Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 2, pp. 170–191
doi: https://doi.org/10.15407/mag15.02.170

# Analog of Hayman's Theorem and its Application to Some System of Linear Partial Differential Equations

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We used the analog of known Hayman's theorem to study the boundedness of **L**-index in joint variables of entire solutions of some linear higher-order systems of PDE's and found sufficient conditions providing the boundedness, where $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $l_j : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function $j \in \{1, \dots, n\}$. Growth estimates of these solutions are also obtained. We proposed the examples of systems of PDE's which prove the exactness of these estimates for entire solutions. The obtained results are new even for the one-dimensional case because of the weakened restrictions imposed on the positive continuous function $l$.

*Key words:* entire function, bounded **L**-index in joint variables, linear higher-order systems of PDE, analytic theory of PDE, entire solution, linear higher-order differential equation.

*Mathematical Subject Classification 2010:* 32W50, 32A15, 32A22, 35G35, 32A40.

## 1. Introduction

W. K. Hayman [19] proved that if $f(z)$ is analytic in $|z| < 2p$, where it satisfies

$$|f^{(p)}(z)| \le \max_{0\le\nu\le p-1} |f^{(\nu)}(z)|, \tag{1.1}$$

then $f(z)$ cannot have more than $(p-1)$ zeros in $|z| < \frac{\sqrt{p}}{e\sqrt{20}}$. Q.I. Rahman, J. Stankiewicz, V. Singh, and R.M. Goel [25, 30] refined this result and enlarged the value $\frac{\sqrt{p}}{e\sqrt{20}}$.

On the other hand, an entire function $f$ is called a function of bounded index [22] if there exists a nonnegative integer $p_0$ such that

$$\frac{|f^{(p)}(z)|}{p!} \le \max_{0\le j\le p_0} \frac{|f^{(j)}(z)|}{j!}$$

for all $z \in \mathbb{C}$ and for all $p \in \mathbb{Z}_+$. In order that an entire function $f$ be of bounded index [19], it is necesssary and sufficient that (1.1) be satisfied for all $z \in \mathbb{C}$.

In the theory of functions of bounded index this necessary and sufficient condition and its generalizations [1, 4, 15, 21, 28] are known as Hayman's Theorem. The criterion is very convenient [6, 11, 14] for studying the boundedness of index of entire solutions of ordinary or partial differential equations. The functions of this class have good properties: sharp growth estimates, uniform distribution of zeros in some sense, certain regular behavior of the solution, etc.

There are two approaches to introduce and study the index boundedness in $\mathbb{C}^n$. The first approach uses a *slice function* $g_{z_0}(t) := F(z^0 + t\mathbf{b})$, $t \in \mathbb{C}$, where $z^0 \in \mathbb{C}^n$ is an arbitrary fixed point, $n \geq 2$, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ is a given direction in $\mathbb{C}^n$, $F : \mathbb{C}^n \to \mathbb{C}$ is an entire function. Applying the slice function and directional derivative, we considered the functions of bounded $L$-index in direction (see the definition and properties in [5–7]). There were obtained sufficient conditions for the boundedness of $L$-index in direction of entire solutions of some linear PDE's [5–7]. The second approach is based on partial derivatives. They are a background for the concept of entire function of bounded index in joint variables (see the definition and inequality (1.2) below).

Let $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $l_j(z)$ are positive continuous functions of $z \in \mathbb{C}^n$, $j \in \{1, 2, \ldots, n\}$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is called *a function of bounded* $\mathbf{L}$*-index in joint variables* [2] if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n_+$:

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \leq \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \ K \in \mathbb{Z}^n_+, \ \|K\| \leq m\right\}, \tag{1.2}$$

where for partial derivatives of the entire function $F(z) = F(z_1, \ldots, z_n)$ we use the notation $F^{(K)}(z) = \frac{\partial^{\|K\|}F}{\partial z^K} = \frac{\partial^{k_1+\cdots+k_n}F}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}}$, and $\mathbf{L}^K(z) = l_1^{k_1}(z)\cdots l_n^{k_n}(z)$, $K! = k_1! \cdot \ldots \cdot k_n!$, $\|K\| = k_1 + \ldots + k_n$, $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$.

If $l_j(z_j) = l_j(|z_j|)$ for every $j \in \{1, 2, \ldots, n\}$, then we obtain a concept of entire functions of bounded $\mathbf{L}$-index in a sense of the definition given in [12, 15]. And if $l_j(z_j) \equiv 1$, then the entire function $F$ is called a *function of bounded index in joint variables* [16–18, 20, 26]. The least integer $m$ for which inequality (1.2) holds is called $\mathbf{L}$*-index in joint variables of the function* $F$ and is denoted by $N(F, \mathbf{L})$.

There are many papers [1, 12, 15–18, 23, 24, 26] devoted to the class of entire functions of bounded index in joint variables. The recent ones are about analytic functions [3, 4, 8, 9] in a ball or a polydisc satisfying (1.2). However, linear higher-order systems of PDE were considered only in two theses [13, 27]. In particular, in [13], there was considered the system

$$a_j(z)f^{(K_j^0)}(z) + \sum\nolimits_{\|K\|\leq s-1} g_{K,j}(z)f^{(K)}(z) = h_j(z), \ j \in \{1, \ldots, m\}, \tag{1.3}$$

where for all $j \in \{1, \ldots, m\}$ $\|K_j^0\| = s$, $\{f^{(K_j^0)}(z) : j = 1, \ldots, m\}$ is a set of all possible $s$-order partial derivatives of the function $f$, the entire functions $a_j$, $g_{K,j}$,

$h_j$ are the functions with separable variables of the form

$$g(z) = \prod_{j=1}^{n} g_j(z_j). \tag{1.4}$$

The author stated the conditions providing the boundedness of $\mathbf{L}$-index in joint variables for every entire solution, where $\mathbf{L}(z) = (l_1(|z_1|), \dots, l_n(|z_n|))$ and each function $l_j : \mathbb{R}_+ \to R_+$ is continuous. Obviously, restriction (1.4) is very strong. Earlier M. Salmassi [27] proved that every entire solution of the system

$$\begin{cases} a_0 f^{(n_1,0)}(z) + a_1 f^{(n_1-1,0)}(z) + \ldots + a_{n_1} f(z) = g(z), \ a_0 \neq 0, \\ b_0 f^{(0,n_2)}(z) + b_2 f^{(0,n_2-1)}(z) + \ldots + b_{n_2} f(z) = h(z), \ \ b_0 \neq 0, \end{cases} \quad z = (z_1, z_2), \tag{1.5}$$

is a function of bounded index in joint variables, where $a_j \in \mathbb{C}$, $b_i \in \mathbb{C}$, $h(z)$ and $g(z)$ are arbitrary entire functions in $\mathbb{C}^2$ of bounded index in joint variables. Unlike in [13], it was not assumed that $h(z)$ and $g(z)$ are functions with separable variables. Therefore, the following natural *question* arises: *Is it possible to deduce sufficient conditions of the boundedness of* $\mathbf{L}$*-index in joint variables for entire solutions of a linear higher-order system of PDE without assumption* (1.4)*?*

This paper gives a positive answer to the posed question for system (3.1) which is more general than (1.5). Theorems 3.1–3.4 are generalizations of Salmassi's results in the following directions:

– we do not assume that the coefficients in system (3.1) are constants;

– we consider a system that may also contain the mixed partial derivatives.

Theorems 3.1–3.4 are also improved analogs of the results from [13] for system (3.1) in the following directions:

– we do not assume that the coefficients in (3.1) are the functions with separable variables;

– the function $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ is of more general form than $\mathbf{L}(z) = (l_1(|z_1|), \dots, l_n(|z_n|))$, where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$;

– we obtain sharp, in general, growth estimates of entire solutions of the system. Note that the growth estimates of solutions are not discussed at all in [13, 27].

