Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 2, pp. 155–169
doi: https://doi.org/10.15407/mag15.02.155

Cohomogeneity One Dynamics on Three Dimensional Minkowski Space

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In the paper, we give a classification of closed and connected Lie groups, up to conjugacy in $\mathrm{Iso}(\mathbb{R}^{1,2})$, acting by cohomogeneity one on the three dimensional Minkowski space $\mathbb{R}^{1,2}$ in both proper and nonproper dynamics. Then we determine causal properties and types of the orbits.

Key words: cohomogeneity one, Minkowski space.

Mathematical Subject Classification 2010: 53C30, 57S25.

1. Introduction

A basic question in most types of modern dynamics is which groups admit actions of the type under consideration. A number of researchers have tried to give a list of groups admitting nonproper (or orbit nonproper) isometric actions on Lorentz manifolds up to local isomorphism or at most up to isomorphism (see, for example, [1, 11, 21, 22]). Their main aim was not to study the induced orbits and they did not consider homogeneity or cohomogeneity assumption as a dynamical restriction. Not many papers are found on cohomogeneity one Lorentzian manifolds with some further dynamical restrictions in the literature (see, for instance, [3, 9, 19, 20]).

Cohomogeneity one Riemannian manifolds have been studied by many mathematicians (see [7, 14, 15, 17, 18]). In most works, finding representations of the acting group in the full isometry group was not given priority. Instead, much emphasis is placed on the study of geometrical properties of the results of the action such as the existence of slices, the induced orbits, the orbit space, etc. In these papers the common hypothesis is that the acting group is a closed Lie subgroup of $\mathrm{Iso}_g(M)$, where g denotes the Riemannian metric on the smooth manifold M. This assumption causes a strong dynamical restriction, that is, the action should be proper. When the metric is indefinite, this assumption in general does not imply that the action is proper, so the study becomes much more difficult. Also, some results and techniques of the definite metric fail for the indefinite metric.

In this paper, we give the list of closed and connected Lie groups, up to conjugacy in $\mathrm{Iso}(\mathbb{R}^{1,2}) = O(1,2) \ltimes \mathbb{R}^3$, acting isometrically and by cohomogeneity one on the three dimensional Minkowski space $\mathbb{R}^{1,2}$. Then we determine, for each group in the list, wether its action is proper or nonproper. Finally, we study causal properties and types of the orbits both for proper and nonproper actions. Also, we specify the orbit space when the action is proper. If the action is nonproper, the orbit space may not be Hausdorff, and so the study seems to be not interesting. Furthermore, we could not use the same definition of principal and singular orbits which was used in [5]. So we use a new definition which is compatible with that of proper actions (see Section 2).

As an interesting result of this paper one gets that if the action is proper, then the linear part of the acting group is either compact or hyperbolic one parameter subgroup, i.e., it has no nilpotent element. Another considerable result is about the existence of exceptional orbits. A well-known result says that for proper actions each exceptional orbit is nonorientable if the G-manifold M is orientable and if M is simply connected, there is no exceptional orbit (see [7, p.185]), but we see in Propositions 4.3–4.5 that for $M = \mathbb{R}^{1,2}$, which is simply connected and orientable, there are orientable exceptional orbits!

2. Preliminaries

Let G be a Lie group which acts on a connected smooth manifold M. The Lie algebra of G is denoted by $\mathfrak{g}$. For each point x in M, $G(x)$ denotes the orbit of x, and G_x is the stabilizer in G of x. The manifold M is called of *cohomogeneity one* under the action of the Lie group G if an orbit has codimension one. The action is said to be *proper* if the mapping $\varphi : G \times M \to M \times M$, $(g, x) \mapsto (g \cdot x, x)$ is proper. Equivalently, for any sequences x_n in M and g_n in G, $g_n x_n \to y$ and $x_n \to x$ imply that g_n has a convergent subsequence. The G-action on M is *nonproper* if it is not proper. Equivalently, there are sequences g_n in G and x_n in M such that x_n and $g_n x_n$ converge in M and $g_n \to \infty$, i.e., g_n leaves compact subsets. For instance, if G is compact, the action is obviously proper. The action of G on M is proper if and only if there is a complete G-invariant Riemannian metric on M (see [4]). This theorem makes a link between proper actions and Riemannian G-manifolds. The orbit space M/G of a proper action of G on M is Hausdorff, the orbits are closed submanifolds, and the stabilizers are compact (see [2]). The orbits $G(x)$ and $G(y)$ have the *same orbit type* if G_x and G_y are conjugate in G. This defines an equivalence relation among the orbits of G on M. Denote by $[G(x)]$ the corresponding equivalence class, which is called the *orbit type* of $G(x)$. A submanifold S of M is called a slice at x if there is a G-invariant open neighborhood U of $G(x)$ and a smooth equivariant retraction $r : U \to G(x)$ such that $S = r^{-1}(x)$. A fundamental feature of proper actions is the existence of a slice (see [16]), which enables one to define a partial ordering on the set of orbit types. The partial ordering on the set of orbit types is defined by $[G(y)] \leq [G(x)]$ if and only if G_x is conjugate in G to some subgroup of G_y. If S is a slice at y, it implies that $[G(y)] \leq [G(x)]$ for all $x \in S$. Since M/G is connected, there is the largest orbit type in the set of orbit types. Each representative of this largest orbit type is called a principal orbit. In other words, an orbit $G(x)$ is principal if and only if for each point $y \in M$ the stabilizer G_x is conjugate to some subgroup of G_y in G. Other orbits are called singular. We say that $x \in M$

is a principal point if $G(x)$ is a principal orbit.

