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Frames in Quaternionic Hilbert Spaces

Sumit Kumar Sharma and Shashank Goel

In this paper, we introduce and study the frames in separable quaternionic Hilbert spaces. The results on the existence of frames in quaternionic Hilbert spaces and a characterization of frames in quaternionic Hilbert spaces in terms of frame operator are given. Finally, a Paley–Wiener type perturbation result for the frames in a quaternionic Hilbert space has been obtained.

Key words: frame, quaternionic Hilbert spaces.

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1. Introduction

Duffin and Schaeffer [11] indroduced *frames for Hilbert spaces* while working on some deep problems in non-harmonic Fourier series. They gave the following definition:

"Let $\mathcal{H}$ be a Hilbert space. Then $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exist finite constants A and B with $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.\text{"} \tag{1.1}$$

The positive constants A and B, respectively, are called the lower and upper frame bounds for the frame $\{x_n\}_{n\in\mathbb{N}}$. The inequality (1.1) is called the *frame inequality* for the frame $\{x_n\}_{n\in\mathbb{N}}$. A frame $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is said to be

- *tight* if it is possible to choose $A = B$,
- *Parseval* if it is a tight frame with $A = B = 1$.

Frame theory began to spread among researchers when in 1986, Daubechies, Grossmann and Meyer published a fundamental paper [10] and observed that frames can also be used to provide the series expansions of functions in $L^2(\mathbb{R})$. The main property of frames which makes them so useful is their redundancy, due to which, representation of signals using frames is advantageous over basis expansions in a variety of practical applications.

Frames work as an important tool in the study of signal and image processing [2], filter bank theory [4], wireless communications [13] and sigma-delta quantization [3]. For more literature on frame theory, one may refer to [5, 6, 9].

Recently, Khokulan, Thirulogasanthar and Srisatkunarajah [14] introduced and studied frames for finite dimensional quaternionic Hilbert spaces. Sharma and Virender [15] studied some different types of dual frames of a given frame in a finite dimensional quaternionic Hilbert space and gave various types of reconstructions with the help of a dual frame. Very recently, Chen, Dang and Qian [7] had studied frames for Hardy spaces in the contexts of the quaternionic space and the Euclidean space in the Clifford algebra. In this paper, we will introduce and study the frames in separable quaternionic Hilbert spaces. The results on the existence of frames in quaternionic Hilbert spaces have been given. Also, a characterization of frames in quaternionic Hilbert spaces in terms of the frame operator is given. Finally, a Paley–Wiener type perturbation result for frames in quaternionic Hilbert space has been obtained.

2. Quaternionic Hilbert space

As the quaternions are non-commutative in nature therefore there are two different types of quaternionic Hilbert spaces, the left quaternionic Hilbert space and the right quaternionic Hilbert space depending on position of quaternions. In this section, we will study some basic notations about the algebra of quaternions, right quaternionic Hilbert space and operators on right quaternionic Hilbert spaces.

Throughout this paper, we will denote $\mathfrak{Q}$ to be the non-commutative field of quaternions, I be a non-empty set of indices, $V_R(\mathfrak{Q})$ be a separable right quaternionic Hilbert space, by the term "right linear operator", we mean a "right $\mathfrak{Q}$-linear operator" and $\mathfrak{B}(V_R(\mathfrak{Q}))$ denotes the set of all bounded (right $\mathfrak{Q}$-linear) operators of $V_R(\mathfrak{Q})$:

$$\mathfrak{B}(V_R(\mathfrak{Q})) := \{T : V_R(\mathfrak{Q}) \to V_R(\mathfrak{Q}) : \ \|T\| < \infty\}.$$

The non-commutative field of quaternions $\mathfrak{Q}$ is a four-dimensional real algebra with unity. In $\mathfrak{Q}$, 0 denotes the null element and 1 denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by i, j, k, i.e.,

$$\mathfrak{Q} = \{x_0 + x_1 i + x_2 j + x_3 k : \ x_0, \ x_1, \ x_2, \ x_3 \in \mathbb{R}\},$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. For each quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathfrak{Q}$, define the conjugate of q denoted by $\overline{q}$ as

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k \in \mathfrak{Q}.$$

If $q = x_0 + x_1 i + x_2 j + x_3 k$ is a quaternion, then x_0 is called the real part of q and $x_1 i + x_2 j + x_3 k$ is called the imaginary part of q. The modulus of a quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is defined as

$$|q| = (\overline{q}q)^{1/2} = (q\overline{q})^{1/2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

For every non-zero quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathfrak{Q}$, there exists a unique inverse q^{-1} in $\mathfrak{Q}$ as

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{x_0 - x_1 i - x_2 j - x_3 k}{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Definition 2.1. A *right quaternionic vector space* $\mathbb{V}_R(\mathfrak{Q})$ is a linear vector space under right scalar multiplication over the field of quaternions $\mathfrak{Q}$, i.e.,

$$\begin{aligned} \mathbb{V}_R(\mathfrak{Q}) \times \mathfrak{Q} &\to \mathbb{V}_R(\mathfrak{Q}) \\ (u, q) &\to uq, \end{aligned} \tag{2.1}$$

and for each $u, v \in \mathbb{V}_R(\mathfrak{Q})$ and $p, q \in \mathfrak{Q}$, the right scalar multiplication (2.1) satisfies the following properties:

$$\begin{aligned} (u + v)q &= uq + vq, \\ u(p + q) &= up + uq, \\ v(pq) &= (vp)q. \end{aligned}$$

Definition 2.2. A *right quaternionic pre-Hilbert space* or *right quaternionic inner product space* $\mathbb{V}_R(\mathfrak{Q})$ is a right quaternionic vector space together with the binary mapping $\langle.|.\rangle : \mathbb{V}_R(\mathfrak{Q}) \times \mathbb{V}_R(\mathfrak{Q}) \to \mathfrak{Q}$ (called the *Hermitian quaternionic inner product*) which satisfies the following properties:

(a) $\overline{\langle v_1|v_2\rangle} = \langle v_2|v_1\rangle$ for all $v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$;

(b) $\langle v|v\rangle > 0$ if $v \neq 0$;

(c) $\langle v|v_1 + v_2\rangle = \langle v|v_1\rangle + \langle v|v_2\rangle$ for all $v, v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$;

(d) $\langle v|uq\rangle = \langle v|u\rangle q$ for all $v, u \in \mathbb{V}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.

