat{C}$, where $\widehat{C} \in \mathbb{R}$. Note that integrating of (3.28) yields $x + y + z = J$, which is also satisfied by the above solution. Hence, by tracing (3.33), we obtain (3.36). If $\widehat{C} \ge 0$, then the only global solution of (3.36) along N-lines is $\tau_1 \equiv 0$ (and hence, $\widehat{C} = 0$). Otherwise, if $\widehat{C} < 0$, any global solution $\tau_1(t)$ $(t \in \mathbb{R})$ is bounded: $\tau_1^2(t) \le |\widehat{C}|$ and given by

$$\begin{aligned} \tau_1(t) &= |\widehat{C}|^{1/2}\left(1 - \frac{2(|\widehat{C}|^{1/2} - \tau_1^0)}{(|\widehat{C}|^{1/2} + \tau_1^0)e^{-2t|\widehat{C}|^{1/2}} + |\widehat{C}|^{1/2} - \tau_1^0}\right), \\ \tau_1(0) &= \tau_1^0 \in \left[-|\widehat{C}|^{1/2}, |\widehat{C}|^{1/2}\right], \end{aligned} \tag{3.37}$$

including constant solutions $\tau_1 \equiv \pm|\widehat{C}|^{1/2}$. Since $z \le 0$, the function $\operatorname{div} \widetilde{H}$ is non-positive somewhere in Ω, and (3.32), (3.33) read as (3.35). □

Codimension-one foliations admit *biregular foliated coordinates* $(x_0, \ldots, x_n)$, see [4, Section 5.1], i.e., the leaves are $\{x_0 = c\}$ and N-curves are given by $\{x_i = c_i\ (i > 0)\}$. Now assume that a foliated pseudo-Riemannian manifold $(M, \mathcal{F}, g)$ admits *orthogonal biregular foliated coordinates* (hence, $g_{ij} = 0$ for $i \ne j$), then $g = g_{00}\, dx_0^2 + \sum_{i>0} g_{ii}\,(dx_i)^2$. Denote by $g_{ii,0}$ the derivative of g_{ii} in the ∂_0-direction. We have $g_{00} = \epsilon_N |g_{00}|$ and $g_{ii} = \epsilon_i |g_{ii}|$.

Lemma 3.13. *For a pseudo-Riemannian metric g in orthogonal biregular foliated coordinates of a codimension-one foliation $\mathcal{F}$, one has*

$$\begin{aligned} &N = \partial_0/\sqrt{|g_{00}|} \quad (\textit{the unit normal}), \\ &h_{ij} = \Gamma^0_{ij}\sqrt{g_{00}} = -\frac{1}{2}\epsilon_N\, \delta_{ij} g_{ii,0}/\sqrt{|g_{00}|} \quad (\textit{the second fundamental form}), \\ &A^j_i = -\Gamma^j_{i0}/\sqrt{|g_{00}|} = -\frac{1}{2\sqrt{|g_{00}|}}\delta^j_i \frac{g_{ii,0}}{g_{ii}} \quad (\textit{the Weingarten operator}), \\ &\tau_1 = -\frac{1}{2\sqrt{|g_{00}|}} \sum_{i>0} \frac{g_{ii,0}}{g_{ii}}, \quad \tau_2 = \frac{1}{4|g_{00}|} \sum_{i>0} \left(\frac{g_{ii,0}}{g_{ii}}\right)^2, \quad \textit{etc.} \end{aligned}$$

Proof. This is similar to the proof of Lemma 2.2 in [14] for the Riemannian case. □

Lemma 3.14. *Let $\mathcal{F}$ be a codimension-one foliation of a pseudo-Riemannian manifold (M,g) with a unit normal field N complete in a domain Ω. Let there exist global orthogonal biregular foliated coordinates $(x_0,x_1,\dots,x_n)$, with the leaves of $\mathcal{F}$ given by $\{x_0=c\}$, and g of the form*

$$g_{ii}=\epsilon_i f_i(x_1,\dots,x_n)\exp\left(-2\int\sqrt{|g_{00}|}\,y_i(t,x_1,\dots,x_n)\,\mathrm{d}t\right),\quad i=1,\dots,n, \tag{3.38}$$

where f_i $(i=1,\dots n)$ are positive functions. Then g is critical for the action (3.20) with respect to all adapted variations if and only if $(3.35)_1$ holds and every function $y_i(t,x_1,\dots,x_n)$ solves the first-order linear ODE along the N-curves:

$$y'(t)-\tau_1(t)\sqrt{|g_{00}(t)|}\,y(t)-\frac{1}{n}\widehat{C}\sqrt{|g_{00}(t)|}=0, \tag{3.39}$$

where $\tau_1(t)$ is given by (3.37), $\widehat{C}\le 0$ is a constant and $g_{00}(t)\neq 0$ is a smooth function.