But for the nonproper action there is no slice in general, so we can not use the same definitions that require the existence of slices, hence we use the definition 2.8.1 of [8] for determining the principal, singular or exceptional orbits. According to it, for the action of a Lie group G on a smooth manifold M, the points $x, y \in M$, are said to be of the same type, with notation $x \approx y$, if there is a G-equivariant diffeomorphism Φ from an open G-invariant neighborhood U of x onto an open G-invariant neighborhood V of y. Clearly, this defines an equivalence relation $\approx$ in M. The equivalence classes will be called *orbit types* in M and denoted by $M_x^{\approx}$. If each stabilizer has only finitely many components, then $x \approx y$ if and only if G_x is conjugate to G_y within G and the actions of G_x and G_y, on $T_xM/T_xG(x)$ and $T_yM/T_yG(y)$, respectively, are equivalent via a linear intertwining isomorphism (see Chapter 2 of [8]). The orbit $G(x)$ of $x \in M$ is *principal* if its type $M_x^{\approx}$ is open in M. Any non-principal orbit is called a *singular* orbit. A nonprincipal orbit with the same dimension as a principal orbit is an *exceptional* orbit.

Throughout the following, $\mathbb{R}^{p,q}$ denotes the $p+q$-dimensional real vector space $\mathbb{R}^{p+q}$ with the scalar product of signature (p,q) given by $\langle x, y\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{p+q} x_i y_i$. If $p = 0$, we get the q-dimensional Euclidean space $\mathbb{E}^q$. Let $\mathrm{Iso}(\mathbb{R}^{p,q})$ denote the group of isometries of $\mathbb{R}^{p,q}$, that is, the group $O(p,q) \ltimes \mathbb{R}^{p+q}$. We may write the natural action of an isometry $(A, a) \in \mathrm{Iso}(\mathbb{R}^{p,q})$ as $(A, a)(x) = A(x) + a$, where $A \in O(p,q)$ is called its *linear part* and $a \in \mathbb{R}^{p+q}$ is called its *translational part*. Denote by $L : G \longrightarrow O(p,q)$ the projection on the linear part of $O(p,q) \ltimes \mathbb{R}^{p+q}$. If $L(G)$ is trivial, then G is called a *pure translation group*. We will restrict our study to the identity component of $O(1,2)$, consisting of orientation and time-orientation preserving isometries, which we denote by $SO_o(1,2)$, a subgroup of $O(1,2)$ of index 4.

In the next section, we use an Iwasawa decomposition of $SO_o(1,2)$ to classify the Lie groups which act isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$. For this, we introduce a fixed Iwasawa decomposition of $SO_o(1,2)$. Let $i, j \in \{1,2,3\}$ and E_{ij} be the 3×3 matrix whose (i,j)-entry is 1 and whose other entries are all 0. Let $\mathfrak{k} = \{t(E_{23} - E_{32}) \mid t \in \mathbb{R}\}$, $\mathfrak{a} = \{s(E_{12} + E_{21}) \mid s \in \mathbb{R}\}$ and $\mathfrak{n} = \{u(E_{13} + E_{23} + E_{31} - E_{32}) \mid u \in \mathbb{R}\}$. Then $\mathfrak{so}(1,2) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (direct sum of vector spaces) is the Iwasawa decomposition of the Lie algebra $\mathfrak{so}(1,2)$ (see [10, p. 372]). Furthermore, $SO_o(1,2) = KAN$ is the Iwasawa decomposition of $SO_o(1,2)$, in which K , A and N are the connected Lie subgroups of $SO_o(1,2)$ associated to $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Clearly, $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(2)$ and using the exponential map, one gets that K and A are the standard embeddings of $SO(2)$ and $SO_o(1,1)$ in $SO_o(1,2)$, respectively. Each $C \in \mathfrak{n}$ is nilpotent, in fact, $C^3 = 0$. It is easy to check that $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$, so $\mathfrak{a} \oplus \mathfrak{n}$ is a Lie subalgebra of $\mathfrak{so}(1,2)$ and $\mathfrak{n}$ is an ideal of $\mathfrak{a} \oplus \mathfrak{n}$. This implies that N is normal in the group corresponding to $\mathfrak{a} \oplus \mathfrak{n}$, so AN is a subgroup (in fact, nonabelian and solvable) of $SO_o(1,2)$. By a well-known result, any two-dimensional Lie subgroup of $SO_o(1,2)$ is conjugate to AN.

We end this section by introducing some notations that are used in the sequel. Let $\{i, j\} \in \{1,2,3\}$. We fix the notations $B_i = E_{i(i+1)} + (-1)^{i+1} E_{(i+1)i}$, where $i \in \{1,2\}$, and $B_3 = E_{13} + E_{31} + B_2$ throughout the paper. Also, $A_t = \exp(tB_1)$

and $N_t = \exp(tB_3)$, where t is a fixed real number. Let $\{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$. Then e_{ij} denotes the vector $e_i + e_j$, and $e_{123} := e_1 + e_2 + e_3$.

3. Lie groups acting by cohomogeneity one on $\mathbb{R}^{1,2}$

In this section, we determine all connected closed Lie subgroups of $\mathrm{Iso}_o(\mathbb{R}^{1,2})$ acting isometrically and, by cohomogeneity one on $\mathbb{R}^{1,2}$, up to conjugacy. If the Lie group G is determined up to conjugacy, an immediate consequence is to specify the orbits up to isometry. If $\dim L(G) = 0$, since $L : G \to SO_o(1,2)$ is a Lie group homomorphism and therefore continuous, $L(G)$ is connected and thus trivial. In this case, G is a two-dimensional pure translation Lie subgroup of $SO_o(1,2) \ltimes \mathbb{R}^3$ whose natural action on $\mathbb{R}^{1,2}$ is obviously proper. If $\dim L(G) = 1$, by a well-known result about one parameter Lie subgroups of $SO_o(1,2)$, the Lie group $L(G)$ is conjugate to one of the groups $SO(2)$ $(= K)$, A or N. The representations of these groups in $SO_o(1,2)$ were introduced in the preceding section. We will study the case of $\dim L(G) = 1$ in the next three lemmas. In all of the lemmas, it is assumed that G is a connected and closed Lie subgroup of $\mathrm{Iso}(\mathbb{R}^{1,2})$, which acts isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$.