In view of Definition 2.2, a right quaternionic inner product space $V_R(\mathfrak{Q})$ also has the property:

(i) $\langle vq|u\rangle = \overline{q}\langle v|u\rangle$ for all $v, u \in \mathbb{V}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.

Let $\mathbb{V}_R(\mathfrak{Q})$ be a right quaternionic inner product space with the Hermitian inner product $\langle.|.\rangle$. Define the quaternionic norm $\|\cdot\| : \mathbb{V}_R(\mathfrak{Q}) \to \mathbb{R}^+$ on $\mathbb{V}_R(\mathfrak{Q})$ by

$$\|u\| = \sqrt{\langle u|u\rangle}, \quad u \in \mathbb{V}_R(\mathfrak{Q}). \tag{2.2}$$

Definition 2.3. The right quaternionic pre-Hilbert space is called a *right quaternionic Hilbert space* if it is complete with respect to the norm (2.2) and is denoted by $V_R(\mathfrak{Q})$.

Theorem 2.4 (The Cauchy–Schwarz inequality [12])**.** *If $V_R(\mathfrak{Q})$ is a right quaternionic Hilbert space, then*

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle \quad \text{for all} \ \ u, v \in V_R(\mathfrak{Q}).$$

Moreover, a norm defined as in (2.2) satisfies the following properties:

(a) $\|uq\| = \|u\||q|$, *for all* $u \in V_R(\mathfrak{Q})$ *and* $q \in \mathfrak{Q}$;

(b) $\|u + v\| \leq \|u\| + \|v\|$, *for all* $u, v \in V_R(\mathfrak{Q})$;

(c) $\|u\| = 0$ *for some* $u \in V_R(\mathfrak{Q})$, *then* $u = 0$.

For the non-commutative field of quaternions $\mathfrak{Q}$, define the space $\ell_2(\mathfrak{Q})$ by

$$\ell_2(\mathfrak{Q}) = \left\{ \{q_i\}_{i\in I} \subset \mathfrak{Q} : \sum_{i\in I} |q_i|^2 < +\infty \right\}$$

under right multiplication by quaternionic scalars together with the quaternionic inner product on $\ell_2(\mathfrak{Q})$ defined as

$$\langle p|q\rangle = \sum_{i\in I} \overline{p_i} q_i, \quad p = \{p_i\}_{i\in I} \text{ and } q = \{q_i\} \in \ell_2(\mathfrak{Q}). \tag{2.3}$$

It is easy to observe that $\ell_2(\mathfrak{Q})$ is a right quaternionic Hilbert space with respect to the quaternionic inner product (2.3).

Definition 2.5 ([12]). Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and S be a subset of $V_R(\mathfrak{Q})$. Then, define the set:

- $S^{\perp} = \{v \in V_R(\mathfrak{Q}) : \langle v|u\rangle = 0 \ \forall \ u \in S\}$,
- $\langle S\rangle$ be the right $\mathfrak{Q}$-linear subspace of $V_R(\mathfrak{Q})$ consisting of all finite right $\mathfrak{Q}$-linear combinations of elements of S.

Let $i \to a_i \in \mathbb{R}^+$, $i \in I$, be a function on I. Then, define $\sum_{i\in I} a_i$ as the following element of $\mathbb{R}^+ \cup \{+\infty\}$:

$$\sum_{i\in I} a_i = \sup \left\{ \sum_{i\in J} a_i : \ J \text{ is a non-empty finite subset of } \ I \right\}.$$

It is clear that if $\sum_{i\in I} a_i < +\infty$, then the set of all $i \in I$ such that $a_i \neq 0$ is at most countable. In view of this, given a quaternionic Hilbert space $V_R(\mathfrak{Q})$ and a map $i \to u_i \in V_R(\mathfrak{Q})$, $i \in I$, the series $\sum_{i\in I} u_i$ is said to be convergent absolutely if $\sum_{i\in I} \|u_i\| < +\infty$. If this happens, then only a finite or countable number of u_i is non-zero and the series $\sum_{i\in I} u_i$ converges to a unique element of $V_R(\mathfrak{Q})$ independently from the ordering of $u_i's$.

Theorem 2.6 ([12]). *Let $V_R(\mathfrak{Q})$ be a quaternionic Hilbert space and let N be a subset of $V_R(\mathfrak{Q})$ such that, for $z, z' \in N$ such that $\langle z|z'\rangle = 0$ if $z \neq z'$ and $\langle z|z\rangle = 1$. Then the following conditions are equivalent:*

(a) *for every $u, v \in V_R(\mathfrak{Q})$, the series $\sum_{z\in N} \langle u|z\rangle\langle z|v\rangle$ converges absolutely and*

$$\langle u|v\rangle = \sum_{z\in N} \langle u|z\rangle\langle z|v\rangle;$$

(b) *for every* $u \in V_R(\mathfrak{Q})$, $\|u\|^2 = \sum_{z\in N} |\langle z|u\rangle|^2$;

(c) $N^{\perp} = 0$;

(d) $\langle N\rangle$ *is dense in* $V_R(Q)$.