Recently, it has been proved in [10] that if $F$ is an entire function of bounded $\mathbf{L}$-index in joint variables $N(F, L)$ and the function $\mathbf{L}$ satisfies some additional assumptions, then

$$\overline{\lim_{|R|\to+\infty}} \frac{\ln \max\{|F(z)|: \ z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}\,(\tau R e^{i\Theta})\rangle d\tau} \leq N(F, L) + 1.$$

Thus, in the paper we will estimate the growth of the same fraction by some constants. The paper uses the methods from [1, 2, 7, 10, 14] to study the entire solutions of system (3.1).

## 2. Auxiliary notations and propositions

We need some standard notations. Let $\mathbb{R}_+ = [0, +\infty)$. Denote $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n_+$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n_+$, $\mathbf{1}_j = (0, \ldots, 0, \underbrace{1}_{j\text{-th place}}, 0, \ldots, 0) \in \mathbb{R}^n_+$.

For $A = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{C}^n$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions:

$$\begin{aligned} A + B &= (a_1 + b_1, \ldots, a_n + b_n), \\ AB &= (a_1 b_1, \cdots, a_n b_n), \\ A/B &= (a_1/b_1, \ldots, a_n/b_n), \\ A^B &= a_1^{b_1} a_2^{b_2} \cdot \ldots a_n^{b_n} \quad (\text{for } B \in \mathbb{Z}^n),\ |z| = \left( \sum_{j=1}^n |z_j|^2 \right)^{1/2}. \end{aligned}$$

If $A$, $B \in \mathbb{R}^n$, the notation $A < B$ means that $a_j < b_j$ ($j \in \{1, \ldots, n\}$). The relation $A \leq B$ is defined in a similar way. For $z$, $w \in \mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$, where $\overline{w}_k$ is the complex conjugate of $w_k$.

For $R = (r_1, \ldots, r_n)$, we denote by

$$\mathbb{D}^n(z^0, R) := \{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j,\ j \in \{1, \ldots, n\}\}$$

the polydisc, by

$$\mathbb{T}^n(z^0, R) := \{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j,\ j \in \{1, \ldots, n\}\}$$

its skeleton and by

$$\mathbb{D}^n[z^0, R] := \{z \in \mathbb{C}^n : \ |z_j - z_j^0| \leq r_j,\ j \in \{1, \ldots, n\}\}$$

the closed polydisc.

By $Q^n$ (in particular, $Q := Q^1$), we denote a class of positive continuous functions $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$ such that

$$\exists R_0 \in \mathbb{R}^n_+ \ \exists C, c \in \mathbb{R}^n_+ \ (\mathbf{0} < c \leq C) \ \forall z_0 \in \mathbb{C}^n \ \forall z \in \mathbb{D}^n[z_0, R_0/L(z_0)] \quad c \leq \mathbf{L}(z)/\mathbf{L}(z_0) \leq C. \tag{2.1}$$

Note that if (2.1) holds for some $R_0$, then (2.1) is valid for all $R \in \mathbb{R}^n_+$. Besides, if for all $z \in \mathbb{C}^n$, $j$, $m \in \{1, 2, \ldots, n\}$,

$$\left| \frac{\partial l_j(z)}{\partial z_m} \right| \leq P(c + |l_j(z)|), \quad P > 0,$$

then $\mathbf{L}^* \in Q^n$, where $\mathbf{L}^*(z) = (c + |l_1(z)|, \ldots, c + |l_n(z)|)$, $c > 0$. It is proved in [10, Lemma 1]. Particularly, if $\mathbf{L}(z) = (l_1(R), \ldots, l_n(R))$, $R = (|z_1|, \ldots, |z_n|)$, for every $j \in \{1, \ldots, n\}$ the function $l_j(R)$ is positive continuously differentiable and $|\nabla \ln l_j(R)| \leq P$ for all $R \in \mathbb{R}^n_+$, then $\mathbf{L} \in Q^n$, where $\nabla l_j(R) = \left( \frac{\partial l_j(R)}{\partial r_1}, \ldots, \frac{\partial l_j(R)}{r_n} \right)$.

Every function $\mathbf{L} \in Q^n$ has the property

$$\max_{\Theta\in[0,2\pi]^n} \int_0^1 \left\langle R, \mathbf{L}\left(\tau R e^{i\Theta}\right)\right\rangle d\tau \to +\infty \quad \text{as } |R| \to +\infty. \tag{2.2}$$

We will need the following analog of Hayman's Theorem.

**Theorem 2.1** ([1]). *Let $\mathbf{L} \in Q^n$. An entire function $F$ has bounded $\mathbf{L}$-index in joint variables if and only if there exists $p \in \mathbb{Z}_+$, $c \in \mathbb{R}_+$ such that for each $z \in \mathbb{C}^n$:*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} : \ \|J\| = p+1\right\} \le c \max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} : \ \|K\| \le p\right\}. \tag{2.3}$$

By $\overline{G}$, we denote the closure of a domain $G \subset \mathbb{C}^n$. Every entire function $F\colon \mathbb{C}^n \to \mathbb{C}$ is a function of bounded $\mathbf{L}$-index in joint variables with arbitrary continuos function $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}_+$ in any bounded domain $G \subset \mathbb{C}^n$.

**Theorem 2.2.** *Let $F(z)$ be an entire function, $G$ be a bounded domain in $\mathbb{C}^n$. If $\mathbf{L}\colon \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function and $F(z)$ is an entire function, then there exists $m \in \mathbb{Z}_+$ such that for all $z \in \overline{G}$ and $J = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}_+^n$:*

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \ K \in \mathbb{Z}_+^n, \ \|K\| \le m\right\}. \tag{2.4}$$

*Proof.* If $F(z) \equiv 0$, then (2.4) is obvious. Let $F(z) \not\equiv 0$. For every fixed $z^0 \in \mathbb{C}^n$, $\frac{|F^{(J)}(z^0)|}{J!}$ is the modulus of a coefficient of power series expansion of the function $F(z)$, $z \in T^n(z^0, \frac{\mathbf{1}}{L(z^0)})$ in the neighborhood of the point $z^0$. Since $F(z)$ is entire, $\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)} \to 0$ as $\|J\| \to \infty$ for every $z^0 \in G$, i.e., there exists $m_0 = m(z^0)$ for which inequality (2.4) holds.

Assume on the contrary that the set of values $m_0$ is not uniformly bounded in $z^0$, i.e., $\sup\limits_{z^0\in G} m_0 = +\infty$. Hence, for every $m \in \mathbb{Z}_+$, there exists $z^m \in \overline{G}$ and $J^m \in \mathbb{Z}_+^n$:

$$\frac{|F^{(J^m)}(z^m)|}{J^m!\mathbf{L}^{J^m}(z^m)} > \max\left\{\frac{|F^{(K)}(z^m)|}{K!\mathbf{L}^K(z^m)} : \ K \in \mathbb{Z}_+^n, \ \|K\| \le m\right\}. \tag{2.5}$$

Since $z^m \in \overline{G}$, there exists the subsequence $z'^m \to z' \in \overline{G}$ as $m \to +\infty$. By Cauchy's integral formula, for any $J \in \mathbb{Z}_+^n$ we have

$$\frac{F^{(J)}(z^0)}{J!} = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n(z^0,R)} \frac{F(z)}{(z-z^0)^{J+\mathbf{1}}}\, dz.$$

We rewrite (2.5) in the form

$$\max\left\{\frac{|F^{(K)}(z^m)|}{K!\mathbf{L}^K(z^m)} : \ K \in \mathbb{Z}_+^n, \ \|K\| \le m\right\}$$

$$\leq \frac{1}{(2\pi)^n \mathbf{L}^{J^m}(z^m)} \int_{\mathbb{T}^n(z^0, R/L(z^m))} \frac{|F(z)|}{|z - z^m|^{J^m + \mathbf{1}}} |dz|$$
$$\leq \frac{1}{R^{J^m}} \max\{|F(z)| \colon z \in G_R\}, \tag{2.6}$$

where $G_R = \bigcup_{z^* \in \overline{G}} \mathbb{D}^n[z^*, R/\mathbf{L}(z^*)]$, $R = (r_1, \dots, r_n) > \mathbf{0}$. We choose $R > \mathbf{1}$. Taking the limit in (2.6) as $m \to \infty$, we deduce

$$\forall K \in \mathbb{Z}^n_+ \quad \frac{|F^{(K)}(z')|}{K!\mathbf{L}^K(z')} \leq \lim_{m \to \infty} \frac{1}{R^{J^m}} \max\{|F(z)| \colon z \in G_R\} = 0$$

as $m \to +\infty$. Thus, all partial derivatives of the function $F$ at the point $z'$ equal 0. By a uniqueness theorem, $F(z) \equiv 0$, which is impossible. □

*Remark* 2.3. A similar proposition to Theorem 2.2 holds for entire functions of bounded $\mathbf{L}$-index in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \mathbf{0}$. This is valid under the additional assumption: $\forall z \in \overline{G}$ $g_z(t) := F(z + t\mathbf{b}) \not\equiv 0$, $t \in \mathbb{C}$ (see [7]).