Lemma 3.1. *If $L(G)$ is conjugate to $SO(2)$, then G is conjugate to one of the following Lie groups within $SO_o(1,2) \ltimes \mathbb{R}^3$:*

(i) *the standard embedding of $SO(2) \times \mathbb{R}$ in $SO_o(1,2) \ltimes \mathbb{R}^3$,*

(ii) *the standard embedding of $\mathrm{Iso}_o(\mathbb{E}^2)$ in $SO_o(1,2) \ltimes \mathbb{R}^3$.*

Proof. By the assumption, $L(G)$ is conjugate to $SO(2)$. Then, up to conjugacy, $l(\mathfrak{g}) = \{tB_2 \mid t \in \mathbb{R}\}$. The action is of cohomogeneity one, so $\ker l$ should be one- or two-dimensional ideal of $\mathfrak{g}$. First, assume that $\dim \ker l = 1$. Then $\mathfrak{g}$, as a vector space, should be $\{(tB_2, (uD + tD')e_{123}) \mid u, t \in \mathbb{R}\}$, where D and D' are two diagonal fixed 3×3 matrices. Choose a vector b in $\mathbb{R}^3$ such that $(B_2, b) \in \mathfrak{g}$. Since $\ker l$ is an ideal in $\mathfrak{g}$, we have $B_2(\ker l) \subseteq \ker l$. Hence $\ker l$ is a subspace of the eigenspace of B_2. The only one dimensional eigendirection of B_2 is $\mathbb{R}e_1$. Hence $D_{22} = D_{33} = 0$. Thus $(C,c)^{-1}\mathfrak{g}(C,c) = \{(tB_2, (uD_{11} + tD'_{11})e_1) | u, t \in \mathbb{R}\}$, where $(C,c) = (I, D'_{33}e_2 - D'_{22}e_3) \in SO_\circ(1,2) \ltimes \mathbb{R}^3$. Thus G is conjugate to $SO(2) \times \mathbb{R}$.

Now suppose that $\dim \ker l = 2$. This implies that $\dim G = 3$. Then $\mathfrak{g}$, as a vector space, should be $\{tB_2, (uD + vD' + tD'')e_{123}) \mid u, v, t \in \mathbb{R}\}$, where D, D' and D'' are three diagonal fixed 3×3 matrices. By rechoosing a basis for $\mathfrak{g}$, we may assume that $D_{33} = D'_{22} = 0$, and up to conjugacy, $D''_{22} = D''_{33} = 0$. We claim that both D_{22} and D'_{33} are nonzero. If $D'_{33} = 0$, then we choose two vectors $X_1 = (E_{23} - E_{32}, D''_{11}e_1)$ and $X_2 = (0, (D + D')e_{123})$ from $\mathfrak{g}$. Then $[X_1, X_2] = (0, -D_{22}e_3)$, the closeness of which under the bracket in $\mathfrak{g}$ implies that $D_{22} = 0$. This implies that $\dim G = 2$, which contradicts to $\dim G = 3$. Hence $D'_{33} = 0$. A similar discussion shows that $D_{22} \neq 0$. Hence, without loss of generality, we may assume that $D_{22} = D'_{33} = 1$. Now choose three vectors X_1 and X_2 as above, and $X_3 = (0, (-A + B)e_{123})$ from $\mathfrak{g}$. By the fact that $[X_1, X_2]$ and $[X_1, X_3]$ belong to

$\mathfrak{g}$, one gets that $D_{11} + D'_{22} = 0$ and $D_{11} - D'_{22} = 0$. So $D_{11} = D'_{22} = 0$. If $D''_{11} \neq 0$, then $G(0) = \mathbb{R}^3$, which is in contradiction to the assumption that the action is of cohomogeneity one. Hence $D''_{11} = 0$. Thus G is conjugate to the standard embedding of $\mathrm{Iso}_o(\mathbb{E}^2)$ in $SO_o(1,2) \ltimes \mathbb{R}^3$. □

Lemma 3.2. *If $L(G)$ is conjugate to A, then $\mathfrak{g}$ is conjugate to one of the following Lie algebras within $\mathfrak{so}(1,2) \oplus_\pi \mathbb{R}^3$, where $\pi : \mathfrak{so}(1,2) \to \mathfrak{Der}(\mathbb{R}^{1,2})$ is the natural representation:*

(i) $\mathfrak{a} \oplus \mathbb{R}$,

(ii) $\{(tB_1, ue_1 + ve_2) \mid u, v \in \mathbb{R}\}$,

(iii) $\{(tB_1, ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$ *or*

(iv) $\{(tB_1, u(e_1 - e_2) + ve_3 \mid u, v \in \mathbb{R}\}$,

(v) $\{(tB_1, s(e_1 \pm e_2) + t\beta e_3) \mid t, s \in \mathbb{R}\}$,

where β is a fixed real number.

Proof. By the assumption, $l(\mathfrak{g}) = \{tB_1 \mid t \in \mathbb{R}\}$ up to conjugacy. The action is of cohomopgeneity one, so $\ker l$ is a one- or two-dimensional ideal of $\mathfrak{g}$. First assume that $\dim \ker l = 1$. Then, by choosing a suitable coordinate, we may assume that $\mathfrak{g}$, as a vector space, is $\{tB_1, (tD + sD')e_{123}\}$, where D and D' are two diagonal matrices. Now we determine the relation between the entries of D and D', to make $\mathfrak{g}$ a Lie algebra. Take the following two vectors in $\mathfrak{g}$, $X_1 = (B_1, De_{123})$ and $X_2 = (0, D'e_{123})$. Then $[X_1, X_2] = (0, D'_{22}e_1 + D'_{11}e_2)$, and so the closeness under the bracket implies the existence of a $s_0 \in \mathbb{R}$ such that $D'_{11} = s_0 D'_{22}$, $D'_{22}s_0 = D'_{11}$ and $D'_{33}s_0 = 0$. Thus, either $D'_{11} = D'_{22} = 0$, $D'_{33} \neq 0$, or $D'_{11} = \pm D'_{22} \neq 0$, $D'_{33} = 0$. The first case shows that $(C,c)^{-1}G(C,c) = A \times \mathbb{R}$, where $(C,c) = (I, -(D_{22}e_1 + D_{11}e_2))$, which is the case (i) of the lemma. The second case implies that $\mathfrak{g}$ is conjugate to one of the four Lie algebras $\{(tB_1, s(e_1 \pm e_2) + tD_{33}e_3) \mid t, s \in \mathbb{R}\}$, depending on whether the real number D_{33} is zero or not, which is the case (v) of the lemma.