Definition 2.7 ([12]). Every quaternionic Hilbert space $V_R(\mathfrak{Q})$ admits a subset N called *Hilbert basis or orthonormal basis* of $V_R(\mathfrak{Q})$, such that for $z, z' \in N$, $\langle z|z'\rangle = 0$ if $z \neq z'$ and $\langle z|z\rangle = 1$, and satisfies all the conditions of Theorem 2.6.

Further, if there are two such sets, then they have the same cardinality. Furthermore, if N is a Hilbert basis of $V_R(\mathfrak{Q})$, then every $u \in V_R(\mathfrak{Q})$ can be uniquely expressed as

$$u = \sum_{z\in N} z\langle z|u\rangle,$$

where the series $\sum_{z\in N} z\langle z|u\rangle$ converges absolutely in $V_R(\mathfrak{Q})$.

Definition 2.8 ([1]). Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and T be an operator on $V_R(\mathfrak{Q})$. Then T is said to be

- *right* $\mathfrak{Q}$*-linear* if $T(v_1\alpha + v_2\beta) = T(v_1)\alpha + T(v_2)\beta$ for all $v_1, v_2 \in V_R(\mathfrak{Q})$ and $\alpha, \beta \in \mathfrak{Q}$,
- *bounded* if there exist $K \geq 0$ such that $\|T(v)\| \leq K\|v\|$ for all $v \in V_R(\mathfrak{Q})$.

Definition 2.9 ([1]). Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and T be an operator on $V_R(\mathfrak{Q})$. Then the *adjoint operator* T^* of T is defined by

$$\langle v|Tu\rangle = \langle T^*v|u\rangle \quad \text{for all } u, v \in V_R(\mathfrak{Q}).$$

Further, T is said to be *self-adjoint* if $T = T^*$.

Theorem 2.10 ([1]). *Let* $V_R(\mathfrak{Q})$ *be a right quaternionic Hilbert space,* S *and* T *be two bounded right linear operators on* $V_R(\mathfrak{Q})$*. Then*

(a) $T + S$ *and* $TS \in \mathfrak{B}(V_R(\mathfrak{Q}))$*, moreover,*

$$\|T + S\| \leq \|T\| + \|S\| \text{ and } \|TS\| \leq \|T\|\|S\|;$$

(b) $\langle Tv|u\rangle = \langle v|T^*u\rangle$;

(c) $(T + S)^* = T^* + S^*$;

(d) $(TS)^* = S^*T^*$;

(e) $(T^*)^* = T$;

(f) $I^* = I$*, where* I *is the identity operator on* $V_R(\mathfrak{Q})$;

(g) *if T is the invertible operator, then* $(T^{-1})^* = (T^*)^{-1}$.

Theorem 2.11 ([12])**.** *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and let $T \in \mathfrak{B}(V_R(\mathfrak{Q}))$ be an operator. If $T \geq 0$, then there exists a unique operator in $\mathfrak{B}(V_R(\mathfrak{Q}))$, indicated by $\sqrt{T}$, such that $\sqrt{T} \geq 0$ and $\sqrt{T}\sqrt{T} = T$. Furthermore, it turns out that $\sqrt{T}$ commutes with every operator which commutes with T and if T is invertible and self-adjoint, then $\sqrt{T}$ is also invertible and self-adjoint.*

Definition 2.12. A sequence $\{q_i\}_{i\in I}$ of quaternions in $V_R(\mathfrak{Q})$ is said to be *semi-normalized* if there are two bounds $b \geq a > 0$ such that

$$a \leq |q_i| \leq b, \quad i \in I.$$

3. Frames in quaternionic Hilbert space

We begin this section with the following definition of frames in right quaternionic Hilbert spaces $V_R(\mathfrak{Q})$.

Definition 3.1. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$. Then $\{u_i\}_{i\in I}$ is said to be a *frame* for $V_R(\mathfrak{Q})$ if there exist two finite constants with $0 < A \leq B$ such that

$$A\|u\|^2 \leq \sum_{i\in I} |\langle u_i|u\rangle|^2 \leq B\|u\|^2 \quad \text{for all } u \in V_R(\mathfrak{Q}). \tag{3.1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{u_i\}_{i\in I}$. The inequality (3.1) is called the frame inequality for the frame $\{u_i\}_{i\in I}$. A sequence $\{u_i\}_{i\in I}$ is called the *Bessel sequence* for the right quaternionic Hilbert space $V_R(\mathfrak{Q})$ with bound B if $\{u_i\}_{i\in I}$ satisfies the right-hand side of inequality (3.1). A frame $\{u_i\}_{i\in I}$ for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$ is said to be

- *tight* if it is possible to choose A and B satisfying inequality (3.1) with $A = B$;
- *Parseval frame* if it is tight with $A = B = 1$;
- *exact* if it ceases to be a frame whenever one of its element is removed.

Regarding the existence of frames in the right quaternionic Hilbert space $V_R(\mathfrak{Q})$, we have the following examples:

Example 3.2. Let N be a Hilbert basis for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$ such that for each $z_i, z_k \in N$, $i, k \in \mathbb{N}$, we have

$$\langle z_i|z_k\rangle = \begin{cases} 0 & \text{for } i \neq k \\ 1 & \text{for } i = k \end{cases}.$$

1. *Tight and non-exact.* Let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$ defined as

$$u_i = u_{2i-1} = z_i, \quad i \in \mathbb{N}.$$

Then $\{u_i\}_{i\in\mathbb{N}}$ is a tight and non-exact frame for $V_R(\mathfrak{Q})$ with bound $A = 2$. Indeed, we have

$$\sum_{i\in\mathbb{N}} |\langle u_i|u\rangle|^2 = 2\sum_{i\in\mathbb{N}} |\langle z_i|u\rangle|^2 = 2\|u\|^2 \quad \text{for all } u \in V_R(\mathfrak{Q}).$$