Using Theorems 2.1 and 2.2, we obtain the following corollary.

**Corollary 2.4.** *Let $\mathbf{L} \in Q^n$, $F$ be an entire function, $G$ be a bounded domain in $\mathbb{C}^n$. The function $F$ is of bounded $\mathbf{L}$-index in joint variables if and only if there exists $p \in \mathbb{Z}_+$ and $c > 0$ such that for all $z \in \mathbb{C}^n \setminus G$ inequality (2.3) holds.*

Let us denote $a^+ = \max\{a, 0\}$, $A^+ = (a_1^+, \dots, a_n^+)$, where $a \in \mathbb{R}$, $A \in \mathbb{R}^n$.

Let $\mathbf{L}(z) = \mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable $r_k$, $k \in \{1, \dots, n\}$, $\Theta \in [0, 2\pi]^n$, $R = (r_1, \dots, r_n) > \mathbf{0}$.

By $QW^n$, we denote the class of functions $\mathbf{L} \in Q^n$ such that

$$\left\langle \frac{\mathbf{1}}{R}, \left( \frac{d}{dt} \frac{\mathbf{1}}{\mathbf{L}(tRe^{i\Theta})} \Big|_{t=1} \right)^+ \right\rangle \to 0 \quad (|R| \to +\infty, R \geq R_0 > \mathbf{0})$$

uniformly in $\Theta \in [0, 2\pi]^n$. For simplicity, we write $QW := QW^1$.

Let $\mathbf{L}(z) = \mathbf{L}(R)$ with $r_k = |z_k|$, $k \in \{1, \dots, n\}$, $R = (r_1, \dots, r_n) > \mathbf{0}$. Every function $\mathbf{L}(R)$ nondecreasing in each variable $r_k$ belongs to the class $QW^n$. In particular, a polynomial

$$\mathbf{L}(R) = \sum\nolimits_{\|J\| \leq p} a_J R^J$$

and an exponent

$$\mathbf{L}(R) = \exp\left( \sum\nolimits_{\|J\| \leq p} a_J R^J \right)$$

belong to the same class with $a_J \in \mathbb{R}^n_+$. The function $\mathbf{L}(R) = (\frac{1}{\ln r_1}, \dots, \frac{1}{\ln r_n})$ $(r_j > 1)$ is nonincreasing in each variable and also belongs to the class $QW^n$.

**Lemma 2.5.** *Let $\mathbf{L} \in QW^n$, $F$ be an entire function. If there exists $R' \in \mathbb{R}^n_+$, $p \in \mathbb{Z}_+$, $c > 0$ such that for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$ inequality (2.3) holds, then*

$$\overline{\lim_{|R|\to\infty}} \frac{\ln \max\{|F(z)|\colon z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \le \max\{1, c\}. \tag{2.7}$$

*Proof.* Let $R = (r_1, \ldots, r_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $R > R'$, $\Theta \in [0, 2\pi]^n$. Denote $r^* = \max_{1\le j\le n} r_j$, $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, \ldots, n\}$ and $A = (\alpha_1, \ldots, \alpha_n)$. We consider the function

$$g(t) = \max\left\{ \frac{\left|F^{(S)}\left(Ate^{i\Theta}\right)\right|}{\mathbf{L}^S\left(Ate^{i\Theta}\right)} : \ \|S\| \le p \right\}, \tag{2.8}$$

where $Ate^{i\Theta} = (\alpha_1 te^{i\theta_1}, \ldots, \alpha_n te^{i\theta_n})$ and $|At| > |R'|$, $t \in \mathbb{R}_+$.

Since the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{\mathbf{L}^S(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0, +\infty)$ outside the zero set of the function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0, +\infty)$ except, perhaps, for a countable set of points.

Let us denote $u_j(t) = u_j(t, R, \Theta) = l_j\left(\frac{tR}{r^*}e^{i\Theta}\right)$, where $t \in \mathbb{R}_+$, $j \in \{1, \ldots, n\}$. Let $\mathbf{L}(Re^{i\Theta})$ be a positive function continuously differentiable in each variable $r_k$, $k \in \{1, \ldots, n\}$, $\Theta \in [0, 2\pi]^n$. It is easy to check that $\mathbf{L} \in QW^n$ if

$$r^*\left(-\left(u_j(t, R, \Theta)\right)'_{t=r^*}\right)^+ / \left(r_j l_j^2\left(Re^{i\Theta}\right)\right) \to 0 \tag{2.9}$$

uniformly in $\Theta \in [0, 2\pi]^n$, $j \in \{1, \ldots, n\}$ as $|R| \to \infty$, $R \ge R_0 > \mathbf{0}$.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \le |g'(r)|$ which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{\mathbf{L}^S(Ate^{i\Theta})}\right) &= \frac{1}{\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}\left|F^{(S)}\left(Ate^{i\Theta}\right)\right| \\
&\qquad + \left|F^{(S)}\left(Ate^{i\Theta}\right)\right| \frac{d}{dt}\frac{1}{\mathbf{L}^S(Ate^{i\Theta})} \\
&\le \frac{1}{\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^n F^{(S+\mathbf{1}_j)}\left(Ate^{i\Theta}\right)\alpha_j e^{i\theta_j}\right| - \frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{s_j u_j'(t)}{l_j(Ate^{i\Theta})} \\
&\le \sum_{j=1}^n \frac{\left|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})\right|}{\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})}\alpha_j l_j(Ate^{i\Theta}) + \frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n \frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}.
\end{aligned} \tag{2.10}$$

For absolutely continuous functions $h_1, h_2, \ldots, h_k$ and $h(x) := \max\{h_j(z) : 1 \le j \le k\}$, $h'(x) \le \max\{h_j'(x) : 1 \le j \le k\}$, $x \in [a, b]$ (see [29, Lemma 4.1, p. 81]). The function $g$ is absolutely continuous, therefore, from (2.3) and (2.10) it follows that

$$g'(t) \le \max\left\{\frac{d}{dt}\left(\frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{\mathbf{L}^S(Ate^{i\Theta})}\right) : \|S\| \le p\right\}$$

$$\leq \max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\frac{\alpha_j l_j(Ate^{i\Theta})\left|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})\right|}{\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})}\right.$$
$$\left.+\frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{\mathbf{L}^{S}(Ate^{i\Theta})}\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}$$
$$\leq g(t)\left(\max\{1,c\}\sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta})+\max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right)$$
$$= g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t)=\max\{1,c\}\sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta}),\quad \gamma(t)=\max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus,

$$\frac{d}{dt}\ln g(t)\leq\beta(t)+\gamma(t)\quad\text{and}\quad g(t)\leq g(t_0)\exp\int_{t_0}^{t}(\beta(\tau)+\gamma(\tau))\,d\tau,$$

where $t_0$ is chosen such that $g(t_0)\neq 0$. The condition $\mathbf{L}\in W^n$ gives

$$\frac{\gamma(t)}{\beta(t)}=\frac{\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{\max\{1,c\}\sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta})}\leq p\sum_{j=1}^{n}\frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})}\leq p\varepsilon,$$

where $\varepsilon=\varepsilon(R)\to 0$ uniformly in $\Theta\in[0,2\pi]^n$, $t=r^*$ as $|R|\to\infty$.