Now suppose that $\dim \ker l = 2$. Choose $b \in \mathbb{R}^3$ such that $(B_1, b) \in \mathfrak{g}$. Then $B_1(\ker l) \subseteq \ker l$. Hence $\ker l$ is a subspace of the eigenspace of B_1. This implies that $\ker l$ is one of the spaces $\{(0, ue_1 + ve_2) \mid u, v \in \mathbb{R}\}$, $\{(0, ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$ or $\{(0, u(e_1 - e_2) + ve_3 \mid u, v \in \mathbb{R}\}$. Thus $\mathfrak{g}$ is conjugate to one of the Lie algebras $\{(tB_1, ue_1 + ve_2) | u, v \in \mathbb{R}\}$, $\{(tB_1, ue_{12} + ve_3 | u, v \in \mathbb{R}\}$ or $\{(tB_1, u(e_1 - e_2) + ve_3 | u, v \in \mathbb{R}\}$, which are the cases from (ii) to (iv) of the lemma. □

Lemma 3.3. *If $L(G)$ is conjugate to N, then $\mathfrak{g}$ is conjugate to one of the following Lie algebras within $\mathfrak{so}(1,2) \oplus_\pi \mathbb{R}^3$:*

(i) $\{(tB_3, r(e_{12}) + \beta te_3) \mid r, t \in \mathbb{R}\}$,

(ii) $\{(tB_3, r(e_{12}) + se_3) \mid r, s, t \in \mathbb{R}\}$,

where β is a fixed real number.

Proof. By the assumption, $l(\mathfrak{g}) = \{tB_3 \mid t \in \mathbb{R}\}$ up to conjugacy. By a similar discussion as in the proof of the previous lemma, one gets that $\ker l$ is a one- or two-dimensional ideal of $\mathfrak{g}$. If $\dim\ker l = 1$, then $\mathfrak{g}$, as a vector space, should be $\{(tB_3, (uD + tD')e_{123}) \mid u, t \in \mathbb{R}\}$, where D and D' are two diagonal fixed matrices. By the fact that the only one-dimensional eigendirection of B_3 is $\mathbb{R}(e_{12})$ and using $B_3(\ker l) \subseteq \ker l$, one gets that $\mathfrak{g} = \{(tB_3, r(e_{12}) + tD'_{33}e_3) \mid r, t \in \mathbb{R}\}$ up to conjugacy. If $\dim\ker l = 2$ then the relation $B_3(\ker l) \subseteq \ker l$ implies that $\ker l = \mathbb{R}(e_{12}) + \mathbb{R}e_3$. Hence the cohomogeneity one assumption implies that $\mathfrak{g} = \{(tB_3, r(e_{12}) + se_3) \mid r, s, t \in \mathbb{R}\}$. □

Any two-dimensional Lie subgroup of $SO_o(1,2)$ is conjugate to AN. Hence, for the case $\dim L(G) = 2$ we have the following Lemma.

Lemma 3.4. *If $L(G)$ is conjugate to AN, then $\mathfrak{g}$ is conjugate to one of the following Lie algebras within $\mathfrak{so}(1,2) \oplus_\pi \mathbb{R}^3$:*

(i) $(\mathfrak{a} \oplus \mathfrak{n}) \times \{0\}$,

(ii) $\{(sB_1 + tB_3, u(\alpha e_{12} + \beta e_3)) \mid s, t, u \in \mathbb{R}\}$,

(iii) $\{(sB_1 + tB_3, ue_{12} + ve_3)) \mid s, t, u, v \in \mathbb{R}\}$,

where α and β are two fixed real numbers.

Proof. If $L(G) = AN$, then we have

$$l(\mathfrak{g}) = \{sB_1 + tB_3 \mid s, t \in \mathbb{R}\}.$$

By the assumption of cohomogeneity one, we get that $0 \leqslant \dim\ker l \leqslant 2$. If $\dim\ker l = 0$, then G is conjugate to AN by a translation. If $\dim\ker l = 1$, then $\ker l$, as a vector space, is $\{uDe_{123} \mid u \in \mathbb{R}\}$, where D is a fixed diagonal matrix. By the fact that $B_i(\ker l) \subseteq \ker l$, $i = 1, 3$, one gets that $D_{11} = D_{22}$. Hence, up to conjugacy,

$$\mathfrak{g} = \{(sB_1 + tB_3, u(D_{11}e_{12} + D_{33}e_3)) \mid s, t, u \in \mathbb{R}\},$$

where D_{11} and D_{33} are fixed real numbers.

If $\dim\ker l = 2$, then $\ker l = \{(uD + vD')e_{123} \mid u, v \in \mathbb{R}\}$, where D and D' are two fixed diagonal matrices. By the same argument as above, one gets that $D_{11} = D'_{11}$ and $D_{22} = D'_{22}$. Hence, by choosing a suitable basis for $\mathfrak{g}$, one gets that

$$\mathfrak{g} = \{(sB_1 + tB_3, ue_{12} + ve_3)) \mid s, t, u, v \in \mathbb{R}\}$$

up to conjugacy. □

Since $\dim L(G) \leqslant 3$, the following lemma ends the classification of Lie groups acting isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$.

Lemma 3.5. *If $L(G) = SO_o(1,2)$, then $G = SO_o(1,2) \times \{0\}$.*

Proof. If $L(G) = SO_o(1,2)$, then $\{B_1, B_2, B_3\}$ is a basis for $l(\mathfrak{g})$. Since $B_i(\ker l) \subseteq \ker l$, where $i = 1,2,3$, then $\ker l$ is either $\{0\}$ or $\mathbb{R}^3$. The action is of cohomogeneity one, so $\ker l = \{0\}$. Thus $G = SO_o(1,2) \times \{0\}$. □

The main theorem of this section is the following.

