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Quasi-Stability Method in Study of Asymptotic Behavior of Dynamical Systems

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In this survey, we have made an attempt to present the contemporary ideas and methods of investigation of qualitative dynamics of infinite dimensional dissipative systems. Essential concepts such as dissipativity and asymptotic smoothness of dynamical systems, global and fractal attractors, determining functionals, regularity of asymptotic dynamics are presented. We place the emphasis on the quasi-stability method developed by I. Chueshov and I. Lasiecka. The method is based on an appropriate decomposition of the difference of the trajectories into a stable and a compact parts. The existence of this decomposition has a lot of important consequences: asymptotic smoothness, existence and finite dimensionality of attractors, existence of a finite set of determining functionals, and (under some additional conditions) existence of a fractal exponential attractor. The rest of the paper shows the application of the abstract theory to specific problems. The main attention is paid to the demonstration of the scope of the quasi-stability method.

Key words: infinite dimensional dynamical systems, asymptotic behavior, global attractors, fractal exponential attractors, determining functionals, finite fractal dimension, quasi-stability, stability, PDEs.

Mathematical Subject Classification 2010: 35-02, 35B40, 35B41, 37-02, 37L05, 37L30.

1. Introduction

The idea of the quasi-stability method was introduced for the first time by I. Chueshov and I. Lasiecka in [35] and was inspired by the so-called stabilizability inequalities, which arise in control theory (see discussions in [39] and [41]). To some extent, this method is a weakened version of the decomposition method (see [2, 126]), which is used for establishing the asymptotic compactness of dissipative dynamical systems. However, in contrast to the standard decomposition method, the quasi-stability method assures the possibility of representation of *the difference* of two trajectories as a sum of compact and exponentially stable parts. Such a decomposition can be not only established for a wide class of infinite-dimensional models, but it also guarantees the existence and finite dimensionality of global attractors. Although the quasi-stability method originates from the theory of hyperbolic equations with nonlinear damping, nowadays it is used for studying many other classes of evolution dissipative systems [39,42]), including delayed systems or systems with memory (see, e.g., [50,51,69,70,113,119] and Theorem 9.3.5 in [41]). The same approach was applied in [33] to the analysis of the asymptotic behavior of a degenerated hyperbolic model. The quantum Zakharov system (see [23] and references therein) and several classes of interactive hydroelastic systems [25, 26, 52, 53] were also successfully studied by the quasi-stability method.

The main goal of the survey is to give a description of the quasi-stability method in the context of general infinite dimensional dissipative dynamical systems theory and to present typical applications of the method to dissipative problems in mathematical physics and solid mechanics.

The survey begins with a short description of general notions and methods in the area of infinite dimensional dissipative dynamics and the presentation of the main ideas of the quasi-stability method. All other sections are devoted to the applications of the method to various problems in mathematical physics.

2. General methods of studying asymptotic dynamics

In this section, we describe basic notions of the dynamical systems theory and introduce several approaches for studying asymptotic behavior of dissipative dynamical systems generated by partial differential equations. We pay significant attention to the quasi-stability method. The reader can find a more detailed description of the methods in monographs [2, 13, 18, 27, 83, 93, 122, 126]. The results presented in this section are rather standard and well-known. Our goal is to formulate these results concisely and make the text self-contained for the readers' convenience.

2.1. Basic notions. A *dynamical system (DS)* is a pair (X, S_t), consisting of a complete metric space X and a family of continuous mappings $\{S_t : t \in \mathbb{R}_+\}$ of X into itself, which satisfies the semigroup property: $S_0 = I$, $S_{t+\tau} = S_t S_\tau$. We also assume that $y(t) = S_t y_0$ is continuous with respect to t for any $y_0 \in X$. In this case, the space X is called a *phase space* (or a state space), and the operator family S_t is said to be an *evolution semigroup* (or an evolution operator, or a (semi)flow).

A closed set $B \subset X$ is called *absorbing* for DS (X, S_t) if for any bounded set $D \subset X$ there exists $t_0(D)$ such that $S_t D \subset B$ for all $t \geq t_0(D)$. DS (X, S_t) is called (bounded, or uniformly) *dissipative* if it possesses a bounded absorbing set B. If X is a Banach space and $B \subset \{x \in X : \|x\|_X \leq R\}$, then the value $R > 0$ is said to be a dissipativity radius of (X, S_t).

DS (X, S_t) is called *asymptotically smooth* if for every bounded set D such that $S_t D \subset D$ for all $t > 0$ there exists a compact K in the closure $\overline{D}$ of D such that

$$\lim_{t\to+\infty} d_X\{S_t D \mid K\} = 0,$$

where $d_X\{A|B\} = \sup_{x\in A} \mathrm{dist}_X(x,B)$. It can be shown (see [27] or [115]) that asymptotic smoothness is equivalent to Ladyzhenskaya condition: *for every bounded set D such that $\bigcup_{t\ge t_0} S_t D$ is bounded for some t_0, every sequence of the form $\{S_{t_n}x_n\}$ is precompact for all $\{x_n\}\subset D$ and $t_n\to+\infty$.*

A set $D\subset X$ is called *positively invariant* if $S_tD\subseteq D$ for all $t\ge 0$, and *negatively invariant* if $S_tD\supseteq D$ for all $t\ge 0$. A set D is called (strictly) *invariant* if it is both positively and negatively invariant, that is, $S_tD = D$ for all $t\ge 0$.

Let $D\subset X$. A set

$$\gamma_D^t \equiv \bigcup_{\tau\ge t} S_\tau D$$

is called a *tail* (from the moment t) of trajectories emanating from D. Evidently, $\gamma_D^t = \gamma_{S_tD}^0$.