2. *Non-tight and non-exact.* Let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$ defined as

$$\begin{cases} u_1 = z_1 \\ u_i = z_{i-1}, \quad i \geq 2,\ i \in \mathbb{N} \end{cases}.$$

Then $\{u_i\}_{i\in\mathbb{N}}$ is a non-tight and non-exact frame for $V_R(\mathfrak{Q})$. Indeed, we have

$$\|u\|^2 \leq \sum_{i\in\mathbb{N}} |\langle u_i|u\rangle|^2 \leq 2\|u\|^2 \quad \text{for all } u \in V_R(\mathfrak{Q}).$$

3. *Parseval.* Let $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$ defined as

$$\begin{cases} u_1 = z_1 \\ u_{i_k} = u_{i_k+1} = u_{i_k+2} = \ldots = u_{i_{k+1}-1} = \frac{z_k}{\sqrt{k}} \end{cases},$$

where $i_k = i_{k-1} + (k-1),\ k \in \mathbb{N},\ i_0 = 1$. Then $\{u_i\}_{i\in\mathbb{N}}$ is a Parseval frame for $V_R(\mathfrak{Q})$. Indeed, we have

$$\sum_{i\in\mathbb{N}} |\langle u_i|u\rangle|^2 = \sum_{i\in\mathbb{N}} i\left|\left\langle \frac{z_i}{\sqrt{i}}\middle| u\right\rangle\right|^2 = \|u\|^2 \quad \text{for all } u \in V_R(\mathfrak{Q}).$$

4. *Exact.* Let $\{z_i\}_{i\in\mathbb{N}}$ be a Hilbert basis of $V_R(\mathfrak{Q})$. Then $\{z_i\}_{i\in\mathbb{N}}$ is an exact frame for $V_R(\mathfrak{Q})$.

Next, we show that for a sequence $\{u_i\}_{i\in I}$ in a right quaternionic Hilbert space $V_R(\mathfrak{Q})$ being a Bessel sequence is a sufficient condition for the series $\sum_{i\in I} u_i q_i$, $\{q_i\}_{i\in I} \subset \ell_2(\mathfrak{Q})$ to converge unconditionally.

Theorem 3.3. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a Bessel sequence for $V_R(\mathfrak{Q})$ with Bessel bound B. Then, for every sequence $\{q_i\}_{i\in I} \in \ell_2(\mathfrak{Q})$, the series $\sum_{i\in I} u_i q_i$ converges unconditionally.*

Proof. Let $i, j \in I$, $i > j$. Then we have

$$\left\|\sum_{k=1}^{i} u_k q_k - \sum_{k=1}^{j} u_k q_k\right\| = \left\|\sum_{k=j+1}^{i} u_k q_k\right\|$$

$$
\begin{aligned}
&= \sup_{\|v\|=1} \left| \left\langle \sum_{k=j+1}^{i} u_k q_k \middle| v \right\rangle \right| = \sup_{\|v\|=1} \sum_{k=j+1}^{i} \left| \overline{q_k} \langle u_k | v \rangle \right| \\
&\leq \left(\sum_{k=j+1}^{i} |q_k|^2 \right)^{1/2} \sup_{\|v\|=1} \left(\sum_{k=j+1}^{i} |\langle u_k | v \rangle|^2 \right)^{1/2} \\
&\leq \sqrt{B} \left(\sum_{k=j+1}^{i} |q_k|^2 \right)^{1/2}.
\end{aligned}
$$

Since $\{q_i\}_{i\in I} \in \ell_2(\mathfrak{Q})$, $\left\{\sum_{k=1}^{i} |q_k|^2\right\}_{i\in I}$ is a Cauchy sequence in $\mathbb{R}$. Therefore $\left\{\sum_{k=1}^{i} u_k q_k\right\}_{i\in I}$ is a Cauchy sequence in $V_R(\mathfrak{Q})$. Hence $\left\{\sum_{i\in I} u_i q_i\right\}$ is unconditionally convergent in $V_R(\mathfrak{Q})$. □

If view of Theorem 3.3, if $\{u_i\}_{i\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$, then the (*right*) *synthesis operator* for $\{u_i\}_{i\in I}$ is a right linear operator $T : \ell_2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ defined by

$$T(\{q_i\}_{i\in I}) = \sum_{i\in I} u_i q_i, \quad \{q_i\} \in \ell_2(\mathfrak{Q}).$$

The adjoint operator T^* of the right synthesis operator T is called the (*right*) *analysis operator*. Further, the analysis operator $T^* : V_R(\mathfrak{Q}) \to \ell_2(\mathfrak{Q})$ is given by

$$T^*(u) = \{\langle u_i | u\rangle\}_{i\in I}, \quad u \in V_R(\mathfrak{Q}).$$

In fact, for $u \in V_R(\mathfrak{Q})$ and $\{q_i\}_{i\in I} \in \ell_2(\mathfrak{Q})$, we have

$$
\begin{aligned}
\langle T^*(u) | \{q_i\}_{i\in I}\rangle = \langle u | T(\{q_i\}_{i\in I})\rangle &= \left\langle u \middle| \sum_{i\in I} u_i q_i \right\rangle \\
&= \sum_{i\in I} \langle u | u_i\rangle q_i = \left\langle \{\langle u_i | u\rangle\}_{i\in I} \middle| \{q_i\}_{i\in I} \right\rangle.
\end{aligned}
$$

Thus,

$$T^*(u) = \{\langle u_i | u\rangle\}_{i\in I}, \quad u \in V_R(\mathfrak{Q}).$$

Next, we give a characterization for a Bessel sequence in a right quaternionic Hilbert space.