Theorem 3.6. *Let G be a closed and connected Lie subgroup of* $\mathrm{Iso}(\mathbb{R}^{1,2})$ *which acts isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$. Then the action is proper if and only if G is conjugate to one of the following Lie groups:*

(i) *a pure translation group,*

(ii) *the standard imbedding of $SO(2) \times \mathbb{R}$ in $SO_o(1,2) \ltimes \mathbb{R}^3$,*

(iii) *the standard imbedding of* $\mathrm{Iso}_o(\mathbb{E}^2)$ *in $SO_o(1,2) \ltimes \mathbb{R}^3$,*

(iv) $\{(A_t, u(e_1 \pm e_2) + \beta t e_3) \mid t, u \in \mathbb{R}\}$, *where β is a fixed nonzero real number.*

Proof. By Lemmas 3.1–3.5, we know the Lie algebras of all Lie groups which act isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$. Hence, to prove the theorem, we need only to study those actings properly. If $\dim L(G) = 0$, then G is a pure translation group and the action is obviously proper. If $L(G) = SO(2)$, then G is conjugate to either $SO(2) \times \mathbb{R}$ or $\mathrm{Iso}_o(\mathbb{E}^2)$ and the action reduces to the action of a closed Lie subgroup of $\mathrm{Iso}(\mathbb{E}^3)$, which is proper clearly. We claim that if $L(G)$ is noncompact, then the case (d) of the theorem occurs. If $L(G)$ is not compact, then $\mathfrak{g}$ is conjugate to one of the Lie algebras stated in Lemmas 3.2–3.5. All the Lie algebras listed in Lemma 3.2 (i) to (iv), Lemma 3.3 (ii), Lemmas 3.4, 3.5 cause a nonproper action since in each case the stabilizer of the origin is not compact. If $\mathfrak{g}$ is conjugate to that of Lemma 3.3 (i), then $\exp(\mathfrak{h})$ is a closed and noncompact subgroup of the stabilizer of each point $(x, x+\beta, 0)^T \in \mathbb{R}^{1,2}$, where $\mathfrak{h} = \{(tB_3, \beta t e_3) \mid t \in \mathbb{R}\}$. This shows that the action is nonproper in case of Lemma 3.3 (i). Thus, to complete the proof of our claim, we only need to verify that the action caused by the Lie algebra stated in Lemma 3.2 (v) is proper if β is nonzero. In that case, by a simple computation, one gets that

$$G = \exp(\mathfrak{g}) = \{(A_t, u(e_1 \pm e_2) + \beta t e_3) \mid t, u \in \mathbb{R}\}.$$

Let $\{t_n\}$ and $\{u_n\}$ be two real sequences. Let $\{X_n = (x_n, y_n, z_n)^T\}$ be a sequence in $\mathbb{R}^{1,2}$. Let $\{g_n = (\cosh t_n(E_{11} + E_{22}) + \sinh t_n(E_{12} + E_{21}) + E_{33}, u_n(e_1 \pm e_2) + \beta t_n e_3)\}$ be a sequence in G. Let $g_n \cdot X_n \to Y$ and $X_n \to X$, when $n \to +\infty$. If $Y = (y_1, y_2, y_3)^T$ and $X = (x_1, x_2, x_3)^T$, then $t_n \to \frac{y_3 - x_3}{\beta}$ and $u_n \to y_1 - x_1 \cosh \frac{y_3 - x_3}{\beta} - x_2 \sinh \frac{y_3 - x_3}{\beta}$. Hence $\{g_n\}$ is a convergent sequence in G, and thus the action is proper. □

As a consequence of Lemmas 3.2–3.4 and Theorem 3.6, one gets the following corollary. The Lie groups in the list are obtained by the exponential map.

Corollary 3.7. *Let G be a closed and connected Lie subgroup of* $\mathrm{Iso}(\mathbb{R}^{1,2})$ *which acts isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$. Then the action is nonproper if and only if G is conjugate to one of the following Lie groups within $SO_o(1,2) \ltimes \mathbb{R}^3$:*

(i) $\{(A_t, se_3) \mid t, s \in \mathbb{R}\}$,

(ii) $\mathrm{Iso}_o(\mathbb{R}^{1,1})$,

(iii) $\{(A_t, ue_{12} + ve_3) \mid t, u, v \in \mathbb{R}\}$,

(iv) $\{A_t, u(e_1 - e_2) + ve_3 \mid t, u, v \in \mathbb{R}\}$,

(v) (v) $\{(A_t, ue_{12}) \mid t, u \in \mathbb{R}\}$,

(vi) $\{(A_t, u(e_1 - e_2) \mid t, u \in \mathbb{R}\}$,

(vii) $\{(N_t, ue_{12} + \beta te_3) \mid t, u \in \mathbb{R}\}$,

(viii) $\{(N_t, ue_{12} + ve_3) \mid t, u, v \in \mathbb{R}\}$,

(ix) AN,

(x) $AN \ltimes_\pi \{u(\alpha e_{12} + \beta e_3) \mid u \in \mathbb{R}\}$,

(xi) $AN \ltimes_\pi \{ue_{12} + ve_3) \mid u, v \in \mathbb{R}\}$,

(xii) $SO_o(1, 2)$,

where α and β are fixed real numbers and $\pi : AN \to \mathbb{R}^3$ is the natural representation.

4. Causal properties of orbits

Assume that the connected and closed Lie subgroup G of $\mathrm{Iso}(\mathbb{R}^{1,2})$ acts isometrically and by cohomogeneity one on $\mathbb{R}^{1,2}$, we determine causal properties of the orbits.

The orbit $G(p)$ is said to be Lorentzian, degenerate or space-like if the induced metric on $G(p)$ is Lorentzian, degenerate or Riemannian, respectively. It is called time-like or light-like if each nonzero tangent vector in $T_pG(p)$ is time-like or null, respectively. The category into which a given orbit falls is called its *causal property*.