If $D=\{v\}$ consists of a single point, then $\gamma_v^+ := \gamma_D^0$ is called a *positive semi-trajectory (or semiorbit)* emanating from v. The continuous curve $\gamma\equiv\{u(t): t\in\mathbb{R}\}$ in X is called a *full trajectory*) if $S_tu(\tau) = u(t+\tau)$ for any $\tau\in\mathbb{R}$ and $t\ge 0$. Since S_t can be a noninvertible operator, full trajectories may not exist. Semi-trajectories are positively invariant sets, full trajectories are (strictly) invariant sets.

For describing the asymptotic behavior of DS, the notion of ω-limit sets is used. A set

$$\omega(D) \equiv \bigcap_{t>0} \overline{\gamma_D^t} = \bigcap_{t>0} \overline{\bigcup_{\tau\ge t} S_\tau D}$$

is said to be an *ω-limit set* of trajectories emanating from D. The bar over a set denotes the closure of the set. Equivalently, $x\in\omega(D)$ if and only if there exist the sequences $t_n\to+\infty$ and $x_n\in D$ such that $S_{t_n}x_n\to x$ when $n\to\infty$. It is clear that the ω-limit sets (if they exist) are positively invariant.

2.2. Asymptotic smoothness criteria. The next assertion is a generalization of the Ceron-Lopez criterion (see [83]). Theorem 2.1 below is rather general and quite simple to apply. In some sense, it uses a weak form of quasi-stability inequality.

Theorem 2.1. *Let (X,S_t) be a dynamical system on a Banach space X. Let for every bounded positively invariant set B in X there exist $T>0$, a continuous nondecreasing function $g:\mathbb{R}_+\mapsto\mathbb{R}_+$, and a pseudometric ϱ_B^T on the space $C(0,T;X)$ of continuous functions with values in X such that*

(i) $g(0)=0$; $g(s)<s$, $s>0$;

(ii) *the pseudometric ϱ_B^T is precompact (with respect to the norm in X) in the following sense: every bounded sequence $\{x_n\}\subset B$ contains a subsequence $\{x_{n_k}\}$ such that the sequence $\{y_k\}\subset C(0,T;X)$ of the form $y_k(\tau)=S_\tau x_{n_k}$ is a Cauchy sequence with respect to the pseudometric ϱ_B^T;*

(iii) *for all $y_1,y_2\in B$, the following estimate holds:*

$$\|S_Ty_1 - S_Ty_2\| \le g\left(\|y_1-y_2\| + \varrho_B^T(\{S_\tau y_1\},\{S_\tau y_2\})\right),$$

where $\{S_\tau y_i\}$ *is an element of* $C(0,T;X)$ *defined by* $y_i(\tau) = S_\tau y_i$.

Then (X, S_t) *is an asymptotically smooth dynamical system.*

We note that the precompact pseudometric is defined for pieces of trajectories S_τ, but not for initial states (as it was in the classical interpretation, see, e.g., [83]). It turns out to be very useful for applications. The details of the proof of Theorem 2.1 can be found in [39]. It relies on the inequality

$$\alpha(S_T B) \le g(\alpha(B)),$$

where $\alpha(B)$ is a Kuratovsky α-measure of non-compactness defined by the formula

$$\alpha(B) = \inf\{\delta : \ B \text{ has a finite cover by sets of diameter } < \delta\} \tag{2.1}$$

for every bounded set B in X. More details about this characteristic can be found in [83] or [122, Lemma 22.2], [27, Chapter 2.2.2].

There exist two other criteria of asymptotic smoothness, which allow us not to use the assumption on the strong compactness of nonlinear terms:

- *the criterion of compensated compactness* is an idea, first introduced by Khanmamedov in [89] and later generalized in [39];
- *Ball's energy method*, see [4] and [106].

The corresponding results are given below.

Theorem 2.2 (The compensated compactness method)**.** *Let* (X, S_t) *be a dynamical system on a complete metric space* X *with a metric* d. *Assume that for every positively invariant set* B *in* X *and every* $\epsilon > 0$ *there exists* $T \equiv T(\epsilon, B)$ *such that*

$$d(S_T y_1, S_T y_2) \le \epsilon + \Psi_{\epsilon,B,T}(y_1, y_2), y_i \in B, \tag{2.2}$$

where $\Psi_{\epsilon,B,T}(y_1, y_2)$ *is a functional defined on* $B \times B$ *such that*

$$\liminf_{m\to\infty} \liminf_{n\to\infty} \Psi_{\epsilon,B,T}(y_n, y_m) = 0 \tag{2.3}$$

for any sequence $\{y_n\}$ *from* B. *Then* (X, S_t) *is an asymptotically smooth dynamical system.*

It follows from (2.2) and (2.3) that $\alpha(S_t B) \to 0$ if $t \to +\infty$, where $\alpha(B)$ is a Kuratovsky α-measure of non-compactness defined by (2.1). This implies the desired result. The details of the proof can be found in [39] or [41].

Note that in the "compact" case, when the functional Ψ is sequentially compact, condition (2.3) is satisfied automatically. The above criterion is used for critical nonlinearities (see [39, 41]).

The second method can be used for the cases of critical and supercritical nonlinearities (see Subsections 4.2 and 8.2), however, there are requirements which restrict the application of this method. The main idea of Ball's method is to construct an appropriate functional of energy type, which can be decomposed into exponentially decaying and compact terms. The idea of the decomposition of a semigroup into uniformly decaying and compact parts lies behind most of criteria of asymptotic smoothness, however, Ball's energy method described below is based on such a decomposition of *functionals*, but not of evolution operators (semigroups).

We introduce Ball's method following [106]. The original method for a dissipative wave equation is described in [4].