But

$$\left|F(Ate^{i\Theta})\right|\leq g(t)\leq g(t_0)\exp\int_{t_0}^{t}(\beta(\tau)+\gamma(\tau))\,d\tau$$

and $r^*A=R$. Then we put $t=r^*$ and obtain

$$\ln\max\{|F(z)|\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\}=\ln\max_{\Theta\in[0,2\pi]^n}\left|F\left(Re^{i\Theta}\right)\right|\leq\ln\max_{\Theta\in[0,2\pi]^n}g(r^*)$$
$$\leq\ln g(t_0)+\max_{\Theta\in[0,2\pi]^n}\int_{t_0}^{r^*}(\beta(\tau)+\gamma(\tau))d\tau$$
$$\leq\ln g(t_0)+\max_{\Theta\in[0,2\pi]^n}\int_{t_0}^{r^*}\max\{1,c\}\sum_{j=1}^{n}\alpha_j l_j\left(A\tau e^{i\Theta}\right)(1+p\varepsilon)\,d\tau$$
$$=\ln g(t_0)+\max\{1,c\}\max_{\Theta\in[0,2\pi]^n}\int_{t_0}^{r^*}\sum_{j=1}^{n}\frac{r_j}{r^*}l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)(1+p\varepsilon)\,d\tau.$$

This implies (2.7). □

**Lemma 2.6.** *Let $\mathbf{L} \in QW^n$, $F$ be an entire function. If there exists $R' \in \mathbb{R}^n_+$, $p \in \mathbb{Z}_+$, $c > 0$ such that for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$ the inequality*

$$\max\left\{\frac{\left|F^{(J)}(z)\right|}{J!\mathbf{L}^J(z)}: \ \|J\| = p+1\right\} \leq c\max\left\{\frac{\left|F^{(K)}(z)\right|}{K!\mathbf{L}^K(z)}: \ \|K\| \leq p\right\} \tag{2.11}$$

*holds, then*

$$\varlimsup_{|R|\to\infty} \frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int_0^1\langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \leq (p+1)\max\{1,c\}. \tag{2.12}$$

*Proof.* The proof of Lemma 2.6 is similar to that of Lemma 2.5. Let $R = (r_1, \ldots, r_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $R > R'$, $\Theta \in [0, 2\pi]^n$. As in the proof of Lemma 2.5, we denote $r^* = \max_{1\leq j\leq n} r_j$, $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, \ldots, n\}$ and $A = (\alpha_1, \ldots, \alpha_n)$. We consider the function

$$g(t) = \max\left\{\frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{S!\mathbf{L}^S(Ate^{i\Theta})}: \ \|S\| \leq p\right\}, \tag{2.13}$$

where $Ate^{i\Theta} = \left(\alpha_1 te^{i\theta_1}, \ldots, \alpha_n te^{i\theta_n}\right)$, $|At| > |R'|$, $t \in \mathbb{R}_+$.

As above, the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0, +\infty)$ outside the zero set of the function $\left|F^{(S)}\left(Ate^{i\Theta}\right)\right|$, the function $g(t)$ is a continuously differentiable function on $[0, +\infty)$ except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \leq |g'(r)|$, which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) \\
&\quad= \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}|F^{(S)}(Ate^{i\Theta})| + |F^{(S)}(Ate^{i\Theta})|\frac{d}{dt}\frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})} \\
&\quad\leq \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^n F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})\alpha_j e^{i\theta_j}\right| - \frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j u_j'(t)}{l_j(Ate^{i\Theta})} \\
&\quad\leq \sum_{j=1}^n \frac{|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)!\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})}\alpha_j(s_j+1)l_j(Ate^{i\Theta}) \\
&\qquad\qquad + \frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}.
\end{aligned} \tag{2.14}$$

For absolutely continuous functions $h_1$, $h_2$, $\ldots$, $h_k$ and $h(x) := \max\{h_j(z) : 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x) : 1 \leq j \leq k\}$, $x \in [a, b]$ (see [29, Lemma 4.1, p. 81]). The function $g$ is absolutely continuous. Therefore, (2.3) and (2.14) yield

$$g'(t) \leq \max\left\{\frac{d}{dt}\left(\frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) : \|S\| \leq N\right\}$$

$$\leq \max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\frac{\alpha_j(s_j+1)l_j(Ate^{i\Theta})\left|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})\right|}{(S+\mathbf{1}_j)!\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})} + \frac{\left|F^{(S)}(Ate^{i\Theta})\right|}{S!\mathbf{L}^{S}(Ate^{i\Theta})}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}$$

$$\leq g(t)\left(\max\{1,c\}\max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\} + \max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right)$$

$$= g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t) = \max\{1,c\}\max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\},$$

$$\gamma(t) = \max_{\|S\|\leq p}\left\{\sum_{j=1}^{n}\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus,

$$\frac{d}{dt}\ln g(t) \leq \beta(t)+\gamma(t) \quad \text{and} \quad g(t)\leq g(t_0)\exp\int_{t_0}^{t}(\beta(\tau)+\gamma(\tau))\,d\tau,$$

where $t_0$ is chosen such that $g(t_0)\neq 0$. Denote $\widetilde{\beta}(t) = \sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta})$. Since $\mathbf{L}\in W^n$, for some $S^*$, $\|S^*\|\leq p$ and $\widetilde{S}$, $\|\widetilde{S}\|\leq p$, we obtain

$$\frac{\gamma(t)}{\widetilde{\beta}(t)} = \frac{\sum_{j=1}^{n}\frac{s_j^*(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{\sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta})} \leq \sum_{j=1}^{n}s_j^*\frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p\sum_{j=1}^{n}\frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \leq p\varepsilon$$

and

$$\frac{\beta(t)}{\widetilde{\beta}(t)} = \frac{\max\{1,c\}\sum_{j=1}^{n}\alpha_j(\tilde{s}_j+1)l_j(Ate^{i\Theta})}{\sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta})}$$

$$= \max\{1,c\}\left(1+\frac{\sum_{j=1}^{n}\alpha_j\tilde{s}_j l_j(Ate^{i\Theta})}{\sum_{j=1}^{n}\alpha_j l_j(Ate^{i\Theta})}\right)$$

$$\leq \max\{1,c\}\left(1+\sum_{j=1}^{n}\tilde{s}_j\right) \leq (1+p)\max\{1,c\},$$

where $\varepsilon=\varepsilon(R)\to 0$ uniformly in $\Theta\in[0,2\pi]^n$, $t=r^*$ as $|R|\to\infty$.

But

$$\left|F(Ate^{i\Theta})\right| \le g(t) \le g(t_0)\exp\int_{t_0}^{t}(\beta(\tau)+\gamma(\tau))\,d\tau$$

and $r^*A = R$. Then we put $t = r^*$ and obtain

$$\begin{aligned}
\ln\max\{|F(z)|\colon\ z\in\mathbb{T}^n(\mathbf{0},R)\} &= \ln\max_{\Theta\in[0,2\pi]^n}|F(Re^{i\Theta})| \le \ln\max_{\Theta\in[0,2\pi]^n} g(r^*)\\
&\le \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int_{t_0}^{r^*}(\beta(\tau)+\gamma(\tau))\,d\tau\\
&\le \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int_{t_0}^{r^*}\sum_{j=1}^{n}\alpha_j l_j(A\tau e^{i\Theta})\,(\max\{1,c\}(1+p)+p\varepsilon)\,d\tau\\
&= \ln g(t_0) + \max_{\Theta\in[0,2\pi]^n}\int_{t_0}^{r^*}\sum_{j=1}^{n}\frac{r_j}{r^*}l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)(\max\{1,c\}(1+p)+p\varepsilon)\,d\tau.
\end{aligned}$$

This implies (2.12). □

*Remark* 2.7. Note that condition (2.11) means that

$$\begin{aligned}
\max\left\{\frac{\left|F^{(J)}(z)\right|}{\mathbf{L}^J(z)}\colon\ \|J\|=p+1\right\} &\le \max\left\{\frac{\left|F^{(J)}(z)\right|}{J!\mathbf{L}^J(z)}\ \colon\ \|J\|=p+1\right\}\max_{\|J\|=p+1}J!\\
&\le c(p+1)!\max\left\{\frac{\left|F^{(K)}(z)\right|}{K!\mathbf{L}^K(z)}\ \colon\ \|K\|\le p\right\}\\
&\le \frac{c(p+1)!}{\min\limits_{\|K\|\le p}K!}\max\left\{\frac{\left|F^{(K)}(z)\right|}{\mathbf{L}^K(z)}\ \colon\ \|K\|\le p\right\}\\
&\le c(p+1)!\max\left\{\frac{\left|F^{(K)}(z)\right|}{\mathbf{L}^K(z)}\ \colon\ \|K\|\le p\right\}.
\end{aligned}$$

Hence, by Lemma 2.5, we have

$$\overline{\lim_{|R|\to\infty}}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \le \max\{1,c(p+1)!\}.$$

Since $c(p+1)! > c(p+1)$ for $p>1$, we see that Lemma 2.6 does not imply Lemma 2.5. Clearly, Lemma 2.5 does not imply Lemma 2.6 as well. Therefore we need both Lemma 2.5 and Lemma 2.6.