4.1. The action is proper. Let a Lie group G act by cohomogeneity one and properly on a smooth manifold M. The results obtained by Mostert (see [13]) for the compact Lie groups, and Berard Bergery (see [6]) for the general case, say that the orbit space M/G is homeomorphic to one of the spaces:

$$\mathbb{R}, \quad S^1, \quad [0, +\infty), \quad [0, 1].$$

In the following theorem we will show that the cases $[0, 1]$ and S^1 can not occur when $M = \mathbb{R}^{1,2}$.

Theorem 4.1. *Let $\mathbb{R}^{1,2}$ be of cohomogeneity one under the isometric action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^{1,2})$. If the action is proper, then one of the following cases occurs:*

(1) *G is a pure translation group. In this case, each orbit is a plane which is obtained by a translation of $G(0)$, and the orbit space is diffeomorphic to $\mathbb{R}$.*

(2) *G is conjugate to $SO(2) \times \mathbb{R}$. In this case, there is a time-like singular orbit, which is a one-dimensional affine subspace, and each other orbit is a cylinder around the singular orbit, and thus the orbit space is diffeomorphic to $[0, +\infty)$. In particular, each principal orbit is a Lorentzian cylinder.*

(3) *G is conjugate to $\mathrm{Iso}_o(\mathbb{E}^2)$. In this case, each orbit is a space-like plane which is obtained by a translation of $G(0)$, and the orbit space is diffeomorphic to $\mathbb{R}$.*

(4) *G is conjugate to $\{((E_{11} + E_{22}) \cosh t + (E_{12} + E_{21}) \sinh t + E_{33}, u(e_1 \pm e_2) + \beta t e_3) \mid t, u \in \mathbb{R}\}$, where β is a fixed nonzero real number. In this case, an orbit is a degenerate plane, and each other orbit is a Lorentzian generalized cylinder. The orbit space is diffeomorphic to $\mathbb{R}$.*

Proof. The theorem is a direct consequence of Theorem 3.6, and only the case (4) needs some explanations. Suppose that $\mathfrak{g} = \{(t(E_{12} + E_{21}), s(e_{12}) + t\beta e_3) \mid t, s \in \mathbb{R}\}$, where $\beta \neq 0$. Let $X_1 = (E_{12} + E_{21}, \beta e_3)$ and $X_2 = (0, e_{12})$. Then $\{X_1, X_2\}$ is a basis for $\mathfrak{g}$. Fix an arbitrary point $p = (x, y, z)^T \in \mathbb{R}^{1,2}$, then

$$\frac{d}{dt} \exp(tX_1)(p)\bigg|_{t=0} = ye_1 + xe_2 + \beta e_3,$$

and

$$\frac{d}{ds} \exp(sX_2)(p)\bigg|_{s=0} = e_{12}.$$

If $x = y$, then the vector $N = e_{12}$ is normal to the above two vectors, and so to $G(p)$. This implies that $G(p)$ is a degenerate principal orbit. It is easily seen that this orbit is a plane.

If $x \neq y$, then the unit space-like vector $N = \frac{\beta}{y-x}(1, 1, \frac{y-x}{\beta})$ is normal to $G(p)$ at p, so $G(p)$ is a Lorentzian orbit, i.e., $T_pG(p)$ is isometric to $\mathbb{R}^{1,1}$, and the shape operator associated to N is represented with respect to the pseudo-orthogonal basis $\{v_1 = \sqrt{2}/2(1, 1)^T,\ v_2 = \sqrt{2}/2(1, -1)^T\}$ as follows:

$$S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Hence, the shape operator is not diagonalizable and $G(p)$ is locally isometric to a generalized cylinder by [12]. $\square$

Corollary 4.2. *Let $\mathbb{R}^{1,2}$ be of cohomogeneity one under the proper action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^{1,2})$. Then one of the followings holds:*

1) *If there is a singular orbit, then it is a time-like one-dimensional affine subspace and each principal orbit is isometric to $\mathbb{R}^{0,1} \times S^1(r)$ for some $r > 0$, where $\mathbb{R}^{0,1}$ denotes $\mathbb{E}^1$ with the reversed negative definite metric.*

2) *If there is a space-like orbit, then each orbit is a space-like hyperplane.*

3) *If there is a Lorentzian orbit isometric to* $\mathbb{R}^{1,1}$*, then each orbit is isometric to* $\mathbb{R}^{1,1}$*.*

4) *If there is more than one degenerate orbit, then each orbit is a degenerate hyperplane.*

5) *If there is exactly one degenerate orbit, then it is a degenerate hyperplane and each other orbit is locally isometric to a generalized cylinder and there is no singular orbit.*

4.2. The action is nonproper. As a consequence of Corollary 3.7, one gets that if the action is not proper, then $L(G)$ is conjugate to one of Lie groups A, N, AN or $SO_o(1,2)$. In the following propositions we will consider the action of each Lie group mentioned in Corollary 3.7, and then study the causal properties of the orbits. Throughout this subsection $p = (x_0, y_0, z_0)^T$ denotes an arbitrary fixed point in $\mathbb{R}^{1,2}$.

Proposition 4.3. *Let* $\mathbb{R}^{1,2}$ *be of cohomogeneity one under the isometric and nonproper action of a connected and closed Lie subgroup* $G \subset \mathrm{Iso}(\mathbb{R}^{1,2})$*. If* $L(G) = A$*, then one of the following cases occurs:*

(i) *There is a one-dimensional space-like affine subspace as a singular orbit. There are four degenerate half-plans which are principal orbits. Each other orbit is a branch of a Lorentzian hyperbolic cylinder which is principal.*

(ii) *Each orbit is a translation of a Lorentzian plane.*

(iii) *There is a degenerate plane as a unique exceptional orbit and there are two open submanifolds as the orbits.*

(iv) *There are infinitely many one-dimensional light-like singular orbits. Each other orbit is a Lorentzian principal orbit, which is not a closed submanifold.*

Proof. By Corollary 3.7, G is conjugate to one of the Lie groups in (i)–(vi).