Theorem 2.3 (Ball's energy method)**.** *Let S_t be a continuous evolution semigroup of operators which are simultaneously strongly and weakly continuous. Let there exist the functionals Φ, Ψ, L, K, defined on the whole phase space, such that the energy inequality*

$$[\Phi(S_t u) + \Psi(S_t u)] + \int_s^t L(S_\tau u) e^{-\omega(t-\tau)} d\tau = [\Phi(S_s u) + \Psi(S_s u)] e^{-\omega(t-s)} + \int_s^t K(S_\tau u) e^{-\omega(t-\tau)} d\tau \quad (2.4)$$

holds for all $u \in X$, and the functionals Φ, Ψ, L and K satisfy the following properties:

- $\Phi : X \mapsto \mathbb{R}^+$ *is a bounded continuous functional, and if $\{U_j\}$ is a bounded sequence in X, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in X and* $\limsup_{n\to\infty} \Phi(S_{t_j} U_j) \le \Phi(U)$*, then $S_{t_j} U_j \to U$ strongly in X.*

- $\Psi : X \mapsto \mathbb{R}$ *is an asymptotically weakly continuous functional in the sense that if $\{U_j\}$ is bounded in X, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in X, then $\Psi(S_{t_j} U_j) \to \Psi(U)$.*

- $K : X \mapsto \mathbb{R}$ *is an asymptotically weakly continuous functional in the sense that if $\{U_j\}$ is bounded in X, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in X, then $K(S_s U) \in L_1(0,t)$, and*

$$\lim_{j\to\infty} \int_0^t e^{-\omega(t-s)} K(S_{s+t_j} U_j) ds = \int_0^t e^{-\omega(t-s)} K(S_s U) ds, \quad \forall t > 0.$$

- L *is an asymptotically weakly semicontinuous from below functional in the sense that if $\{U_j\}$ is bounded in X, $t_j \to +\infty$, $S_{t_j} U_j \rightharpoonup U$ weakly in X, then $L(S_s U) \in L_1(0,t)$, and*

$$\liminf_{j\to\infty} \int_0^t e^{-\omega(t-s)} L(S_{s+t_j} U_j) ds \ge \int_0^t e^{-\omega(t-s)} L(S_s U) ds, \quad \forall t > 0.$$

Then the semigroup S_t is asymptotically smooth.

Concerning the result above, we make the following remarks.

1. The existence of decomposition (2.4) depends also on whether the energy *equality* holds for weak solutions. This is quite a restrictive condition which may be difficult to verify (in contrast to the energy *inequality*). Besides, the proof of the *equality* in (2.4) for second-order equations requires the linearity of damping.

2. The functional Φ (usually convex) plays a role of the energy of the linearised system and is a good topological measure for the solution. In the uniformly convex spaces X the assumptions on Φ are satisfied automatically. Indeed, the weak convergence and the convergence of norms to the norm of the weak limit implies strong convergence.

3. The assumptions on the functionals Ψ, K, and L represent some properties of compactness of the nonlinear part of the energy of the system, which often take place even for supercritical nonlinearities. In fact, these terms allow us to work with noncompact sources in the equation in the case when the corresponding nonlinear part of the energy is sequentiallly compact. A three-dimensional wave equation with the source $|f(s)| \le C(1 + |s|^p)$ with $p < 5$ is a typical example, for which Ball's method can be applied due to the embedding $H^1(\Omega) \subset L_q(\Omega)$, $q \le 6$.

2.3. Global attractors. Attractors are principal objects which appear in the analysis of the asymptotic behavior of infinite-dimensional dynamical systems. Investigation of attractors allows us to answer a number of questions on the properties of limit regimes, which may appear in the system under consideration. Nowadays, there exist several general approaches and methods of proving the existence and finite dimensionality of global attractors for a wide range of dynamical systems generated by nonlinear PDEs (see, e.g., [2, 18, 27, 83, 93, 126] and references therein).

Definition 2.4. A bounded closed set $A \subset X$ is said to be a *global attractor* of a dynamical system (X, S_t) if the following properties hold:

(i) A is a strictly invariant set, i.e., $S_t A = A$ for all $t \ge 0$.

(ii) A is a uniformly attracting set, i.e., for every bounded set $D \subset X$,

$$\lim_{t\to+\infty} d_X\{S_t D \,|\, A\} = 0,$$

where $d_X\{A|B\} = \sup_{x\in A} \operatorname{dist}_X(x, B)$ is the Hausdorff semidistance.

It turns out that the dissipativity together with the asymptotic smoothness implies the existence of a global attractor. This result is well known (see [83], and [2, 93, 126]).

Theorem 2.5. *Every dissipative and asymptotically smooth DS (X, S_t) on a Banach space X possesses a unique global attractor A. The attractor is a connected set and can be described as a set of all bounded full trajectories. Besides, $A = \omega(B)$ for any bounded absorbing set B of the DS (X, S_t).*

In the case when a DS possesses a specific structure, namely, when the DS is *gradient*, the dissipativity is unnecessary (in the explicit form) for proving the existence of a global attractor. This is a very useful property, especially when it is technically complicated to prove dissipativity.

Below we introduce the notions of the Lyapunov function, the gradient DS and the unstable manifold, which are common in dynamical systems theory (see, e.g., [2, 18, 83, 93, 126]).

Definition 2.6. Let $Y \subseteq X$ be a positively invariant set of a DS (X, S_t).

- A continuous functional $\Phi(y)$, defined on Y, is said to be a *Lyapunov function* of the DS (X, S_t) on the set Y if a function $t \mapsto \Phi(S_t y)$ is non-increasing for any $y \in Y$.
- The Lyapunov function $\Phi(y)$ is said to be *strict* on Y if the equality $\Phi(S_t y) = \Phi(y)$ *for all* $t > 0$ implies $S_t y = y$ for all $t > 0$; that is, y is a stationary point of (X, S_t).
- A dynamical system (X, S_t) is said to be *gradient* if it possesses a strict Lyapunov function on the whole phase space X.