Proof. Let $x_0=t$ and $N=|g_{00}|^{-1/2}\partial_t$. Let $g\in\mathrm{Riem}(M,T\mathcal{F},\mathcal{D})$ be a critical point of the action (3.20) with respect to all adapted variations supported in Ω. Then $\widehat{C}\le 0$, see Lemma 3.12, and τ_1 is a bounded function: $\tau_1^2\le|\widehat{C}|$, see (3.37). Using Lemma 3.13, we obtain

$$(\nabla_N h_{\rm sc})_{ii}=-\frac{\epsilon_N}{2|g_{00}|}\left(g_{ii,00}-\frac{1}{2}g_{ii,0}\frac{|g_{00}|_{,0}}{|g_{00}|}-(g_{ii,0})^2/g_{ii}\right). \tag{3.40}$$

By (3.40), Euler–Lagrange equation $(3.35)_2$ becomes the system for $i=1,\dots,n$:

$$g_{ii,00}-\frac{1}{g_{ii}}(g_{ii,0})^2-g_{ii,0}\left(\frac{1}{2}\frac{|g_{00}|_{,0}}{|g_{00}|}+\tau_1\sqrt{|g_{00}|}\right)+\frac{2}{n}\widehat{C}|g_{00}|g_{ii}=0. \tag{3.41}$$

Substituting (3.38) in (3.41) yields (3.39). The proof of the opposite statement is similar. □

Note that from Lemma 3.13 it follows that for the metric given by (3.38) the Weingarten operator is diagonal in biregular orthogonal foliated coordinates and the functions $y_1,\dots,y_n$ are its eigenvalues (i.e., the principal curvatures of the leaves). Hence, they must satisfy

$$y_1+\dots+y_n=\tau_1,\quad y_1^2+\dots+y_n^2=\tau_2. \tag{3.42}$$

Corollary 3.15. *Let a metric g of the form (3.38) be critical for the action (3.20) with respect to all adapted variations. If $\tau_1=0$ (i.e., $\widetilde{\mathcal{D}}$ is harmonic), then all principal curvatures y_i $(i=1,\dots,n)$ are constant along the N-curves. Also, if $\mathrm{Ric}_N(\Omega,g)=0$, then all y_i vanish.*

Proof. Taking the sum of the left-hand sides of equations (3.39) for $i=1,\dots,n$, we obtain $\widehat{C}=0$, substituting this result together with assumption $\tau_1=0$ to every equation (3.39) yields $\partial_t y_i=0$ for every $i=1,\dots,n$. This proves the first claim. To prove the second one, note that y_i satisfy equation $(3.42)_2$, from which it follows that $\mathrm{Ric}_N(\Omega,g)\le 0$, and if $\mathrm{Ric}_N(\Omega,g)=0$, the only solution of $(3.42)_2$ is $y_i=0$, i.e., a totally geodesic foliation. □

For $n = 2$, in global biregular orthogonal foliated coordinates, we can explicitly (for N complete in Ω) solve the Euler–Lagrange equations for a pseudo-Riemannian metric.

Theorem 3.16. *Let (M, g) be a 3-dimensional pseudo-Riemannian manifold with a 2-dimensional integrable distribution $\widetilde{\mathcal{D}}$, admitting global biregular orthogonal foliated coordinates. Let the unit normal field N of $\widetilde{\mathcal{D}}$ be complete in a domain $\Omega \subset M$. Then g is critical for the action (3.20) with respect to all adapted variations preserving the volume of Ω if and only if τ_1 is constant and one of the following holds:*

$$\begin{gathered}\tau_1 \neq 0, \quad \mathrm{Ric}_N(\Omega, g) = -(3/2)\,(\tau_1)^2,\\ g_{11} = \epsilon_1 f_1 \exp\left(\tau_1 \int \sqrt{|g_{00}|}\,\mathrm{d}t\right), \quad g_{22} = \epsilon_2 f_2 \exp\left(\tau_1 \int \sqrt{|g_{00}|}\,\mathrm{d}t\right),\end{gathered} \tag{3.43}$$