(i) Let $G = \{(A_t, se_3) \mid t, s \in \mathbb{R}\}$. If $x_0 = y_0 = 0$, then $G(p) = \{se_3 \mid s \in \mathbb{R}\}$ and so $G(p)$ is a one dimensional space-like singular orbit. Let $x_0 \neq 0$ or $y_0 \neq 0$. Then $G(p)$ is a principal orbit since G_p is the trivial subgroup. It is easily seen that if $(u, v, w)^T$ belongs to $G(p)$, then $u^2 - v^2 = x_0^2 - y_0^2$. Hence, there are four degenerate principal orbits for the cases $x_0 = \pm y_0$ that depend on the sign of x_0. If $x_0 \neq \pm y_0$, then $G(p)$ is a branch of a Lorentzian hyperbolic cylinder $u^2 - v^2 = x_0^2 - y_0^2$.

(ii) Let $G = \mathrm{Iso}_o(\mathbb{R}^{1,1})$. It is easily seen that each orbit is a translation of the Lorentzian plane $z = 0$.

(iii) Let $G = \{(A_t, ue_{12} + ve_3) \mid t, u, v \in \mathbb{R}\}$. If $x_0 = y_0$, then $G(p) = \{ue_{12} + ve_3 \mid\ u, v \in \mathbb{R}\}$ and $G_p = \{(A_t, x_0(e^t - 1)e_{12}) \mid t \in \mathbb{R}\}$, and so $G(p)$ is a degenerate plane as an exceptional orbit. If $x_0 \neq y_0$, then G_p is the trivial subgroup and $G(p)$ is an open submanifold. So, there are two open orbits corresponding to the cases $x_0 > y_0$ and $x_0 < y_0$. A similar discussion about the action of the Lie group $G = \{A_t, u(e_1 - e_2) + ve_3 \mid t, u, v \in \mathbb{R}\}$, stated in Corollary 3.7 (iv), yields the same result as in Proposition 4.3 (iii).

(iv) Let $G = \{(A_t, ue_{12}) \mid t, u, v \in \mathbb{R}\}$. If $x_0 = y_0$, then $G(p) = \{ue_{12} + z_0e_3 \mid u \in \mathbb{R}\}$. Hence $(x, y, z)^T$ belongs to $G(p)$ if and only if $x = y$ and $z = z_0$. Hence, there are infinitely many one-dimensional light-like singular orbits. If $x_0 \neq y_0$, then G_p is the trivial subgroup and $G(p) = \{ue_1 + ve_2 + z_0e_3 \mid (u - v)(x_0 - y_0) > 0\}$. This shows that $G(p)$ is a Lorentzian principal orbit, which is not a closed submanifold. A similar discussion about the action of the Lie group $G = \{(A_t, u(e_1 - e_2) \mid t, u \in \mathbb{R}\}$, stated in Corollary 3.7 (vi), yields the same result as in Proposition 4.3 (iv). □

Proposition 4.4. *Let $\mathbb{R}^{1,2}$ be of cohomogeneity one under the isometric and nonproper action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^{1,2})$. If $L(G) = N$, then one of the following cases occurs:*

(i) *The acting group is conjugate to $\{(N_t, ue_{12} + \beta te_3) \mid t, u \in \mathbb{R}\}$. There are two cases. If $\beta \neq 0$, then each orbit is principal, which is obtained by a translation of a fixed degenerate plane. If $\beta = 0$, then there are infinitely many light-like one-dimensional singular orbits of the same type, where the union of them is a degenerate plane. Each principal orbit is obtained by a translation of a fixed degenerate plane. In both cases all the principal orbits are of the same type.*

(ii) *The acting group is conjugate to $\{(N_t, ue_{12} + ve_3) \mid t, u, v \in \mathbb{R}\}$. Each orbit is a degenerate plane as a principal orbit and the set of orbits is a foliation of $\mathbb{R}^{1,2}$. All orbits are of the same type.*

Proof. By Corollary 3.7, G is conjugate to one of Lie groups in (vii) and (viii). So, we have the two cases.

(i) Let $G = \{(N_t, ue_{12} + \beta te_3) \mid t, u \in \mathbb{R}\}$. If $x_0 \neq y_0$, then G_p is trivial and $G(p) = \{p + ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$, which is a degenerate plane. Since the union of these orbits is an open subset of $\mathbb{R}^{1,2}$, then each of them is principal. Let $x_0 = y_0$. If $\beta = 0$, then $G_p = \{(N_t, (-tz_0)e_{12}) \mid t \in \mathbb{R}\}$. This shows that $G(p)$, which is equal to $\{p + ue_{12} \mid u \in \mathbb{R}\}$, is a one-dimensional light-like subspace as a singular orbit. Let $p' = (x, y, z)^T$. It is easily seen that p' belongs to $G(p)$ if and only if $x = y$ and $z = z_0$. Furthermore, if $p' \notin G(p)$ and $x = y$, then $G_{p'} = g^{-1}G_pg$, where $g = (I, (z + z_0)e_3)$. Thus, there are infinitely many singular orbits of the same type. Obviously, the union of the singular orbits is $\{ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$, which is a degenerate plane. If $\beta \neq 0$ (and $x_0 = y_0$), then G_p is the trivial subgroup and $G(p)$ is a degenerate plane as a principal orbit.

(ii) Let $G = \{(N_t, ue_{12} + ve_3) \mid t, u, v \in \mathbb{R}\}$. Then $G_p = \{N_t, ((y_0 - x_0)\frac{t^2}{2} - tz_0)e_{12} + (y_0 - x_0)te_3\}$, and $G(p) = \{p + ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$. Hence each orbit is a degenerate plane as a principal orbit. All orbits are of the same type since each of the stabilizers is conjugate to $\{(N_t, 0) \mid t \in \mathbb{R}\}$. In fact, $g^{-1}G_pg = \{(N_t, 0) \mid t \in \mathbb{R}\}$, where $g = (I, (y_0 - x_0)e_2 + z_0e_3)$. □

Proposition 4.5. *Let $\mathbb{R}^{1,2}$ be of cohomogeneity one under the isometric and nonproper action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^{1,2})$. If $L(G) = AN$, then one of the following cases occurs:*

(i) *The acting group is conjugate to AN. Then there is one principal orbit type and two singular orbit types. Each principal orbit is either a Lorentzian or a space-like surface. There is a zero-dimensional singular orbit and infinitely many one-dimensional light-like singular orbits.*