## 3. Growth and boundedness of L-index in joint variables of entire solutions of system of PDE's

Using the proved lemmas, we will formulate and prove the propositions that provide the growth estimates of entire solutions of the following system of partial differential equations:

$$G_{p_j\mathbf{1}_j}(z)F^{(p_j\mathbf{1}_j)}(z) + \sum_{\|S_j\|\le p_j-1}G_{S_j}(z)F^{(S_j)}(z) = H_j(z),\ j\in\{1,\dots,n\}, \quad (3.1)$$

$p_j \in \mathbb{N}$, $S_j \in \mathbb{Z}_+^n$, $H_j$ and $G_{S_j}$ are entire functions. Note if $\mathbf{L} \in Q^n$, then $\mathbf{L}$ satisfies (2.2). It is proved in (Lemma 2, [10]).

We will say that nonhomogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{H}, \mathbf{L})$, if $\mathbf{L} \in QW^n$, for all $z \in \mathbb{C}^n$ and for every $j \in \{1, \dots, n\}$ the entire functions $H_j$ and $G_{S_j}$ satisfy the following conditions:

1) for every $\|S_j\| \le p_j - 1$ and $M \in \mathbb{Z}_+^n$,

$$\|M\| \le 1 + \sum_{\substack{k=1\\k\neq j}}^n p_k, \quad \left|G_{S_j}^{(M)}(z)\right| \mathbf{L}^{S_j - M}(z) \le B_{S_j,M} l_j^{p_j}(z) \left|G_{p_j \mathbf{1}_j}(z)\right|,$$

and

$$\left|G_{p_j\mathbf{1}_j}^{(M)}(z)\right| \le B_{p_j\mathbf{1}_j,M} \mathbf{L}^M(z) \left|G_{p_j\mathbf{1}_j}(z)\right|,$$

2) for every $I \in \mathbb{Z}_+^n$,

$$\|I\| = 1 + \sum_{k=1, k\neq j}^n p_k, \quad \left|H_j^{(I)}(z)\right| \le D_{I,j} \mathbf{L}^I(z) \left|H_j(z)\right|,$$

3) $G_{p_j\mathbf{1}_j}(z) \neq 0$,

where $B_{S_j,M}$, $D_{I,j}$, $B_{p_j\mathbf{1}_j,M}$ are positive constants, $\mathbf{H}(z) = (H_1(z), \dots, H_n(z))$, $\mathbf{G}(z)$ is a matrix consisting of the coefficients $G_{S_J}(z)$ of system (3.1).

A homogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G}, \mathbf{0}, \mathbf{L})$ if condition 1) holds for $M \in \mathbb{Z}_+^n$ such that $\|M\| \le \sum_{k=1,k\neq j}^n p_k$ and $G_{p_j\mathbf{1}_j}(z) \neq 0$. Condition 2) is not required.

Instead of the condition $G_{p_j\mathbf{1}_j}(z) \neq 0$, we can require the validity of conditions 1) and 2) for all $z \in \mathbb{C}^n \setminus \mathbb{D}^n(\mathbf{0}, R')$. It is possible in view of Theorem 2.2. If for some $M \in \mathbb{Z}_+^n$ $G_{S_j}^{(M)}(z) \equiv 0$ or $H_j^{(M)}(z) \equiv 0$, then we suppose $B_{S_j,M} = 0$ or $D_{M,j} = 0$, respectively.

**Theorem 3.1.** *If nonhomogeneous system of PDE's* (3.1) *belongs to the class* $\mathcal{A}(\mathbf{G}, \mathbf{H}, \mathbf{L})$ *and an entire function* $F(z)$ *satisfies* (3.1)*, then* $F$ *has bounded* $\mathbf{L}$*-index in joint variables, and*

$$\varlimsup_{|R|\to\infty} \frac{\ln \max\{|F(z)|\colon z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle \, d\tau} \le \max\{1, c\}, \tag{3.2}$$

*where* $c$ *is defined in* (3.8)*.*

*Proof.* Taking into account that the function $F(z)$ satisfies system (3.1), we calculate the partial derivative $I \in \mathbb{Z}_+^n$ in each equation of the system

$$\sum_{\mathbf{0}\le M\le I} C_I^M \Bigg( G_{p_j\mathbf{1}_j}^{(M)}(z) F^{(p_j\mathbf{1}_j + I - M)}(z)$$

$$+\sum_{\|S_j\|\le p_j-1} G_{S_j}^{(M)}(z)F^{(S_j+I-M)}(z)\Bigg) = H_j^{(I)}(z), \tag{3.3}$$

where $C_I^M = \frac{i_1!\dots i_n!}{m_1!(i_1-m_1)!\dots m_n!(i_n-m_n)!}$ and $\|I\| = 1-p_j+\sum_{k=1}^n p_k = 1+\sum_{k=1,k\ne j}^n p_k$. Using condition 2) of the theorem, we obtain

$$\begin{aligned}\left|H_j^{(I)}(z)\right| &\le D_{I,j}\mathbf{L}^I(z)|H_j(z)|\\ &\le D_{I,j}\mathbf{L}^I(z)\left(\left|G_{p_j\mathbf{1}_j}(z)\right|\left|F^{(p_j\mathbf{1}_j)}(z)\right| + \sum_{\|S_j\|\le p_j-1}\left|G_{S_j}(z)\right|\left|F^{(S_j)}(z)\right|\right).\end{aligned} \tag{3.4}$$

Equation (3.3) yields

$$\begin{aligned}F^{(p_j\mathbf{1}_j+I)}(z) = \frac{1}{G_{p_j\mathbf{1}_j}(z)}\Bigg(H_j^{(I)}(z) &- \sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M G_{p_j\mathbf{1}_j}^{(M)}(z)F^{(p_j\mathbf{1}_j+I-M)}(z)\\ &- \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} G_{S_j}^{(M)}(z)F^{(S_j+I-M)}(z)\Bigg).\end{aligned} \tag{3.5}$$

From (3.5) and condition 2) it follows that

$$\begin{aligned}\left|F^{(p_j\mathbf{1}_j+I)}(z)\right| \le \Bigg(&D_{I,j}\mathbf{L}^I(z)\left(\left|F^{(p_j\mathbf{1}_j)}(z)\right| + \sum_{\|S_j\|\le p_j-1}\frac{|G_{S_j}(z)|}{|G_{p_j\mathbf{1}_j}(z)|}\left|F^{(S_j)}(z)\right|\right)\\ &+ \sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M \frac{\left|G_{p_j\mathbf{1}_j}^{(M)}(z)\right|}{|G_{p_j\mathbf{1}_j}(z)|}\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right|\\ &+ \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1}\frac{\left|G_{S_j}^{(M)}(z)\right|}{|G_{p_j\mathbf{1}_j}(z)|}|F^{(S_j+I-M)}(z)|\Bigg)\\ \le\ &D_{I,j}\mathbf{L}^I(z)\left(\left|F^{(p_j\mathbf{1}_j)}(z)\right| + \sum_{\|S_j\|\le p_j-1} B_{S_j,\mathbf{0}} l_j^{p_j}(z)L^{-S_j}(z)\left|F^{(S_j)}(z)\right|\right)\\ &+ \sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M}\mathbf{L}^M(z)\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right|\\ &+ \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} B_{S_j,M} l_j^{p_j}(z)\mathbf{L}^{M-S_j}(z)\left|F^{(S_j+I-M)}(z)\right|.\end{aligned} \tag{3.6}$$