Theorem 3.4. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$. Then $\{u_i\}_{i\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$ with bound B if and only if the right linear operator $T : \ell_2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ defined by*

$$T(\{q_i\}_{i\in I}) = \sum_{i\in I} u_i q_i, \quad \{q_i\}_{i\in I} \in \ell_2(\mathfrak{Q})$$

is a well-defined and bounded operator with $\|T\| \leq \sqrt{B}$.

Proof. Let $\{u_i\}_{i\in I}$ be a Bessel sequence for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$. Then, by Theorem 3.3, T is a well-defined and bounded operator with $\|T\| \leq \sqrt{B}$.

Conversely, let T be a well-defined and bounded right linear operator with $\|T\| \leq \sqrt{B}$. Then the adjoint of a bounded right linear operator T is itself bounded and $\|T\| = \|T^*\|$. Since, for $u \in V_R(\mathfrak{Q})$ we have

$$\sum_{i\in I} |\langle u_i|u\rangle|^2 = \|T^*(u)\|^2 \leq \|T^*\|^2\|u\|^2 = \|T\|^2\|u\|^2,$$

it follows that $\{u_i\}_{i\in I}$ is the Bessel sequence for the right quaternionic Hilbert space $V_R(\mathfrak{Q})$ with bound B. $\square$

Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$. Then the *(right) frame operator* $S : V_R(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ for the frame $\{u_i\}_{i\in I}$ is the right linear operator given by

$$S(u) = TT^*(u) = T(\{\langle u_i|u\rangle\}_{i\in I}) = \sum_{i\in I} u_i\langle u_i|u\rangle, \quad u \in V_R(\mathfrak{Q}).$$

In the next result, we discuss some properties of the frame operator for a frame in a right quaternionic Hilbert space.

Theorem 3.5. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively, and frame operator S. Then S is a positive bounded invertible and self-adjoint right linear operator on $V_R(\mathfrak{Q})$.*

Proof. For any $u \in V_R(\mathfrak{Q})$, we have

$$\langle Su|u\rangle = \left\langle \sum_{i\in I} u_i\langle u_i|u\rangle|u \right\rangle = \sum_{i\in I} \overline{\langle u_i|u\rangle}\langle u_i|u\rangle = \sum_{i\in I} |\langle u_i|u\rangle|^2.$$

This gives

$$A\|u\|^2 \leq \langle Su|u\rangle \leq B\|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

Thus,

$$AI \leq S \leq BI. \tag{3.2}$$

Hence S is a positive and bounded right linear operator on $V_R(\mathfrak{Q})$. Also, $0 \leq I - B^{-1}S \leq \frac{B-A}{B}I$ and, consequently,

$$\|I - B^{-1}S\| = \sup_{\|v\|=1} \left|\left\langle (I - B^{-1}S)v\middle|v\right\rangle\right| \leq \frac{B-A}{B} < 1.$$

Then S is invertible. Further, for any $u, v \in V_R(\mathfrak{Q})$, we have

$$\langle Su|v\rangle = \left\langle \sum_{i\in I} u_i\langle u_i|u\rangle \middle| v \right\rangle = \sum_{i\in I} \overline{\langle u_i|u\rangle}\langle u_i|v\rangle$$

$$= \sum_{i\in I}\langle u|u_i\rangle\langle u_i|v\rangle = \left\langle u \,\middle|\, \sum_{i\in I} u_i\langle u_i|v\rangle \right\rangle = \langle u|Sv\rangle.$$

Thus S is also a self-adjoint right linear operator on $V_R(\mathfrak{Q})$. □

Corollary 3.6 (The reconstruction formula)**.** *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with frame operator S. Then every $u \in V_R(\mathfrak{Q})$ can be expressed as*

$$u = \sum_{i\in I} S^{-1}u_i\langle u_i|u\rangle.$$

Proof. S is invertible. Therefore, for $u \in V_R(\mathfrak{Q})$, we have

$$u = S^{-1}S(u) = \sum_{i\in I} S^{-1}u_i\langle u_i|u\rangle.$$ □

In the next result, we construct a new frame starting from a given frame in a right quaternionic Hilbert space.

Theorem 3.7. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively, and frame operator S. Then $\{S^{-1}u_i\}_{i\in I}$ is also a frame for $V_R(\mathfrak{Q})$ with bounds B^{-1} and A^{-1} and right frame operator S^{-1}.*

Proof. For $u \in V_R(\mathfrak{Q})$, we have

$$\sum_{i\in I}|\langle S^{-1}u_i|u\rangle|^2 = \sum_{i\in I}|\langle u_i|S^{-1}u\rangle|^2 \leq B\|S^{-1}u\|^2 \leq B\|S^{-1}\|^2\|u\|^2.$$

Hence, $\{S^{-1}u_i\}_{i\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$. It follows that the right frame operator for $\{S^{-1}u_i\}_{i\in I}$ is well-defined. Therefore, we have

$$\begin{aligned}\sum_{i\in I} S^{-1}u_i\langle S^{-1}u_i|u\rangle &= S^{-1}\Big(\sum_{i\in I} u_i\langle S^{-1}u_i|u\rangle\Big)\\ &= S^{-1}S(S^{-1}u) = S^{-1}u,\ \ u \in V_R(\mathfrak{Q}). \end{aligned} \tag{3.3}$$

Thus, the right frame operator for $\{S^{-1}u_i\}_{i\in I}$ is S^{-1}. The operator S^{-1} commutes with both S and $I : V_R(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ (the identity operator on $V_R(\mathfrak{Q})$). Therefore, multiplying the inequality (3.2) with S^{-1}, we have

$$B^{-1}I \leq S^{-1} \leq A^{-1}I,$$

i.e.,

$$B^{-1}\|u\|^2 \leq \langle S^{-1}u|u\rangle \leq A^{-1}\|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

By using (3.3), we get

$$B^{-1}\|u\|^2 \leq \sum_{i\in I}|\langle S^{-1}u_i|u\rangle|^2 \leq A^{-1}\|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

Therefore $\{S^{-1}u_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ with bounds $\frac{1}{B}$ and $\frac{1}{A}$ and right frame operator S^{-1}. □

In view of Theorem 3.7, we have the following definition:

Definition 3.8. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with frame operator S. Then the frame $\{S^{-1}u_i\}_{i\in I}$ is called the *canonical dual* of the frame $\{u_i\}_{i\in I}$.