Definition 2.7. Let $\mathcal{N}$ be a set of stationary points of the DS (X, S_t):

$$\mathcal{N} = \{v \in X : \; S_t v = v \text{ for all } t \geq 0\}.$$

The unstable manifold $\mathcal{M}^u(\mathcal{N})$, emanating from the set $\mathcal{N}$, is a set of all $y \in X$ such that there exists a full trajectory $\gamma = \{u(t) : t \in \mathbb{R}\}$ for which the following properties hold:

$$u(0) = y \quad \text{and} \quad \lim_{t \to -\infty} \mathrm{dist}_X(u(t), \mathcal{N}) = 0.$$

The main result on the existence and properties of a global attractor for gradient systems is the following theorem (the proof can be found in [27, 39, 41]; see also [115, Theorem 4.6], where a similar result is proved).

Theorem 2.8. *Let (X, S_t) be a gradient asymptotically smooth DS. Assume that its Lyapunov function $\Phi(x)$ is bounded from above on every bounded subset of X and the sets $\Phi_R = \{x : \; \Phi(x) \leq R\}$ are bounded for every R. If, in addition, the set of the stationary points $\mathcal{N}$ of the DS (X, S_t) is bounded, then (X, S_t) possesses a compact global attractor $A = \mathcal{M}^u(\mathcal{N})$ with the following properties:*

- *the global attractor A consists of full trajectories $\gamma = \{u(t) : \; t \in \mathbb{R}\}$ such that*
$$\lim_{t \to -\infty} \mathrm{dist}_X(u(t), \mathcal{N}) = 0 \quad \textit{and} \quad \lim_{t \to +\infty} \mathrm{dist}_X(u(t), \mathcal{N}) = 0;$$
- *for any $x \in X$,*
$$\lim_{t \to +\infty} \mathrm{dist}_X(S_t x, \mathcal{N}) = 0,$$
i.e., every trajectory tends to the set of the stationary points of the DS $\mathcal{N}$;

- *if* $\mathcal{N} = \{z_1, \dots, z_n\}$ *is a finite set, then* $A = \cup_{i=1}^n \mathcal{M}^u(z_i)$*, where* $\mathcal{M}^u(z_i)$ *is an unstable manifold emanating from a stationary point* z_i*, and, moreover,*

  (i) *the global attractor* A *consists of full trajectories* $\gamma = \{u(t) :\ t \in \mathbb{R}\}$*, connecting pairs of stationary points: every* $u \in A$ *belongs to a full trajectory* γ *and for every* $\gamma \subset A$ *there exists a pair* $\{z, z^*\} \subset \mathcal{N}$ *such that*

$$u(t) \to z \quad \textit{as } t \to -\infty \qquad \textit{and} \qquad u(t) \to z^* \quad \textit{as } t \to +\infty;$$

  (ii) *for every* $v \in X$ *there exits a stationary point* z *such that* $S_t v \to z$ *when* $t \to +\infty$*.*

In many applications a question of stability of the attractor with respect to the parameters of the system is important. To describe this phenomenon at the abstract level, we consider a family of dynamical systems (X, S_t^λ) with the same phase space X and evolution operators S_t^λ depending on a parameter λ belonging to a complete metric space Λ. The following assertion is proved by Kapitansky and Kostin [88] (similar results can be found in [2] and [83]).

Theorem 2.9 (Upper semicontinuity)**.** *Let the dynamical system* (X, S_t^λ) *on the complete metric space* X *possess a compact global attractor* $\mathfrak{A}^\lambda$ *for every* $\lambda \in \Lambda$*. Let also the following conditions hold:*

(i) *there exists a compact set* $K \subset X$ *such that* $\mathfrak{A}^\lambda \subset K$*;*

(ii) *if* $\lambda_k \to \lambda_0$*,* $x_k \to x_0$ *and* $x_k \in \mathfrak{A}^{\lambda_k}$*, then*

$$S_\tau^{\lambda_k} x_k \to S_\tau^{\lambda_0} x_0 \quad \textit{for some } \tau > 0. \tag{2.5}$$

Then the family of attractors $\{\mathfrak{A}^\lambda\}$ *is upper semicontinuous in the point* λ_0*, i.e.,*

$$d_X\left\{\mathfrak{A}^\lambda \mid \mathfrak{A}^{\lambda_0}\right\} \equiv \sup\left\{\mathrm{dist}_X(x, \mathfrak{A}^{\lambda_0}) :\ x \in \mathfrak{A}^\lambda\right\} \to 0 \quad \textit{when } \lambda \to \lambda_0.$$

Moreover, if (2.5) *holds for every* $\tau > 0$*, then the upper limit* $\mathfrak{A}(\lambda_0, \Lambda)$ *of the attractor family* $\mathfrak{A}^\lambda$ *in* λ_0 *is defined by the formula*

$$\mathfrak{A}(\lambda_0, \Lambda) = \bigcap_{\delta>0} \overline{\bigcup \{\mathfrak{A}^\lambda :\ \lambda \in \Lambda,\ 0 < \mathrm{dist}(\lambda, \lambda_0) < \delta\}}$$

and is a non-empty compact strictly invariant set contained in the attractor $\mathfrak{A}^{\lambda_0}$ *and possesses the property*

$$d_X\left\{\mathfrak{A}^\lambda \mid \mathfrak{A}(\lambda_0, \Lambda)\right\} \to 0 \quad \textit{when } \lambda \to \lambda_0.$$

The finite dimensionality is an important property of an attractor which can be established for many important from the point of view of applications dynamical systems. There exist several methods of efficient estimation of the dimension of attractors of dynamical systems generated by PDEs. The survey of the approaches can be found in monograph [27].

Here we present a method that can be applied to locally Lipschitz flows. This method is a generalization of the Ladyzhenskaya theorem [93], but it does not require explicit construction of projectors. There is a similar approach [114], based on a squeezing property. However, the dimension estimates obtained with the help of this theorem (as well as the estimates based on the Ladyzhenskaya theorem) are often too pessimistic.

Definition 2.10. Let M be a compact set in a metric space X. *The fractal dimension* $\dim_f M$ of the set M is defined as

$$\dim_f M = \limsup_{\varepsilon\to 0} \frac{\ln n(M,\varepsilon)}{\ln(1/\varepsilon)} ,$$

where $n(M,\varepsilon)$ is the minimal number of closed balls of radius ε covering the set M.