or

$$\begin{gathered}\tau_1 = 0, \quad \mathrm{Ric}_N(\Omega, g) = -\tau_2,\\ g_{11} = \epsilon_1 f_1 \exp\left(\pm 2 \int \sqrt{|g_{00}|}(-\frac{1}{2}\,\mathrm{Ric}_N(\Omega, g))^{1/2}\mathrm{d}t\right),\\ g_{22} = \epsilon_2 f_2 \exp\left(\mp 2 \int \sqrt{|g_{00}|}(-\frac{1}{2}\,\mathrm{Ric}_N(\Omega, g))^{1/2}\mathrm{d}t\right),\end{gathered} \tag{3.44}$$

where $f_i = f_i(x_1, x_2)$ $(i = 1, 2)$ are positive functions and $g_{00}(t) \neq 0$ is a smooth function.

Proof. First we show that for $n = 2$ we must have $\partial_t \tau_1 = 0$. From $(3.42)_1$ and $(3.35)_1$, we obtain a quadratic equation for y, from which it follows that

$$y_{1,2} = \frac{1}{2}\tau_1 \pm \frac{1}{2}\left(\tau_1^2 - 4|\widehat{C}| - 2\,\mathrm{Ric}_N(\Omega, g)\right)^{1/2}, \tag{3.45}$$

where $\widehat{C} \leq 0$ is a constant. Substituting (3.45) into (3.39) yields the following two equations (for the upper and the lower signs) relating τ_1 with g_{00}:

$$\sqrt{|g_{00}|} = \mp \frac{\partial_t \tau_1 \left(\tau_1 \pm \sqrt{\tau_1^2 + G}\right)}{\left(|\widehat{C}| - \tau_1^2 \mp \tau_1 \sqrt{\tau_1^2 + G}\right) \sqrt{\tau_1^2 + G}}, \tag{3.46}$$

where $G = -4\,|\widehat{C}| - 2\,\mathrm{Ric}_N(\Omega, g)$. Since τ_1 satisfies (3.37), for $\partial_t \tau_1 \neq 0$, we obtain from (3.46) two different values, for $|g_{00}|$ we obtain a contradiction. For $\partial_t \tau_1 \equiv 0$, (3.46) also seems to yield a contradiction: $g_{00} = 0$. However, we cannot use (3.46) if $\tau_1^2 + G = 0$ or $-\tau_1^2 \mp \tau_1 \sqrt{\tau_1^2 + G} + |\widehat{C}| = 0$, and we shall see that this is exactly what happens, when we treat the cases $\tau_1 = 0$ and $\tau_1(t, x_1, \ldots, x_n) = \tau_1(x_1, \ldots, x_n) \neq 0$ separately.

1. Let τ_1 be nonzero and constant along the N-curves. According to Lemma 3.12, it is only possible for $\tau_1 = \pm\,|\widehat{C}|^{\frac{1}{2}}$, with $\widehat{C} < 0$. Then from $(3.35)_1$ we obtain

$\tau_2 = -|\widehat{C}| - \operatorname{Ric}_N(\Omega, g) \geq 0$. The principal curvatures of the leaves obey $y_1 + y_2 = \pm|\widehat{C}|^{\frac{1}{2}}$ and $y_1 y_2 = |\widehat{C}| + \frac{1}{2}\operatorname{Ric}_N(\Omega, g)$, and therefore are constant

$$y_{1,2} = \frac{1}{2}\tau_1 \pm \frac{1}{2}\big(- 2\operatorname{Ric}_N(\Omega, g) - 3|\widehat{C}|\big)^{\frac{1}{2}}. \tag{3.47}$$

For g_{11} and g_{22} as in (3.38), we get

$$g_{11} = \epsilon_1 f_1(x_1, x_2) \exp\left(-2y_1 \int \sqrt{|g_{00}|}\,\mathrm{d}t\right),$$
$$g_{22} = \epsilon_2 f_2(x_1, x_2) \exp\left(-2y_2 \int \sqrt{|g_{00}|}\,\mathrm{d}t\right). \tag{3.48}$$