Dividing this inequality by $l_j^{p_j}(z)\mathbf{L}^I(z)$, we obtain that for every $I$, $\|I\| = 1+\sum_{k=1,k\ne j}^n p_k$ and $j\in\{1,\dots,n\}$:

$$\frac{\left|F^{(p_j\mathbf{1}_j+I)}(z)\right|}{\mathbf{L}^{p_j\mathbf{1}_j+I}(z)} \le D_{I,j}\left(\frac{\left|F^{(p_j\mathbf{1}_j)}(z)\right|}{l_j^{p_j}(z)} + \sum_{\|S_j\|\le p_j-1} B_{S_j,\mathbf{0}}\frac{\left|F^{(S_j)}(z)\right|}{\mathbf{L}^{S_j}(z)}\right)$$

$$
\begin{aligned}
&+\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M}\frac{\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right|}{\mathbf{L}^{p_j\mathbf{1}_j+I-M}(z)}\\
&+\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} B_{S_j,M}\frac{\left|F^{(S_j+I-M)}(z)\right|}{\mathbf{L}^{S_j+I-M}(z)}\\
&\le\Bigg(D_{I,j}\Bigg(1+\sum_{\|S_j\|\le p_j-1}B_{S_j,\mathbf{0}}\Bigg)+\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M}\\
&+\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} B_{S_j,M}\Bigg)\max\left\{\frac{\left|F^{(S)}(z)\right|}{\mathbf{L}^{S}(z)}:\|S\|\le\sum_{j=1}^n p_j\right\}.
\end{aligned}
$$

Obviously, $\|p_j\mathbf{1}_j+I\|=1+\sum_{j=1}^n p_j$. This implies

$$
\begin{aligned}
\max&\left\{\frac{\left|F^{(K)}(z)\right|}{\mathbf{L}^{K}(z)}:\|K\|=1+\sum_{j=1}^n p_j\right\}\\
&\le\max\{1,c\}\max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^{S}(z)}:\|S\|\le\sum_{j=1}^n p_j\right\}, \qquad (3.7)
\end{aligned}
$$

where

$$
\begin{aligned}
c=\max_{\substack{\|I\|=1-p_j+\sum_{k=1}^n p_k,\\ j\in\{1,\dots,n\}}}&\Bigg(D_{I,j}\Bigg(1+\sum_{\|S_j\|\le p_j-1}B_{S_j,\mathbf{0}}\Bigg)\\
&+\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M}+\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} B_{S_j,M}\Bigg) \qquad (3.8)
\end{aligned}
$$

for all $z\in\mathbb{C}^n\setminus\mathbb{D}^n(\mathbf{0},R')$. Thus, by Lemma 2.5, estimate (3.2) holds, and by Corollary 2.4, the entire function $F$ has bounded $\mathbf{L}$-index in joint variables. □

If system (3.1) is homogeneous ($H_j(z)\equiv 0$), the previous theorem can be simplified.

**Theorem 3.2.** *If homogeneous system of PDE's* (3.1) *belongs to the class* $\mathcal{A}(\mathbf{G},\mathbf{0},\mathbf{L})$ *and an entire function* $F$ *is a solution of the system, then* $F$ *has bounded* $\mathbf{L}$*-index in joint variables, and*

$$
\overline{\lim_{|R|\to\infty}}\frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n}\int_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau}\le\max\{1,c\}, \qquad (3.9)
$$

*where* $c$ *is defined in* (3.8) *with* $D_{I,j}=0$ *and* $\|I\|=-p_j+\sum_{k=1}^n p_k$ *instead of* $\|I\|=1-p_j+\sum_{k=1}^n p_k$.

*Proof.* If $H_j(z)\equiv 0$, then (3.5) implies

$$F^{(p_j\mathbf{1}_j+I)}(z)=\frac{1}{G_{p_j\mathbf{1}_j}(z)}\Bigg(-\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M G^{(M)}_{p_j\mathbf{1}_j}(z)F^{(p_j\mathbf{1}_j+I-M)}(z) - \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} G^{(M)}_{S_j}(z)F^{(S_j+I-M)}(z)\Bigg). \tag{3.10}$$

Hence we obtain

$$\left|F^{(p_j\mathbf{1}_j+I)}(z)\right|\le\frac{1}{|G_{p_j\mathbf{1}_j}(z)|}\Bigg(\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M \left|G^{(M)}_{p_j\mathbf{1}_j}(z)\right|\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right| + \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} \left|G^{(M)}_{S_j}(z)\right|\left|F^{(S_j+I-M)}(z)\right|\Bigg).$$

Dividing the obtained inequality by $\mathbf{L}^{p_j\mathbf{1}_j+I}(z)$ and using the assumptions of the theorem on the functions $G_{S_j}$, we deduce

$$\begin{aligned}\frac{\left|F^{(p_j\mathbf{1}_j+I)}(z)\right|}{\mathbf{L}^{p_j\mathbf{1}_j+I}(z)} &\le \sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M}\frac{\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right|}{\mathbf{L}^{p_j\mathbf{1}_j+I-M}(z)} \\ &\quad + \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} B_{S_j,M}\frac{\left|F^{(S_j+I-M)}(z)\right|}{\mathbf{L}^{S_j+I-M}(z)} \\ &\le \Bigg(\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M} + \sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1} B_{S_j,M}\Bigg) \\ &\quad \times \max\left\{\frac{\left|F^{(S)}(z)\right|}{\mathbf{L}^S(z)} : \|S\|\le -1+\sum_{j=1}^n p_j\right\}.\end{aligned}$$

Obviously, $\|p_j\mathbf{1}_j+I\|=\sum_{j=1}^n p_j$. Therefore,

$$\max\left\{\frac{\left|F^{(K)}(z)\right|}{\mathbf{L}^K(z)} : \|K\|=\sum_{j=1}^n p_j\right\} \le \max\{1,c\}\max\left\{\frac{|F^{(S)}(z)|}{\mathbf{L}^S(z)} : \|S\|\le -1+\sum_{j=1}^n p_j\right\}$$

for all $z\in\mathbb{C}^n\setminus\mathbb{D}^n(\mathbf{0},R')$. Thus, all conditions of Corollary 2.4 are satisfied. Hence the function $F$ has a bounded $\mathbf{L}$-index in joint variables and, by Lemma 2.5, estimate (3.9) holds. □

Note that estimates (3.2) and (3.9) cannot be improved (see examples for $n = 1$ in [14]). Let us consider the case $n = 2$. For example, the entire function $w = 1 + e^{-z_1-z_2}$ is a solution of the nonhomogeneous system

$$\begin{cases} w'_{z_1} + w = 1, \\ w'_{z_2} + w = 1. \end{cases} \tag{3.11}$$

In view of (3.1) and (3.11), we have $H_1(z) = H_2(z) = 1$, $G_{1,1}(z) = G_{1,2} \equiv 1$, $G_{2,1}(z) = G_{2,2} \equiv 1$, where $G_{i,j}$ is the coefficient in the $i$-th equation of system (3.11) at $j$-th order partial derivative in variable $z_i$ of the function $w$. Obviously, the entire function $w$ has bounded index in joint variables, that is, $\mathbf{L}(z_1, z_2) = (1, 1)$. Validating the assumptions of Theorem 3.1, we deduce $B_{i,j,M} \equiv 0$ for $M \neq \mathbf{0}$ and $B_{i,j,\mathbf{0}} = 1$. Also, all second order partial derivatives of $H_1$ and $H_2$ equal 0. Then $D_{I,1} = D_{I,2} = 0$, where $I \in \mathcal{I} := \{(2, 0), (1, 1), (0, 2)\}$. Hence, in view of (3.8), $c = 1$. Thus, by Theorem 3.1,

$$\varlimsup_{|R|\to\infty} \frac{\ln \max\{|w(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 \sum_{j=1}^2 r_j \, d\tau} \leq 1.$$

But

$$\varlimsup_{|R|\to\infty} \frac{\ln \max\{|1 + \exp(-z_1 - z_2)|\colon (z_1, z_2) \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 (r_1 + r_2) \, d\tau} = \varlimsup_{|R|\to\infty} \frac{r_1 + r_2}{(r_1 + r_2)\tau\big|_0^1} = 1.$$

Therefore, estimate (3.2) is sharp.