To describe the complexity and properties of the embedding of compact sets one can also use the Hausdorff dimension $\dim_H$. We do not give formal definition of this notion here (see, e.g., [68]), but note that

(i) the Hausdorff dimension does not exceed (but in general case, is not equal to) the fractal dimension;

(ii) the fractal dimension is more convenient for calculations.

The following result is a generalization of [93] and it was established in [27]. Similar results were obtained earlier in [35] (see also [39]).

Theorem 2.11 (On the finite dimensionality of an attractor)**.** *Let H be a separable Banach space and M be a bounded closed set in H. Let there exist a map $V : M \mapsto H$ such that $M \subseteq VM$, a Lipschitz map K from M to a Banach space Z and a compact seminorm $n_Z(x)$ on Z such that*

$$\|Vv_1 - Vv_2\| \le \eta\|v_1 - v_2\| + n_z(Kv_1 - Kv_2)$$

for every $v_1, v_2 \in M$, where $0 < \eta < 1$ is a constant.

Then M is a compact in H set of finite fractal dimension, which can be estimated as

$$\dim_f M \le \ln m_z\left(\frac{4L_K}{1-\eta}\right)\left[\ln\frac{2}{1+\eta}\right]^{-1},$$

where L_K is the Lipschitz constant for K and $m_Z(R)$ is a minimal number of elements z_i in the ball $\{z \in Z : ||z|| \le R\}$ such that $||z_i - z_j||_Z > 1$ for $i \ne j$.

Seminorm $n(x)$ on a Banach space H is said to be compact if for every sequence $\{x_m\} \subset H$ such that $x_m \rightharpoonup 0$ weakly in H.

The main idea of the proof is to establish that the mapping V is an α-contraction in the sense of [83] (see also Lemma 2.18 in [39]) provided the assumptions of Theorem 2.11 hold.

2.4. Quasi-stable systems. In this section, we give a brief description (a more detailed presentation can be found in monograph [27]) of a special criterion which gives the existence of an attractor as well as its important properties: finite dimensionality, smoothness, existence of a fractal exponential attractor and so on. We will define a class of the so-called quasi-stable systems, which satisfy some stabilizability estimates given in general form. These inequalities imply a number of consequences which describe different properties of attractors. At the first time this type of stability attracted attention in the work by I. Chueshov and I. Lasiecka [35, Theorem 3.11], devoted to the dynamics of evolution equations of the second order in-time. Later this method was developed for many other nonlinear problems of mathematical physics, which we will consider in subsequent chapters. Recently the notion of quasi-stability was generalized for application to "parabolic-like" models. We will consider two classes of quasi-stable systems, which are motivated by different types of dynamics and demonstrate its additional properties. The first class is designed mainly for the study of semilinear parabolic problems, the second one is developed to deal with systems generated by second-order in-time equations.

2.4.1. General concept of quasi-stability. We begin with quasi-stability at a fixed time moment. This notion is motivated by several classes of PDEs, both parabolic and hyperbolic. The general idea serving as a basis of this notion can be applied to many other classes of problems.

Definition 2.12. Let (X, S_t) be a dynamical system on a Banach space X. The system is said to be *quasi-stable (QS)* on a set $\mathcal{B} \subset X$ at the moment t_* if there exist:

(a) a moment $t_* > 0$,

(b) a Banach space Z,

(c) a globally Lipschitz mapping $K : \mathcal{B} \mapsto Z$ and

(d) a compact seminorm $n_Z(\cdot)$ on the space Z such that

$$\|S_{t_*}y_1 - S_{t_*}y_2\|_X \leq q\|y_1 - y_2\|_X + n_Z(Ky_1 - Ky_2) \tag{2.6}$$

for all $y_1, y_2 \in \mathcal{B}$ with $0 \leq q < 1$.

Note that the space Z, the mapping K, the seminorm n_Z and the moment t_* may depend on $\mathcal{B}$.

The definition of *quasi-stability* is natural from the point of view of asymptotic behavior. In some sense, it reflects decomposition of the flow into compact and exponentially stable parts (see. (2.6)), and is, to some extent, an analog of the decomposition methods according to Babin–Vishik [2] and Temam [126]. However, our decomposition is performed for a difference of the trajectories, not for an individual trajectory. It is worth mentioning that in the degenerate case, if $n_Z \equiv 0$, the expression in (2.6) transforms into

$$\|S_{t_*}y_1 - S_{t_*}y_2\|_X \le q\|y_1 - y_2\|_X \quad \text{for all } y_1, y_2 \in \mathcal{B}. \tag{2.7}$$

Thus, S_{t_*} is a contraction on $\overline{\mathcal{B}}$ (closure of $\mathcal{B}$). Provided $\mathcal{B}$ is positively invariant, there exists a unique stationary point $\tilde{y}$ of the mapping S_{t_*} in $\overline{\mathcal{B}}$. The invariance of $\overline{\mathcal{B}}$ implies that $S_t\tilde{y}$ is a stationary point of S_t for every $t > 0$. Thus, $S_t\tilde{y} = \tilde{y}$ for all $t > 0$, that is, $\tilde{y}$ is a unique equilibrium in $\overline{\mathcal{B}}$. Moreover, (2.7) implies that this equilibrium is exponentially stable in $\overline{\mathcal{B}}$, namely,

$$\|S_t y - \tilde{y}\|_X \le Ce^{-\alpha t} \sup_{\tau \in [0, t_*]} \|S_\tau y - \tilde{y}\|_X \quad \text{for all } y \in \overline{\mathcal{B}}$$

for some $\alpha > 0$. This observation explains the term "quasi-stability" for property (2.6).

Quasi-stable systems possess a number of remarkable properties, including the existence of global attractors of finite fractal dimension and global exponential attractors. First, we formulate consequences of quasi-stability in the most general form and then represent particular forms of quasi-stability.