The metric (3.48) is critical for $J_{\rm mix}$ with respect to all adapted variations if and only if $(3.35)_2$ holds, i.e., y_1 and y_2 are both solutions of (3.39). The only solution of (3.39) constant along the N-curves is $y = \frac{|\widehat{C}|}{2\tau_1} = \frac{1}{2}\tau_1$. Comparing this result with (3.47), we see that there exists a metric of the form (3.48) critical with respect to all adapted variations if and only if

$$\operatorname{Ric}_N(\Omega, g) = -\frac{3}{2}|\widehat{C}| = -\frac{3}{2}{\tau_1}^2. \tag{3.49}$$

In this case, we have $y_{1,2} = \frac{1}{2}\tau_1$, and from (3.48) we obtain $(3.16)_2$ and $(3.16)_3$. Note also that from (3.49) it follows that

$$\tau_1^2 + G = |\widehat{C}| - 4|\widehat{C}| - 2\operatorname{Ric}_{N,N}(\Omega, g) = 0,$$

thus, we cannot use (3.46).

2. Consider now the case $\tau_1 = 0$. According to Lemma 3.12, it is only possible for $\widehat{C} = 0$. The system (3.35) reads

$$\tau_2 = -\operatorname{Ric}_N(\Omega, g), \qquad \nabla_N h_{\rm sc} = 0 \tag{3.50}$$

with $\operatorname{Ric}_N(\Omega, g) \leq 0$, while the system (3.41) has the following form:

$$g_{ii,00} - \frac{1}{g_{ii}}(g_{ii,0})^2 - \frac{1}{2}g_{ii,0}(\log|g_{00}|)_{,0} = 0 \quad (i = 1, 2),$$

and $y' = 0$, see (3.39). Thus, $y_1 = -y_2$ are constant along the N-curves. In view of $(3.50)_1$ and assumption $\tau_1 = 0$, the principal curvatures of the leaves are $y_{1,2} = \pm(-\operatorname{Ric}_N(\Omega, g)/2)^{1/2}$. Note that we cannot use equation (3.46) because $|\widehat{C}| - \tau_1^2 \mp \tau_1\sqrt{\tau_1^2 + G} = 0$. □

Corollary 3.17. *Let conditions of Theorem 3.16 hold. If g is critical for the action (3.20) with respect to all adapted variations, then the leaves of the foliation tangent to $\widetilde{\mathcal{D}}$ are either totally umbilical or minimal, with the principal curvatures constant along the N-curves.*

A critical metric for the action (3.20) satisfies (3.35), but we are free to make various assumptions about its behaviour along the leaves of the foliation. In particular, we can assume this foliation to be *isoparametric* (i.e., composed of the level sets of an isoparametric function).

Recall (see [15, Chap. 8]) that a smooth function f without critical points on a pseudo-Riemannian manifold (M, g) is called *isoparametric* if for any vector X tangent to a level hypersurface of f the following conditions are satisfied:

$$X(g(\nabla f, \nabla f)) = 0, \qquad X(\Delta f) = 0. \tag{3.51}$$

Proposition 3.18. *In biregular foliated coordinates $(x_0, x_1, \ldots, x_n)$, the function $f = x_0$ is isoparametric if and only if for any X tangent to a level hypersurface of f we have*

$$X(|g_{00}|) = 0, \quad X(\tau_1) = 0, \tag{3.52}$$

where τ_1 is the mean curvature of the level sets of f.

Proof. We have

$$\nabla f = \epsilon_N\, g(\nabla f, N) N = \epsilon_N \frac{1}{\sqrt{|g_{00}|}} N, \qquad g(\nabla f, \nabla f) = \frac{g_{00}}{|g_{00}|^2} = \frac{\epsilon_N}{|g_{00}|}.$$

It follows that $(3.51)_1$ and $(3.52)_1$ are equivalent. Let X be a vector tangent to a level set of f, satisfying $(3.52)_1$. Then similar computations, as in the proof of Theorem 8.22 in [15], lead to $X(\Delta f) = -\epsilon_N |\, g_{00}|^{-1/2} X(\tau_1)$; hence, $(3.51)_2$ and $(3.52)_2$ are equivalent. □

Since the derivatives of $|g_{00}|$ and τ_1 in the directions of $\widetilde{\mathcal{D}}$ do not appear in equations $(3.35)_1$ and (3.41) (the second being $(3.35)_2$ in the biregular orthogonal foliated coordinates), there exist metrics critical for the action (3.20) with $\widetilde{\mathcal{D}}$ tangent to an isoparametric foliation. To obtain such an example, it is enough to find a metric satisfying $(3.35)_1$ and (3.41) along one N-curve and then to extend its coefficients to leafwise constant functions on M.