By analogy, we can consider a homogeneous system

$$\begin{cases} w'_{z_1} - z_2 w = 0, \\ w'_{z_2} - z_1 w = 0. \end{cases}$$

The function $w(z_1, z_2) = \exp(z_1 z_2)$ is its entire solution of bounded $\mathbf{L}$-index in joint variables, where $\mathbf{L}(z_1, z_2) = (|z_2|, |z_1|)$. Using Theorem 3.2, it is easy to show that

$$\varlimsup_{|R|\to\infty} \frac{\ln \max\{|w(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R)\}}{\int_0^1 (r_1 r_2 \tau + r_2 r_1 \tau) \, d\tau} \leq 1.$$

Direct calculations prove that this estimate is exact too. Thus, (3.9) is non-improvable.

Moreover, using Corollary 2.4 and Lemma 2.6, we can supplement two previous Theorems 3.1 and 3.2 with the propositions that contain the estimates $\max\{|F(z)|\colon z \in \mathbb{T}^n(\mathbf{0}, R)\}$ which sometimes can be better than (3.9) and (3.2).

Two following theorems have the proofs similar to those of Theorems 3.1 and 3.2.

**Theorem 3.3.** *If nonhomogeneous system of PDE's* (3.1) *belongs to the class* $\mathcal{A}(\mathbf{G}, \mathbf{H}, \mathbf{L})$ *and an entire function* $F(z)$ *satisfies* (3.1), *then* $F$ *has bounded* $\mathbf{L}$-*index in joint variables, and*

$$\varlimsup_{|R|\to\infty} \frac{\ln \max\{|F(z)|\colon z \in \mathbb{T}^n(\mathbf{0}, R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}\,(\tau R e^{i\Theta})\rangle \, d\tau} \leq \left(1 + \sum_{j=1}^n p_j\right) \max\{1, c'\}, \tag{3.12}$$

*where* $c'$ *is defined in* (3.13)*.*

*Proof.* As in the proof of Theorem 3.1, dividing (3.6) by $(p_j\mathbf{1}_j+I)!L^{p_j\mathbf{1}_j+I}(z)$, we obtain that for every $\|I\|=1+\sum_{\substack{k=1\\k\neq j}}^n p_k$ and $j\in\{1,\ldots,n\}$:

$$\begin{aligned}
\frac{\left|F^{(p_j\mathbf{1}_j+I)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j+I}(z)} &\leq \frac{D_{I,j}\left|F^{(p_j\mathbf{1}_j)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j}(z)} + \sum_{\|S_j\|\leq p_j-1}\frac{D_{I,j}B_{S_j,\mathbf{0}}\left|F^{(S_j)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j-S_j}(z)} \\
&\quad + \sum_{\mathbf{0}\leq M\leq I,\, M\neq\mathbf{0}} C_I^M B_{p_j\mathbf{1}_j,M}\frac{\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j+I-M}(z)} \\
&\quad + \sum_{\mathbf{0}\leq M\leq I} C_I^M \sum_{\|S_j\|\leq p_j-1} B_{S_j,M}\frac{\left|F^{(S_j+I-M)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{S_j+I-M}(z)} \\
&\leq D_{I,j}\left(\frac{\left|F^{(p_j\mathbf{1}_j)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j}(z)}\right. \\
&\quad \left. + B\sum_{\|S_j\|\leq p_j-1}\frac{\left|F^{(S_j)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j-S_j}(z)}\right) \\
&\quad + \sum_{\mathbf{0}\leq M\leq I, M\neq\mathbf{0}}\frac{C_I^M B\left|F^{(p_j\mathbf{1}_j+I-M)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{p_j\mathbf{1}_j+I-M}(z)} \\
&\quad + \sum_{\mathbf{0}\leq M\leq I} C_I^M \sum_{\|S_j\|\leq p_j-1}\frac{B\left|F^{(S_j+I-M)}(z)\right|}{(p_j\mathbf{1}_j+I)!\mathbf{L}^{S_j+I-M}(z)} \\
&\leq \left(D_{I,j}\left(\frac{p_j!}{(p_j\mathbf{1}_j+I)!} + B\sum_{\|S_j\|\leq p_j-1}\frac{(p_j\mathbf{1}_j-S_j)!}{(p_j\mathbf{1}_j+I)!}\right)\right. \\
&\quad + B\sum_{\substack{\mathbf{0}\leq M\leq I\\ M\neq\mathbf{0}}} C_I^M\frac{(p_j\mathbf{1}_j+I-M)!|}{(p_j\mathbf{1}_j+I)!} \\
&\quad \left. + B\sum_{\mathbf{0}\leq M\leq I} C_I^M\sum_{\|S_j\|\leq p_j-1}\frac{(S_j+I-M)!}{(p_j\mathbf{1}_j+I)!}\right) \\
&\qquad \times\max\left\{\frac{\left|F^{(S)}(z)\right|}{\mathbf{L}^S(z)} : \|S\|\leq\sum_{j=1}^n p_j\right\},
\end{aligned}$$

where

$$B=\max\left\{B_{S_j,M}, B_{p_j\mathbf{1}_j,M} : j\in\{1,\ldots,n\}, \mathbf{0}\leq M\leq I, \|I\|=1+\sum_{k=1,\,k\neq j}^n p_k\right\}.$$

Obviously, $\|p_j\mathbf{1}_j+I\|=1+\sum_{j=1}^n p_j$. For all $z\in\mathbb{C}^n\setminus\mathbb{D}^n(\mathbf{0},R')$, it implies

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}: \|K\| = 1+\sum_{j=1}^n p_j\right\} \le \max\{1,c'\}\max\left\{\frac{|F^{(S)}(z)|}{S!\mathbf{L}^S(z)}: \|S\| \le \sum_{j=1}^n p_j\right\},$$

where

$$c' = \max_{\substack{\|I\|=1-p_j+\sum_{k=1}^n p_k,\\ j\in\{1,\dots,n\}}} \left(D_{I,j}\left(\frac{p_j!}{(p_j\mathbf{1}_j+I)!} + B\sum_{\|S_j\|\le p_j-1}\frac{(p_j\mathbf{1}_j - S_j)!}{(p_j\mathbf{1}_j+I)!}\right) + B\sum_{\mathbf{0}\le M\le I, M\ne\mathbf{0}} C_I^M \frac{(p_j\mathbf{1}_j+I-M)!|}{(p_j\mathbf{1}_j+I)!} + B\sum_{\mathbf{0}\le M\le I} C_I^M \sum_{\|S_j\|\le p_j-1}\frac{(S_j+I-M)!}{(p_j\mathbf{1}_j+I)!}\right). \tag{3.13}$$

In view of Theorem 2.2, the entire function $F$ has bounded $\mathbf{L}$-index in joint variables. And by Lemma 2.6, estimate (3.12) holds. □

By analogy to the proofs of Theorems 3.2 and 3.3, the following assertion can be proved.

**Theorem 3.4.** *If homogeneous system of PDE's (3.1) belongs to the class $\mathcal{A}(\mathbf{G},\mathbf{0},\mathbf{L})$ and an entire function $F$ is a solution of the system, then $F$ has bounded $\mathbf{L}$-index in joint variables, and*

$$\varlimsup_{|R|\to\infty}\frac{\ln\max\{|F(z)|: z\in\mathbb{T}^n(\mathbf{0},R)\}}{\max_{\Theta\in[0,2\pi]^n}\int_0^1\langle R,\mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \le \sum_{j=1}^n p_j\max\{1,c\},$$

*where $c'$ is defined in (3.13) with $D_{I,j}=0$ and $\|I\| = -p_j+\sum_{k=1}^n p_k$ instead of $\|I\| = 1-p_j+\sum_{k=1}^n p_k$.*

If $\mathbf{L}(z)\equiv\mathbf{1}$, then Theorem 3.1 implies the following corollary.