Next, we construct a Parseval frame with the help of a given frame in a right quaternionic Hilbert space.

Theorem 3.9. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with frame operator S. Then $\{S^{-1/2}u_i\}_{i\in I}$ is a Parseval frame for $V_R(\mathfrak{Q})$.*

Proof. By Theorem 2.11, for any $u \in V_R(\mathfrak{Q})$, we have

$$u = S^{-1/2}SS^{-1/2}u = S^{-1/2}\sum_{i\in I} u_i\langle u_i|S^{-1/2}u\rangle = \sum_{i\in I} S^{-1/2}u_i\langle u_i|S^{-1/2}u\rangle. \quad (3.4)$$

Therefore, for any $u \in V_R(\mathfrak{Q})$, by using (3.4), we obtain

$$\begin{aligned}\|u\|^2 = \langle u|u\rangle &= \left\langle \sum_{i\in I} S^{-1/2}u_i\langle u_i|S^{-1/2}u\rangle \middle| u\right\rangle \\ &= \sum_{i\in I} \overline{\langle u_i|S^{-1/2}u\rangle}\langle S^{-1/2}u_i|u\rangle = \sum_{i\in I} |\langle S^{-1/2}u_i|u\rangle|^2.\end{aligned}$$

Thus $\{S^{-1/2}u_i\}_{i\in I}$ is a Parseval frame for $V_R(\mathfrak{Q})$. □

Next, we give a characterization of Parseval frames $\{u_i\}_{i\in I}$ for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$.

Theorem 3.10. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with frame operator S. Then $\{u_i\}_{i\in I}$ is a Parseval frame for $V_R(\mathfrak{Q})$ if and only if S is the identity operator on $V_R(\mathfrak{Q})$.*

Proof. Let $\{u_i\}_{i\in I}$ be a Parseval frame for $V_R(\mathfrak{Q})$. Then, for all $u \in V_R(\mathfrak{Q})$, we have

$$\sum_{i\in I} |\langle u_i|u\rangle|^2 = \|u\|^2$$

which implies

$$\langle Su|u\rangle = \langle u|u\rangle.$$

Thus, S is the identity operator on $V_R(\mathfrak{Q})$.

Conversely, let S be the identity operator on $V_R(\mathfrak{Q})$. Thus, for $u \in V_R(\mathfrak{Q})$, we have

$$u = S(u) = \sum_{i\in I} u_i\langle u_i|u\rangle. \quad (3.5)$$

So, we have

$$\|u\|^2 = \langle u|u\rangle = \left\langle \sum_{i\in I} u_i\langle u_i|u\rangle|u \right\rangle = \sum_{i\in I} |\langle u_i|u\rangle|^2.$$

Hence $\{u_i\}_{i\in I}$ is a Parseval frame for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$. □

In the next theorem, we give a necessary condition for a Parseval frame $\{u_i\}_{i\in I}$ for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$.

Theorem 3.11. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{q_i\}_{i\in I}$ be a semi-normalized sequence of quaternions in $\mathfrak{Q}$ with bounds a and b. If $\{u_iq_i\}_{i\in I}$ is a Parseval frame for $V_R(\mathfrak{Q})$, then $\{u_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ with bounds b^{-2} and a^{-2}.*

Proof. Let $\{u_iq_i\}_{i\in I}$ be a Parseval frame for $V_R(\mathfrak{Q})$. Then, for any $u \in V_R(\mathfrak{Q})$, we have

$$\sum_{i\in I} |\langle u_iq_i|u\rangle|^2 = \|u\|^2.$$

This gives

$$\sum_{i\in I} |q_i|^2|\langle u_i|u\rangle|^2 = \|u\|^2. \tag{3.6}$$

Since $\{q_i\}_{i\in I}$ is a normalized sequence with bounds a and b, we get

$$a^2\sum_{i\in I} |\langle u_i|u\rangle|^2 \le \sum_{i\in I} |q_i|^2|\langle u_i|u\rangle|^2 \le b^2\sum_{i\in I} |\langle u_i|u\rangle|^2, \quad u \in V_R(\mathfrak{Q}).$$

Therefore, by using (3.6), we have

$$\frac{1}{b^2}\|u\|^2 \le \sum_{i\in I} |\langle u_i|u\rangle|^2 \le \frac{1}{a^2}\|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

Hence $\{u_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ with bounds b^{-2} and a^{-2}. □

Finally, in this section we give a characterization of frames for a right quaternionic Hilbert space in terms of operators.

Theorem 3.12. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$. Then $\{u_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ if and only if the right linear operator $T : \ell_2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$,*

$$T(\{q_i\}_{i\in I}) = \sum_{i\in I} u_iq_i, \quad \{q_i\}_{i\in I} \in V_R(\mathfrak{Q}),$$

is a well-defined and bounded mapping from $\ell_2(\mathfrak{Q})$ onto $V_R(\mathfrak{Q})$.

Proof. Let $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$. Then the frame operator $S = TT^*$ for $\{u_i\}_{i\in I}$ is invertible on $V_R(\mathfrak{Q})$. So T is an onto mapping. Also, by Theorem 3.4, T is a well-defined and bounded mapping from $\ell_2(\mathfrak{Q})$ to $V_R(\mathfrak{Q})$.