3.4. Conformal submersions. Conformal submersions form an important class of mappings, which were investigated also in relation with Einstein equations, see survey in [5].

Definition 3.19. Let (M, g), $(\widehat{M}, \widehat{g})$ be smooth pseudo-Riemannian manifolds. A differentiable mapping $\pi : (M, g) \to (\widehat{M}, \widehat{g})$ is called a *conformal* (or: *horizontally conformal*) *submersion* if

1. π is a submersion, i.e., it is surjective and has maximal rank,
2. $\mathrm{d}\pi$ restricted to the distribution orthogonal to the fibers of π is a conformal mapping, i.e., there exists a smooth function $f : M \to \mathbb{R}$ (called *dilation* of the submersion) such that for all vectors X, Y orthogonal to the fibers of π we have
$$e^{-2f} g(X, Y) = \widehat{g}(\mathrm{d}\pi(X), \mathrm{d}\pi(Y)).$$

Note that for $p = 1$ any submersion is conformal.

In this subsection, we denote by $\widetilde{\mathcal{D}}$ the distribution tangent to the fibers of conformal submersion and we assume that both $\widetilde{\mathcal{D}}$ and its orthogonal complement $\mathcal{D}$ are non-degenerate. It follows that $\widetilde{\mathcal{D}}$ is integrable, and one can also show [8] that $\mathcal{D}$ is totally umbilical with the second fundamental form satisfying

$$\tilde{h} = -(\nabla^{\top} f) g^{\perp}. \tag{3.53}$$

Among conformal submersions, those with totally umbilical fibers are an example of particularly interesting geometry. While the adapted variations (2.11), (2.12) preserve the orthogonality of two distributions, we can consider their particular class which preserves the structure of conformal submersion with totally umbilical fibers.

Definition 3.20. We say that a variation $g_t \in \mathrm{Riem}(M, \mathcal{D}, \widetilde{\mathcal{D}})$ is *$\mathcal{D}$-conformal* if $\partial_t g_t^{\perp} = s g_0^{\perp}$ for some $s \in C^\infty(M)$. We define *$\widetilde{\mathcal{D}}$-conformal* variations in a similar way and say that variation g_t is *biconformal* if it is both $\mathcal{D}$-conformal and $\widetilde{\mathcal{D}}$-conformal.

A tensor $B \in \mathfrak{M}_{\mathcal{D}}$ is *$\mathcal{D}$-conformal* if $B = s g^{\perp}$ for some $s \in C^\infty(M, \mathbb{R})$. Given $g \in \mathrm{Riem}(M, \widetilde{\mathcal{D}}, \mathcal{D})$, the subspace of $\mathfrak{M}$, consisting of *biconformal* adapted tensors, splits into the direct sum of $\mathcal{D}$- and $\widetilde{\mathcal{D}}$-conformal components.

Proposition 3.21. *Let $\pi : (M^{n+p}, g) \to (\widehat{M}^p, \widehat{g})$ be a conformal submersion with totally umbilical fibers, and g_t be an adapted variation of g. Then all mappings $\pi : (M, g_t) \to (\widehat{M}, \widehat{g})$ are conformal submersions with totally umbilical fibers if and only if variation g_t is $\mathcal{D}$-conformal and*

$$\nabla \left(B - \frac{1}{n} (\operatorname{Tr} B^\sharp) \tilde{g} \right) = 0. \tag{3.54}$$

Proof. If all the mappings $\pi : (M, g_t) \to (\widehat{M}, \widehat{g})$ are conformal submersions, then we have $e^{-2f_t} g_t^{\perp} = \pi^*(\widehat{g})$ for some $f_t \in C^\infty(M)$. Differentiating the above, we obtain

$$e^{-2f_t} \partial_t g_t^{\perp} - 2\partial_t f_t e^{-2f_t} g_t^{\perp} = 0.$$

Hence, $\partial_t g_t^{\perp} = s g_0^{\perp}$ for $s = 2\,\partial_t f_t$, that is, our variation is $\mathcal{D}$-conformal.