Proposition 2.13 (On asymptotic smoothness of QSDS)**.** *Let the DS (X, S_t) be quasi-stable on every bounded positively invariant set $\mathcal{B}$ in X. Then (X, S_t) is an asymptotically smooth dynamical system.*

One can apply Theorem 2.1 with $g(s) = qs$, $T = t_*$, and $\varrho_{\mathcal{B}}^T(\{S_\tau y_1\}, \{S_\tau y_2\}) = n_Z(Ky_1 - Ky_2)$ to prove the proposition.

Corollary 2.14 (On a global attractor of QSDS)**.** *Let the DS (X, S_t) be dissipative and let it satisfy Assumption 2.13. Then the DS possesses a compact global attractor.*

The following assertion concerns the finiteness of dimension of global attractors. One can prove it using Theorem 2.11 with $V = S_{t_*}$.

Theorem 2.15 (Finite dimensionality)**.** *Let the DS (X, S_t) possess a compact global attractor $\mathfrak{A}$ and be quasi-stable on $\mathfrak{A}$ at some moment $t_* > 0$. Then the attractor $\mathfrak{A}$ is of finite fractal dimension $\dim_f \mathfrak{A}$ in X. Moreover, the dimension can be estimated from above by*

$$\dim_f \mathfrak{A} \le \left[\ln \frac{2}{1+q}\right]^{-1} \ln m_Z\left(\frac{4L_K}{1-q}\right),$$

where $L_K > 0$ is the Lipschitz constant for K and $m_Z(R)$ is the maximal number of elements z_i in the ball $\{z \in Z : \|z_i\|_Z \le R\}$ such that $n_Z(z_i - z_j) > 1$ for $i \ne j$.

There are several results on the (generalized) fractal exponential attractors of quasi-stable systems. We remind the following definition [67].

Definition 2.16. A compact set $A_{\rm exp} \subset X$ is said to be an *inertial manifold* or a *fractal exponential attractor* of DS (X, S_t) if $A_{\rm exp}$ is a positive invariant set of finite fractal dimension and for every bounded set $D \subset X$ there exist positive constants t_D, C_D and γ_D such that

$$d_X\{S_t D \,|\, A_{\rm exp}\} \equiv \sup_{x\in D} {\rm dist}_X(S_t x,\, A_{\rm exp}) \le C_D e^{-\gamma_D(t-t_D)}, \quad t \ge t_D.$$

If an exponential attractor has a finite fractal dimension in some extended space $\widetilde{X} \supset X$, it is often called an *exponentially attracting set* or a *generalized fractal exponential attractor*.

Theory of fractal exponential attractors is well presented in [67] and also in the survey [105]. We emphasize that

(i) a global attractor may be not exponentially attracting, and

(ii) a fractal exponential attractor may be not unique, however, it necessary contains a global attractor.

We also note that the squeezing property in the sense of Foias–Temam is a standard tool (see, e.g., [67, 105]) for the construction of fractal exponential attractors (see also discussions in [64, 126]). This property says, roughly speaking, that higher modes of the flow are controlled by lower modes or the semigroup is exponentially contracting (some generalizations of this method are described in the survey [105]). The approach developed in [39] and [27] is based on the quasi-stability property. It means that the semigroup is an asymptotically contracting modulo homogeneous compact term.

Theorem 2.17 (On a fractal exponential attractor of QSDS)**.** *Let the DS (X, S_t) is dissipative and quasi-stable on a bounded absorbing set $\mathcal{B}$ at a moment $t_* > 0$. We also assume that*

$$\|S_t y_1 - S_t y_2\|_X \le C_{\mathcal{B}} \|y_1 - y_2\|_X \quad \text{for all } y_1, y_2 \in \mathcal{B} \text{ and } t \in [0, t_*],$$

and there exits a space $\widetilde{X} \supseteq X$ such that the map $t \mapsto S_t y$ is Hölder continuous in $\widetilde{X}$ for all $y \in \mathcal{B}$ in the sense that there exist $0 < \gamma \le 1$ and $C_{\mathcal{B},t_*} > 0$ such that

$$\|S_{t_1} y - S_{t_2} y\|_{\widetilde{X}} \le C_{\mathcal{B}} |t_1 - t_2|^{\gamma}, \quad t_1, t_2 \in [0, t_*],\ y \in \mathcal{B}. \tag{2.8}$$

Then the dynamical system (X, S_t) possesses a (generalized) fractal exponential attractor of finite in $\widetilde{X}$ fractal dimension.

The idea of the proof is to apply Theorem 3.2.1 from [27] with $V := S_{t_*}$ that gives us the existence of a fractal exponential attractor $\mathcal{A} \subset \mathcal{B}$ for the discrete DS (X, V^k). Setting $A_{\rm exp} = \cup\{S_t\mathcal{A} :\ t \in [0, t_*]\}$, we obtain the existence of a fractal exponential attractor for (X, S_t). Its dimension in the corresponding space is finite due to the Hölder continuity (2.8). The details of the proof can be found in [27]. We do not know whether the finiteness of the fractal dimension $\dim_f^{\widetilde{X}} A_{\rm exp}$ holds without an assumption of the Hölder's continuity (2.8) in a neighborhood of $A_{\exp}$. It is a consequence of the fact that $A_{\exp}$ is an uncountable union of finite dimensional sets $S_t\mathcal{A}$. We also note that the fractal dimension depends on the topology.

Now we proceed to two special cases which demonstrate some additional properties of dynamics.

2.4.2. Quasi-stable systems: parabolic-like dynamics. We consider the quasi-stability estimates with particular choice of the space Z, the seminorm n_Z, and the operator K. This choice is motivated by several classes of parabolic problems with partial derivatives (see discussion in [27]). We also discuss the existence of a finite set of determining functionals.