**Corollary 3.5.** *For all $z\in\mathbb{C}^n\setminus\mathbb{D}^n(\mathbf{0},R')$ and for every $j\in\{1,\dots,n\}$, the entire functions $H_j$ and $G_{S_j}$ satisfy the following conditions:*

1) *for every $\|S_j\|\le p_j-1$ and for each $M\in\mathbb{Z}_+^n$, $\|M\|\le 1+\sum_{\substack{k=1\\k\ne j}}^n p_k$,*
$\left|G_{S_j}^{(M)}(z)\right| \le B_{S_j,M}|G_{p_j\mathbf{1}_j}(z)|$ *and* $\left|G_{p_j\mathbf{1}_j}^{(M)}(z)\right| \le B_{p_j\mathbf{1}_j,M}|G_{p_j\mathbf{1}_j}(z)|$,

2) *for every $I\in\mathbb{Z}_+^n$, $\|I\| = 1+\sum_{\substack{k=1\\k\ne j}}^n p_k$,* $\left|H_j^{(I)}(z)\right|\le D_{I,j}|H_j(z)|$,

3) $G_{p_j\mathbf{1}_j}(z)\ne 0$,

*where $B_{S_j,M}$, $D_{I,j}$, $B_{p_j\mathbf{1}_j,M}$ are positive constants. If an entire function $F(z)$ satisfies (3.1), then $F$ has bounded index in joint variables, and*

$$\overline{\lim_{|R|\to\infty}} \frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{r_1+\ldots+r_n} \le \max\{1,c\},$$

*where $c$ is defined in (3.8).*

Similar corollaries also can be obtained from Theorems 3.2–3.4.

Suppose that all coefficients in (3.1) are polynomials. Let $l_0(z)=\max\limits_{1\le j\le n}|z_j|$, $\deg G_{S_j}(z)$ be the degree of a polynomial $G_{S_j}(z)$ that is the highest degree of its terms. Then Theorem 3.1 implies the following corollary.

**Corollary 3.6.** *Let $H_j(z)$ be a monomial, $G_{S_j}(z)$ be a polynomial, $G_{p_j\mathbf{1}_j}(z)\equiv 1$ in (3.1) for every $j\in\{1,\ldots,n\}$ and for each $\|S_j\|\le p_j-1$, $s=\max_{j,S_j}\frac{\deg G_{S_j}(z)}{\|p_j\mathbf{1}_j-S_j\|}$, $\mathbf{L}(z)=(l_0^s(z),\ldots,l_0^s(z))$. Then every entire solution $F$ of (3.1) has bounded $\mathbf{L}$-index in joint variables, and the inequality*

$$\overline{\lim_{|R|\to\infty}} \frac{\ln\max\{|F(z)|\colon z\in\mathbb{T}^n(\mathbf{0},R)\}}{\frac{(r^*)^s}{s+1}\sum_{j=1}^n r_j} \le \max\{1,c\}$$

*holds, where $c$ is defined in (3.8).*

*Remark* 3.7. We should like to note that the obtained propositions are improvements of the corresponding theorems for $n=1$ in [14]. Indeed, the author considered a positive continuous function $l=l(|z|)$ such that $l'(t)=o(l^2(t))$ as $t\to+\infty$. But our restrictions on the function $l$ are weaker. We study a positive continuous function $l=l(z)$ such that $(-(u(r,\theta))'_r)^+/l^2(r,\theta)\to 0$ uniformly in $\theta\in[0,2\pi]$ as $r\to\infty$, where $u(r,\theta)=l(re^{i\theta})$.

For example, if $n=1$, then system (3.1) reduces to the following equation:

$$g_p(z)f^{(p)}(z)+\sum_{j=0}^{p-1} g_j(z)f^{(j)}(z)=h(z), \tag{3.14}$$

where $h$ and $g_j$ are entire functions. Theorem 3.1 implies the corollary for $n=1$.

**Corollary 3.8.** *Let $l\in QW$ and for all $z\in\mathbb{C}$ such that $|z|>r'$ entire functions $h$ and $g_j$ satisfy the following conditions:*

1) $\left|g_j^{(m)}(z)\right|\le B_{j,m}l^{p-j+m}(z)|g_p(z)|$ *for every $j\in\{0,\ldots,p\}$, $m\in\{0,1\}$,*

2) $|h'(z)|\le Dl(z)|h(z)|$,

3) $g_p(z)\ne 0$,

*where $B_{j,m}$ and $D$ are positive constants. If an entire function $f$ satisfies (3.14), then $f$ has bounded $l$-index in joint variables, and*

$$\varlimsup_{r\to\infty}\frac{\ln\max\{|f(z)|\colon |z|=r\}}{\max\limits_{\theta\in[0,2\pi]}\int_0^r l\left(\tau e^{i\theta}\right)d\tau}\le \max\{1,c\},$$

*where* $c=D(1+\sum_{j=0}^{p-1}B_{j,0})+\sum_{j=0}^{p}B_{j,1}+\sum_{j=0}^{p-1}B_{j,0}$.

Similar corollaries can be obtained from Theorems 3.2–3.4 for $n=1$. Particularly, in a corollary from Theorem 3.2 the constant $c$ equals $\sum_{j=0}^{p-1}B_{j,0}$, but in a corollary from Theorem 3.4 the constant $c'$ equals $\max_j\{p,2,2B_{j,0}\}$, where $p$ is the order of differential equation. Each of these constants may be greater than or lesser than the other [14].

Suppose that all coefficients in (3.14) are polynomials.

**Corollary 3.9.** *Let $h(z)$, $g_j(z)$ be polynomials for every $j\in\{0,\dots,p-1\}$, $g_p(z)\equiv 1$ in (3.14), $s=\max\limits_{0\le j\le p-1}\frac{\deg g_j(z)}{p-j}$, $l(z)=|z|^s$, $z\in\mathbb{C}$. Then every entire solution $f$ of (3.14) has bounded $l$-index, and*

$$\varlimsup_{r\to\infty}\frac{\ln\max\{|f(z)|\colon |z|=r\}}{r^{s+1}/(s+1)}\le \max\{1,c\},$$

*where* $\mathcal{J}=\left\{j\colon \frac{\deg g_j}{p-j}=s,\ 0\le j\le p-1\right\}$, $g_j(z)=b_{j,m}z^m+\dots+b_{j,0}$, $c=\sum\limits_{j\in\mathcal{J}}|b_{j,\deg g_j}|$.

Note that Theorems 3.1–3.4 are proved for system (3.1). Perhaps, the following conjecture is true.

**Conjecture 3.10** (O.B. Skaskiv)**.** *The counterparts of Theorems 3.1–3.4 for entire solutions of system (1.3) are valid.*

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Received October 28, 2017, revised November 6, 2017.

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## Аналог теореми Хеймана та його застосування до однієї системи лінійних рівнянь з частинними похідними

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Аналог відомої теореми Хеймана застосовується до дослідження обмеженості **L**-індексу за сукупністю змінних цілих розв'язків деяких лінійних систем рівнянь з частинними похідними вищих порядків та знайдено достатні умови, які гарантують цю обмеженість, де $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, $l_j : \mathbb{C}^n \to \mathbb{R}_+$ — неперервна функція, $j \in \{1, \ldots, n\}$. Також отримано оцінки зростання цих розв'язків. Наведено приклади систем РЧП, які доводять точність встановлених оцінок для цілих розв'язків. Отримані результати також є новими в одновимірному випадку, бо послаблено додаткові умови на додатну неперервну функцію $l$.

*Ключові слова:* ціла функція, обмежений **L**-індекс за сукупністю змінних, лінійна система РЧП вищих порядків, аналітична теорія РЧП, цілий розв'язок, лінійне диференціальне рівняння вищого порядку.