If $\widetilde{\mathcal{D}}$ is totally umbilical for all g_t, then $h = \frac{1}{n} H \tilde{g}_t$, and from (2.15), we obtain

$$\frac{2}{n}\big(B(X,Y) H + g(X,Y)\, \partial_t H\big) = \frac{2}{n} B(X,Y)\, H - \nabla B(X,Y)$$

for all $X, Y \in \widetilde{\mathcal{D}}$. Using $(2.16)_1$ yields the equality $\frac{1}{n} g(X,Y) \nabla(\operatorname{Tr} B^\sharp) = \nabla B(X,Y)$.

On the other hand, if (3.54) is satisfied and the variation is $\mathcal{D}$-conformal, then from the uniqueness of the solution of ODE it follows that $h = \frac{1}{n} H \tilde{g}_t$ and $e^{-2f_t} g_t^{\perp} = \pi^* \widehat{g}$ for all t; hence, all $\pi : (M, g_t) \to (\widehat{M}, \widehat{g})$ are conformal submersions with totally umbilical fibers. □

Note that the condition (3.54) is satisfied, in particular, by biconformal variations.

We examine the metrics critical for the action (2.2) with respect to $\mathcal{D}$-conformal variations. The Euler–Lagrange equation for these metrics is a scalar equation. To find it, we can use our equation (2.32), with $B = sg^\perp$; by demanding it to be satisfied for all $s \in C^\infty(M)$, we obtain

$$(p-1)\operatorname{div}\widetilde{H} + \frac{p-2}{2}\big(\mathrm{S}_{\rm ex} + \langle\widetilde{T},\widetilde{T}\rangle\big) + \frac{p}{2}\big(\widetilde{\mathrm{S}}_{\rm ex} + \langle T,T\rangle - \mathrm{S}^*_{\rm mix}(\Omega,g)\big) = 0, \quad (3.55)$$

where $\mathrm{S}^*_{\rm mix} = \mathrm{S}_{\rm mix} - \frac{2}{p}\big(\mathrm{S}_{\rm ex} + 2\langle\widetilde{T},\widetilde{T}\rangle - \langle T,T\rangle\big)$.

The mixed scalar curvature is an important tool in investigation of conformal submersions with totally umbilical fibers. In [18], it was used to obtain some integral formulas and existence conditions for such mappings. There, the following formula was established:

$$\mathrm{S}_{\rm mix} = -p\widetilde{\Delta} f - pg\big(\nabla^\top f, \nabla^\top f\big) + \big\langle\widetilde{T},\widetilde{T}\big\rangle + \operatorname{div} H + \frac{n-1}{n} g(H,H), \qquad (3.56)$$

which is just a particular case of (2.10) expressed in terms of f and H. We can present in a similar way the Euler–Lagrange equations for biconformal variations on the domains of conformal submersions with totally umbilical fibers.

Proposition 3.22 (Euler–Lagrange equations)**.** *Let* $\pi : \big(M^{n+p}, g\big) \to \big(\widehat{M}^p, \widehat{g}\big)$, *where* $p > 1$, *be a conformal submersion with totally umbilical fibers. Then* g *is critical for the action* (2.2) *with respect to biconformal variations if and only if*

$$\begin{aligned}
&-2p(p-1)\widetilde{\Delta} f - p^2(p-1) g\big(\nabla^\top f, \nabla^\top f\big) + \frac{(p-2)(n-1)}{n} g(H,H) \\
&\qquad + (p-2)\big\langle\widetilde{T},\widetilde{T}\big\rangle = p\mathrm{S}^*_{\rm mix}(\Omega,g) \quad (\textit{for } \mathcal{D}\textit{-conformal variations}), \quad (3.57)\\
&p(p-1)(n-2) g\big(\nabla^\top f, \nabla^\top f\big) + 2(n-1)\operatorname{div} H + (n-1) g(H,H) \\
&\qquad + n\big\langle\widetilde{T},\widetilde{T}\big\rangle = n\widetilde{\mathrm{S}}^*_{\rm mix}(\Omega,g) \qquad (\textit{for } \widetilde{\mathcal{D}}\textit{-conformal variations}), \quad (3.58)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{S}^*_{\rm mix} &= -p\left(\widetilde{\Delta} f + g\big(\nabla^\top f, \nabla^\top f\big)\right) + \frac{p-4}{p}\langle\widetilde{T},\widetilde{T}\rangle \\
&\quad + \frac{(n-1)(p-2)}{np} g(H,H) + \frac{p-2}{p}\operatorname{div} H, \\
\widetilde{\mathrm{S}}^*_{\rm mix} &= -p\frac{n-2}{n}\left(\widetilde{\Delta} f + g\big(\nabla^\top f, \nabla^\top f\big)\right) + \frac{n+2}{n}\langle\widetilde{T},\widetilde{T}\rangle \\
&\quad + \operatorname{div} H + \frac{n-1}{n} g(H,H). \qquad (3.59)
\end{aligned}$$