(ii) *The acting group is conjugate to $G = AN \ltimes_\pi \{u(\alpha e_{12} + \beta e_3) \mid u \in \mathbb{R}\}$. If $\alpha = 0$, then there is a degenerate exceptional orbit and there are two orbits which are open submanifolds of $\mathbb{R}^{1,2}$ (the orbit space consists of three points). If $\alpha \neq 0$, then there are infinitely many one-dimensional light-like singular orbits of the same type and two orbits which are open submanifolds of $\mathbb{R}^{1,2}$.*

(iii) *The acting group is conjugate to $G = AN \ltimes_\pi \{ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$. Then there is a degenerate exceptional orbit and there are two orbits which are open submanifolds of $\mathbb{R}^{1,2}$ (the orbit space consists of three points).*

Proof. By Corollary 3.7, G is conjugate to one of Lie groups in (ix)–(xi). So, we have the two cases.

(i) Let $G = AN$. Then G fixes the origin, and thus the origin is a singular orbit. Let p be not the origin. The set $\{B_1, B_3\}$ is a basis for the Lie algebra $\mathfrak{g}$. To determine causal properties of the orbits, let

$$\Phi_p(t) = \exp((t\alpha B_1 + B_3))p,$$

where α is an arbitrary real number. Then

$$\langle \frac{d\Phi_p}{dt}(0), \frac{d\Phi_p}{dt}(0) \rangle = \alpha^2(x_0^2 - y_0^2) + 2\alpha z_0(x_0 - y_0) + (x_0 - y_0)^2. \tag{4.1}$$

This implies that if $x_0 \neq y_0$ and p, as a vector, is a nonzero space-like (respectively, time-like) vector, then the polynomial (4.1) has two roots (respectively, has no root). Hence the orbit $G(p)$ is a Lorentzian (respectively, space-like) orbit. Since G_p is trivial and the union of these orbits is an open subset of $\mathbb{R}^{1,2}$, all these orbits are of the same type and principal. If $x_0 = y_0$, then $G(p) = \{ue_{12} + z_0 e_3 \mid ux_0 > 0\}$, and so it is a one-dimensional light-like singular orbit. All these orbits are of the same type since their stabilizers are conjugate to N.

(ii) Let $G = AN \ltimes_\pi \{u(\alpha e_{12} + \beta e_3) \mid u \in \mathbb{R}\}$.

Let $\alpha = 0$. If $x_0 = y_0$, then $G(p) = \{ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$, which is a two-dimensional degenerate orbit. If $x_0 \neq y_0$, then $\dim G(p) = 3$ and $(u, v, w)^T \in G(p)$ if and only if $(u - v)(x_0 - y_0) > 0$. Hence, there are two open submanifolds as three-dimensional orbits. This implies that the two-dimensional degenerate orbit is an exceptional orbit.

Let $\beta = 0$ (and so $\alpha \neq 0$). If $x_0 = y_0$, then $G(p) = \{p + ue_{12} \mid u \in \mathbb{R}\}$, which is a one-dimensional light-like singular orbit. Furthermore, $(u, v, w)^T \in G((x, y, z)^T)$ if and only if $w = z$. This implies that there are infinitely many singular orbits. Since their stabilizers are conjugate to AN, all of them are of the same type. If $x_0 \neq y_0$, then $\dim G(p) = 3$, and $(u,v,w)^T \in G(p)$ if and only if $(u-v)(x_0-y_0) > 0$. Hence, there are two open submanifolds as three-dimensional orbits.

Now let α and β be nonzero. If $x_0 = y_0$, then $G(p) = \{p + \alpha u e_{12} + \beta u e_3 \mid u \in \mathbb{R}\}$, which is a one-dimensional light-like singular orbit. As in the previous case, there are infinitely many singular orbits of the same type and two open orbits in $\mathbb{R}^{1,2}$.

(iii) Let $G = AN \ltimes_\pi \{ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$. If $x_0 = y_0$, then $G(p) = \{ue_{12} + ve_3 \mid u, v \in \mathbb{R}\}$. If $x_0 \neq y_0$ then $\dim G(p) = 3$. In the later case, $(u,v,w)^T \in G((x,y,z)^T)$ if and only if $(u-v)(x-y) > 0$. Hence, there is a degenerate exceptional orbit and two orbits which are open submanifolds of $\mathbb{R}^{1,2}$. $\square$

By Corollary 3.7, the only case that we have not studied in three previous propositions is the case where G is conjugate to $SO_o(1,2)$. Let $G = SO_o(1,2)$ and let G act on $\mathbb{R}^{1,2}$ naturally. Then the origin is a zero-dimensional singular orbit, each component of the light-cone is an exceptional orbit, and each pseudo-sphere and each pseudo-hyperbolic space is a principal orbit. Hence, there is one singular orbit type, one exceptional orbit type and two principal orbit types. By reviewing Propositions 4.3–4.5, we get that there are at most two exceptional orbits and we obtain the following corollary.

Corollary 4.6. *Let $\mathbb{R}^{1,2}$ be of cohomogeneity one under the isometric and nonproper action of a connected and closed Lie subgroup $G \subset \mathrm{Iso}(\mathbb{R}^{1,2})$. If there is a unique exceptional orbit, then it is a degenerate plane and there are two orbits which are open submanifolds. In particular, the orbit space, which consists of three points, is not Housdorff.*

Acknowledgments. The author would like to thank the referee for invaluable suggestions for the revision of the paper.

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Received July 26, 2017, revised February 18, 2018.

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Динаміки кооднорідності один на тривимірному просторі Мінковського

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У роботі надається класифікація замкнутих і зв'язних груп Лі, з точністю до спряженості в $\mathrm{Iso}(\mathbb{R}^{1,2})$, що діють з кооднорідністю один на тривимірному просторі Мінковського $\mathbb{R}^{1,2}$ як для власної, так і невласної динаміки. Потім визначаються причинно-наслідкові властивості і типи орбіт.

Ключові слова: кооднорідність один, простір Мінковського.