Conversely, let T be a well-defined bounded right linear operator from $\ell_2(\mathfrak{Q})$ onto $V_R(\mathfrak{Q})$. Then, by Theorem 3.4, $\{u_i\}_{i\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$. Since T is an onto mapping, there exists a right linear operator $T^\dagger : V_R(\mathfrak{Q}) \to \ell_2(\mathfrak{Q})$ such that

$$u = TT^\dagger u = \sum_{i\in I} u_i (T^\dagger u)_i, \qquad u \in V_R(\mathfrak{Q}),$$

where $(T^\dagger u)_i$ denotes the i-th coordinate of $T^\dagger u$. Therefore, for $u \in V_R(\mathfrak{Q})$, we have

$$\begin{aligned}\|u\|^4 = |\langle u|u\rangle|^2 &= \left|\left\langle \sum_{i\in I} u_i (T^\dagger u)_i \middle| u \right\rangle\right|^2 \\ &\le \sum_{i\in I} |(T^\dagger u)_i|^2 \sum_{i\in I} |\langle u_i|u\rangle|^2 \le \|T^\dagger\|^2 \|u\|^2 \sum_{i\in I} |\langle u_i|u\rangle|^2.\end{aligned}$$

Hence $\{u_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ with bounds $\|T^\dagger\|^{-2}$ and $\|T\|^2$. □

4. Stability of frames in right quaternionic Hilbert space

Christensen [8] gave a version of Paley–Wiener theorem for frames in Hilbert spaces. In this section, we give a similar version for frames in right quaternionic Hilbert spaces.

Theorem 4.1. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively, and with frame operator S. Let $\{v_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$ and assume that there exist λ, $\mu \ge 0$ such that $\left(\lambda + \frac{\mu}{\sqrt{A}}\right) < 1$ and*

$$\left\| \sum_{i\in J} (u_i - v_i) q_i \right\| \le \lambda \left\| \sum_{i\in J} u_i q_i \right\| + \mu \left(\sum_{i\in J} |q_i|^2 \right)^{1/2} \tag{4.1}$$

for all finite sets of quaternions $q_i \in \mathfrak{Q}$, $i \in J \subseteq I$ with $|J| < +\infty$. Then $\{v_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ with bounds $A\left(1 - \left(\lambda + \frac{\mu}{\sqrt{A}}\right)\right)^2$ and $B\left(1 + \left(\lambda + \frac{\mu}{\sqrt{B}}\right)\right)^2$.

Proof. Let $J \subseteq I$ with $|J| < +\infty$. Then

$$\begin{aligned}\left\| \sum_{i\in J} v_i q_i \right\| &\le \left\| \sum_{i\in J} (u_i - v_i) q_i \right\| + \left\| \sum_{i\in J} u_i q_i \right\| \\ &\le (1+\lambda) \left\| \sum_{i\in J} u_i q_i \right\| + \mu \left(\sum_{i\in J} |q_i|^2 \right)^{1/2}.\end{aligned}$$

It holds

$$\Big\|\sum_{i\in J}u_iq_i\Big\| = \sup_{\|v\|=1}\Big|\Big\langle\sum_{i\in J}u_iq_i\Big|v\Big\rangle\Big|$$
$$\leq\Big(\sum_{i\in J}|q_i|^2\Big)^{1/2}\sup_{\|v\|=1}\Big(\sum_{i\in J}|\langle u_i|v\rangle|^2\Big)^{1/2}\leq\sqrt{B}\Big(\sum_{i\in J}|q_i|^2\Big)^{1/2}.$$

This gives

$$\Big\|\sum_{i\in J}v_iq_i\Big\|\leq\Big((1+\lambda)\sqrt{B}+\mu\Big)\Big(\sum_{i\in J}|q_i|^2\Big)^{1/2}. \tag{4.2}$$

Let $T:\ell_2(\mathfrak{Q})\to V_R(\mathfrak{Q})$ be the right linear operator defined by

$$T(\{q_i\}_{i\in I})=\sum_{i\in I}v_iq_i,\quad \{q_i\}_{i\in I}\in\ell_2(\mathfrak{Q}).$$

Then, by (4.2), T is well-defined and bounded with $\|T\|\leq\Big((1+\lambda)\sqrt{B}+\mu\Big)$. Therefore, by Theorem 3.4, $\{v_i\}_{i\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$ with the upper bound $B\Big(1+\Big(\lambda+\frac{\mu}{\sqrt{B}}\Big)\Big)^2$.

Let U be the right synthesis operator for the frame $\{u_i\}_{i\in I}$. Define an operator $W:V_R(\mathfrak{Q})\to\ell_2(\mathfrak{Q})$ by

$$W(u)=U^*(UU^*)^{-1}u,\quad u\in V_R(\mathfrak{Q}).$$

Since $\{(UU^*)^{-1}u_i\}_{i\in I}$ is also a frame for $V_R(\mathfrak{Q})$ with bounds $\frac{1}{B}$ and $\frac{1}{A}$, then

$$\|W(u)\|^2=\sum_{i\in I}|\langle S^{-1}u_i|u\rangle|^2\leq\frac{1}{A}\|u\|^2,\quad u\in V_R(\mathfrak{Q}).$$

Using (4.1) with $\{q_i\}_{i\in I}=Wu$, $u\in V_R(\mathfrak{Q})$, we get

$$\|u-TWu\|\leq\lambda\|u\|+\mu\|Wu\|\leq\Big(\lambda+\frac{\mu}{\sqrt{A}}\Big)\|u\|,\ u\in V_R(\mathfrak{Q}).$$