Assumption 2.18. *Let (X, S_t) be a dynamical system on a Banach space X and $\mathcal{B} \subset X$. Assume that there exist:*

(a) *compact seminorms $n_1(\cdot)$ and $n_2(\cdot)$ on the space X;*

(b) *constants $a_*, t_* > 0$ and $0 \le q < 1$ such that*

$$\|S_t y_1 - S_t y_2\|_X \le a_* \|y_1 - y_2\|_X \quad \text{for all } y_1, y_2 \in \mathcal{B} \text{ and } t \in [0, t_*] \tag{2.9}$$

and

$$\|S_{t_*} y_1 - S_{t_*} y_2\|_X \le q\|y_1 - y_2\|_X + n_1(y_1 - y_2) + n_2(S_{t_*} y_1 - S_{t_*} y_2) \tag{2.10}$$

for all $y_1, y_2 \in \mathcal{B}$.

Proposition 2.19. *Under Assumption 2.18, the dynamical system (X, S_t) is quasi-stable on $\mathcal{B} \subset X$.*

In order to prove this, we set $Z = X \times X$, $n_z(x, y) = n_1(x) + n_2(y)$ in Definition 2.12 and define $K : X \mapsto Z$ as $Kx = (x; S_{t_*} x)$.

Thus, dissipative systems satisfying Assumption 2.18 possess a compact global attractor of finite fractal dimension. Under additional assumption of the Hölder continuity of $t \mapsto S_t y$, we also obtain the existence of a fractal exponential attractor of finite fractal dimension.

Using the structure of quasi-stability estimate (2.10), we can establish some assertions on the existence of a finite number of determining functionals. For many applications, it is important to find minimal (or close to minimal) set of natural parameters of the problem, which determine its asymptotic behavior in the unique way. The question was first raised in [73] and [91] for the 2D Navier–Stokes equations. Later it was studied for many other models (see, e.g., [63, 72, 74, 75, 92, 124] and references therein). The concept of determining nodes was introduced in [72,74,124] and the concept of determining local means in [75,86,87]. The general concept of determining functionals in the framework of interpolation theory was introduced in [60,61]. Such functional sets can be interpreted as some measurements of the system state.

In fact, we speak not about the existence of such sets of functionals (which can be not of much use), but rather about a criterion which allows us to verify whether the finite set of functionals of prescribed type (measurements) is determining. We introduce a criterion which uses the estimate of completeness defect of a functional set via seminorms and constants from quasi-stability inequality.

The reader can find the details in survey [16] (see also [18, Chapter 5]). The theory of determining functionals was applied for the (discrete) problem of data assimilation, originated from weather prediction (see, e.g., [24, 85]).

Definition 2.20. A family of linearly independent functionals on X $\mathcal{L} = \{l_j : j = 1, \ldots, N\}$ is said to be a family of asymptotically determining functionals if the relation

$$\lim_{t\to\infty} l_j(S_t y_1 - S_t y_2) = 0, \quad j = 1, 2, \ldots, N,$$

implies $\lim_{t\to\infty} \|S_t y_1 - S_t y_2\|_X = 0$.

To characterize a set of asymptotically determining functionals, it is convenient to use the notion of the completeness defect suggested in [15, 16] for the case of two embedded Banach spaces. See also [27] for the case of one space endowed with additional seminorm.

Definition 2.21 (Completeness defect)**.** Let V be a Banach space and μ be a seminorm on V. The *completeness defect* of a set of linear functionals $\mathcal{L}$ on V with respect to the seminorm μ is said to be a quantity

$$\epsilon_{\mathcal{L}}(V, \mu) = \sup\left\{\mu(w) : w \in V,\ l(w) = 0,\ l \in \mathcal{L},\ \|w\|_V \le 1\right\}.$$

In the case when V is continuously and densely embedded in another Banach space X (such that $\|\cdot\|_X \le c\|\cdot\|_V$), the quantity

$$\begin{aligned}\epsilon_{\mathcal{L}}(V, X) &\equiv \epsilon_{\mathcal{L}}(V, \|\cdot\|_X)\\ &= \sup\{\|w\|_X :\ w \in V,\ l(w) = 0,\ l \in \mathcal{L},\ \|w\|_V \le 1\}\end{aligned}$$

is said to be a completeness defect of the set of linear functionals $\mathcal{L}$ on V with respect to X [15, 16]).

Theorem 2.22 (On the determining functionals of QSDS)**.** *Let the DS (X, S_t) be dissipative and satisfy Assumption 2.18 on some bounded absorbing set. Let $\mathcal{L} = \{l_j :\ j = 1, \ldots, N\}$ be a set of linearly independent functionals on X such that*

$$\epsilon_{\mathcal{L}}(n_1) + \epsilon_{\mathcal{L}}(n_2) < 1 - q,$$

where $\epsilon_{\mathcal{L}}(n_j) \equiv \epsilon_{\mathcal{L}}(X, n_j)$ is a completeness defect of the functional family $\mathcal{L}$ with respect to the seminorm n_j and the constant $q < 1$ and the seminorms n_i are taken from estimate (2.9). Then $\mathcal{L}$ is a set of asymptotically determining functionals in the sense of Definition 2.20.

The details of the proof and further discussion can be found in [27].

Under the assumptions of Theorem 2.22, it is possible to prove that the family of functionals $\mathcal{L}$ is determining in the sense of Ladyzhenskaya: for any two full trajectories $\gamma_j = \{u_j(t) : t \in \mathbb{R}\}$ from $\mathcal{B}$ (the set for which quasi-stability holds) the property

$$\exists\, t_* \in \mathbb{R} : \quad l(u_1(t)) = l(u_2(t)) \quad \text{for all } \ t < t_*,\ l \in \mathcal{L},$$

implies $u_1(t) \equiv u_2(t)$ for all $t \in \mathbb{R}$.

2.4.3. Asymptotically quasi-stable systems. Now we discuss the properties of quasi-stable systems with phase spaces of special structure. The results given below were obtained by I. Chueshov and I. Lasiecka in [41, Section 7.9] (see also [27]).