Proof. For conformal submersions with totally umbilical fibers, we have

$$T = 0, \quad \mathrm{S}_{\rm ex} = \frac{n-1}{n} g\big(H,H\big), \quad \mathrm{S}_{\rm ex} = \frac{p-1}{p} g\big(\widetilde{H},\widetilde{H}\big),$$

and from (3.53) we obtain $\widetilde{H} = -p\nabla^\top f$. Using this, we rewrite (3.55) as (3.57). For $\widetilde{\mathcal{D}}$-conformal variations of metrics on the domain of conformal submersion with umbilical fibers, a formula analogous to (3.55) yields (3.58). Using (3.56) in (2.27), we get remaining formulas (3.59). □

We examine the above equations in a particular case of totally geodesic fibers, i.e., $H = 0$.

Proposition 3.23. *Let $\pi : (M^{n+p}, g) \to (\widehat{M}^p, \widehat{g})$, where $p > 1$ and $g|_{\widetilde{\mathcal{D}}} > 0$, be a conformal submersion with complete totally geodesic fibers. Then g is critical for the action (2.2) with respect to biconformal variations if and only if $e^{\lambda f}$, where $\lambda = \frac{1}{2n}(pn + (p-2)(n-2)) > 0$, is a fiberwise harmonic function.*

Proof. From (3.57) and (3.58), we obtain

$$p(p-1)\Big(\widetilde{\Delta} f + \lambda g\big(\nabla^\top f, \nabla^\top f\big)\Big) = \frac{p-2}{2}\widetilde{\mathrm{S}}^*_{\mathrm{mix}}(\Omega, g) - \frac{p}{2}\mathrm{S}^*_{\mathrm{mix}}(\Omega, g).$$

Using the identity $\widetilde{\Delta} f + \lambda g\big(\nabla^\top f, \nabla^\top f\big) = \frac{1}{\lambda}e^{-\lambda f}\widetilde{\Delta} e^{\lambda f}$ in the above yields

$$\widetilde{\Delta} e^{\lambda f} = G\lambda e^{\lambda f}, \tag{3.60}$$

where $G = \frac{1}{p(p-1)}\Big(\frac{p-2}{2}, \widetilde{\mathrm{S}}^*_{\mathrm{mix}} - \frac{p}{2}\mathrm{S}^*_{\mathrm{mix}}\Big)(\Omega, g)$. Equation (3.60) is a closed eigenvalue problem of operator $\widetilde{\Delta}$ on every fiber, and $e^{\lambda f}$ is its positive solution; hence, $G = 0$ and $e^{\lambda f}$ is fiber-wise harmonic. For closed fibers, (3.60) admits only fiber-wise constant solutions f. If we allow our variations not to preserve the volume of Ω, then again $G = 0$ and (3.60) becomes the fiberwise Laplace equation for $e^{\lambda f}$. □

Remark 3.24. The set of bounded (or positive) harmonic functions on open manifolds with nonnegative curvature was described in [9]: in particular, positive harmonic functions correspond to “large ends” of the manifold (in our case, the fiber of the submersion).

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Дія типу Ейнштейна–Гільберта на псевдоріманових многовидах-майже-добутках

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Ми досліджуємо варіаційні формули для зовнішньо геометричних величин псевдоріманових многовидів-майже-добутків і ми розглядаємо варіації метрики, які зберігають ортогональність розподілів. Ці формули застосовано для вивчення дій типу Ейнштейна–Гільберта для змішаної скалярної кривини та зовнішньої скалярної кривини розподілу. Рівняння Ейлера–Лагранжа одержано у повній загальності та в декількох окремих випадках (розшарувань, які є інтегровними пласкими полями, конформних субмерсій та ін.). Одержані рівняння Ейлера–Лагранжа узагальнюють результати для розшарувань ковимірності один на випадок довільної ковимірності та допускають багато розв'язків, тобто скручених добутків та ізопараметричних розшарувань.

Ключові слова: псевдоріманова метрика, многовид-майже-добуток, розшарування, друга фундаментальна форма, адаптована варіація, змішана скалярна кривина, конформна субмерсія.