This gives $\|TW\|\leq 1+\lambda+\frac{\mu}{\sqrt{A}}$. Since $\Big(\lambda+\frac{\mu}{\sqrt{A}}\Big)<1$, then the operator TW is invertible and $\|(TW)^{-1}\|\leq\frac{1}{1-\Big(\lambda+\frac{\mu}{\sqrt{A}}\Big)}$. Now, for any $u\in V_R(\mathfrak{Q})$, we have

$$u=(TW)(TW)^{-1}u=\sum_{i\in I}v_i\langle(UU^*)^{-1}u_i|(TW)^{-1}u\rangle.$$

Therefore, for each $u\in V_R(\mathfrak{Q})$, we have

$$\|u\|^4=\Big|\Big\langle\sum_{i\in I}v_i\langle(UU^*)^{-1}u_i|(TW)^{-1}u\rangle\Big|u\Big\rangle\Big|^2$$

$$\leq \sum_{i\in I} \left|\left\langle (UU^*)^{-1}u_i|(TW)^{-1}u\right\rangle\right|^2 \left(\sum_{i\in I} |\langle v_i|u\rangle|^2\right)$$
$$\leq \frac{1}{A}\left[\frac{1}{1-\left(\lambda+\frac{\mu}{\sqrt{A}}\right)}\right]^2 \|u\|^2\left(\sum_{i\in I} |\langle v_i|u\rangle|^2\right).$$

This gives

$$\sum_{i\in I} |\langle v_i|u\rangle|^2 \geq A\left(1-\left(\lambda+\frac{\mu}{\sqrt{A}}\right)\right)^2 \|u\|^2, \quad u \in V_R(\mathfrak{Q}).$$

Hence $\{v_i\}_{i\in I}$ is a frame for $V_R(\mathfrak{Q})$ with the desired bounds. □

Corollary 4.2. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in I}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively. Let $\{v_i\}_{i\in I}$ be a sequence in $V_R(\mathfrak{Q})$ and assume that there exist $0 < R < A$ such that*

$$\|\sum_{i\in I}(u_i - v_i)q_i\| \leq \sqrt{R}\left(\sum_{i\in I}|q_i|^2\right)^{1/2} \quad \textit{for all } \{q_i\}_{i\in I} \in \ell_2(\mathfrak{Q}). \tag{4.3}$$

Then $\{v_i\}_{i\in I}$ is also a frame for $V_R(\mathfrak{Q})$ with bounds

$$\left(\sqrt{A}-\sqrt{R}\right)^2 \quad \textit{and} \quad \left(\sqrt{B}+\sqrt{R}\right)^2.$$

Proof. Take $\lambda = 0$ and $\mu = \sqrt{R}$ in Theorem 4.1. □

Remark 4.3. The condition $\left(\lambda + \frac{\mu}{\sqrt{A}}\right) < 1$ can not be dropped in Theorem 4.1. In this regard, we give the following example:

Example 4.4. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space, $\{z_i\}_{i\in\mathbb{N}}$ be a Hilbert basis of $V_R(\mathfrak{Q})$, $\{p_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{Q}$ and $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$ such that

$$v_i = z_i + z_{i+1}p_i, \quad i \in \mathbb{N}.$$

Then, for $J \subseteq \mathbb{N}$ with $|J| < \infty$ and $\{q_i\}_{i\in\mathbb{N}} \in \ell_2(\mathfrak{Q})$, we have

$$\left\|\sum_{i\in J}(v_i - z_i)q_i\right\| = \left\|\sum_{i\in J} z_{i+1}p_iq_i\right\| \leq \sup_i |p_i|\left(\sum_{i\in I}|q_i|^2\right)^{1/2}.$$

Thus, if $p = \sup_i |p_i| < 1$, then, by Theorem 4.1, $\{u_i\}_{i\in\mathbb{N}}$ is a frame for $V_R(\mathfrak{Q})$ with bounds $(1-p)^2$ and $(1+p)^2$. By taking $p_i = 1$, for all $i \in \mathbb{N}$, we get

$$v_i = z_i + z_{i+1}, \quad i \in \mathbb{N}. \tag{4.4}$$

Then, for any sequence $\{q_i\}_{i\in\mathbb{N}} \in \ell_2(\mathfrak{Q})$ and for $J \subseteq \mathbb{N}$ with $|J| < \infty$,

$$\left\| \sum_{i\in J}(v_i - z_i)q_i \right\| = \left\| \sum_{i\in J} z_{i+1}q_i \right\| \leq \left(\sum_{i\in I} |q_i|^2 \right)^{1/2}.$$

Thus, the condition (4.1) is satisfied with either $(\lambda,\mu) = (1,0)$ or $(0,1)$. So, $\left(\lambda + \frac{\mu}{\sqrt{A}}\right) \not< 1$ and $\{v_i\}_{i\in\mathbb{N}}$ is not a frame for $V_R(\mathfrak{Q})$.

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Sumit Kumar Sharma,
Kirori Mal College, University of Delhi, Delhi-110007, India,
E-mail: sumitkumarsharma@gmail.com

Shashank Goel,
Amity Institute of Applied Sciences, Amity University, Noida, U.P-201301, India,
E-mail: sgoel54@amity.edu

Фрейми в кватерніонних просторах Гільберта

Sumit Kumar Sharma and Shashank Goel

У статті представлено та вивчено фрейми в сепарабельних кватерніонних гільбертових просторах. Надано результати щодо існування фреймів у кватерніонних гільбертових просторах та представлено характеризацію фреймів у кватерніонних гільбертових просторах в термінах оператора фрейма. Нарешті, одержано результат щодо збурення типу Пелі–Вінера для фреймів у кватерніонному просторі Гільберта.

Ключові слова: фрейм, кватерніонні простори Гільберта.