We set the following assumptions on the structure of the model under consideration.

Assumption 2.23 (Structural assumption)**.** *Let X, Y, and Θ be reflexive Banach spaces, and X be compactly embedded into Y. The phase space $H = X \times Y \times \Theta$ is endowed with the norm*

$$\|y\|_H^2 = \|u_0\|_X^2 + \|u_1\|_Y^2 + \|\theta\|_\Theta^2, \quad y = (u_0; u_1; \theta_0).$$

The trivial case $\Theta = \{0\}$ is allowed. Assume that (H, S_t) is a dynamical system on $H = X \times Y \times \Theta$ with the evolution operator of the form

$$S_t y = (u(t); u_t(t); \theta(t)), \quad y = (u_0; u_1; \theta_0) \in H, \tag{2.11}$$

where the functions $u(t)$ and $\theta(t)$ are such that

$$u \in C(\mathbb{R}_+, X) \cap C^1(\mathbb{R}_+, Y), \quad \theta \in C(\mathbb{R}_+, \Theta).$$

Such a structure of the phase space H and the evolution operator S_t arises from many nonlinear PDEs of second order in time on $X \times Y$, possibly interacting with parabolic equations on the space Θ. This type of interaction is observed, for example, in problems of thermoelasticity (see., e.g, [40] and [41, Chapters 5 and 11]) and in problems of interaction of plates and viscous fluid (e.g, [21, 52]). Further, we will use the following definition.

Definition 2.24 (Asymptotic quasi-stability)**.** The DS of the form (2.11) is said to be asymptotically quasi-stable on a set $\mathcal{B} \subset H$ if there exists a compact seminorm $\mu_X(\cdot)$ on the space X and non-negative scalar functions $a(t)$, $b(t)$ and $c(t)$ on $\mathbb{R}_+$ such that

(i) $a(t)$ and $c(t)$ are locally bounded on $[0, \infty)$;

(ii) $b(t) \in L_1(\mathbb{R}_+)$ is such that $\lim_{t\to\infty} b(t) = 0$;

(iii) for every $y_1, y_2 \in \mathcal{B}$ and $t > 0$, the following estimates take place:

$$\|S_t y_1 - S_t y_2\|_H^2 \le a(t)\|y_1 - y_2\|_H^2,$$
$$\|S_t y_1 - S_t y_2\|_H^2 \le b(t)\|y_1 - y_2\|_H^2 + c(t) \sup_{0 \le s \le t} \left[\mu_X(u^1(s) - u^2(s))\right]^2. \tag{2.12}$$

Here we denote $S_t y_i = (u^i(t); u^i_t(t); \theta^i(t))$, $i = 1, 2$.

In the framework of asymptotic dynamics, relation (2.12) was first introduced in [35] (see also [32] and discussion in [39]). Roughly speaking, it means asymptotic stability modulo compact terms. Inequality in (2.12) was named a stabilizability estimate. It was mentioned in [39] that derivation of such estimates may be technically complicated (especially for critical problems) and require rather sophisticated tools of the PDE theory.

Proposition 2.25 (On quasi-stability)**.** *Let the structural hypothesis 2.23 take place. Assume that the dynamical system (H, S_t) is asymptotically quasi-stable on a set $\mathcal{B}$ in H. Then the system is quasi-stable on $\mathcal{B}$ at every moment $T > 0$ such that $b(T) < 1$.*

The proof can be found in [27].

Thus, Proposition 2.25, Corollary 2.14 and Theorems 2.15, 2.17 give us the following result.

Theorem 2.26 (On attractors of AQSDS)**.** *Let the structural Assumption 2.23 take place, and the dynamical system (H, S_t) be dissipative and asymptotically quasi-stable on a bounded invariant absorbing set $\mathcal{B}$ in H. Then the system (H, S_t) possesses a compact global attractor A of finite fractal dimension $dim^H_f A$.*

If there exists a space $\widetilde{H} \supseteq H$ such that the map $t \mapsto S_t y$ is Hölder continuous in $\widetilde{H}$ for every $y \in \mathcal{B}$ (see (2.8)), then the dynamical system (H, S_t) possesses a (generalized) fractal exponential attractor of finite in $\widetilde{H}$ fractal dimension.

This theorem was first proved in [39, 41] by using the method of "short" trajectories, originally suggested in [102] and [103].

Quasi-stability allows us to prove additional regularity of trajectories lying in an attractor. The theorem below gives us regularity of time derivatives. Additional "spatial" regularity usually follows from elliptic PDEs theory (see corresponding results for von Karman plates in [41]).

Theorem 2.27 (On regularity of attractors of AQSDS)**.** *Let the structural Assumption 2.23 hold, and let the DS (H, S_t) possess a compact global attractor $\mathfrak{A}$ and be asymptotically quasi-stable on $\mathfrak{A}$. Besides, let (2.12) take place with a function $c(t)$ such that $c_\infty = \sup_{t\in\mathbb{R}_+} c(t) < \infty$ (global boundedness). Then every full trajectory $\{(u(t); u_t(t); \theta(t)) : t \in \mathbb{R}\}$ lying in the global attractor possesses the following regularity properties:*

(i) $u_t \in L_\infty(\mathbb{R}; X) \cap C(\mathbb{R}; Y)$, $u_{tt} \in L_\infty(\mathbb{R}; Y)$, $\theta_t \in L_\infty(\mathbb{R}; \Theta)$*;*

(ii) *there exists $R > 0$ such that*

$$\|u_t(t)\|^2_X + \|u_{tt}(t)\|^2_Y + \|\theta_t(t)\|^2_\Theta \le R^2, \quad t \in \mathbb{R},$$

with R depending only on the constant c_∞, the seminorm μ_X in Definition 2.24 and the properties of embedding of X in Y.

It is necessary to estimate a norm of the difference of two trajectories $\gamma_h = \{y(t+h) : t \in \mathbb{R}\}$ from the attractor for small $|h|$. The details of the proof can be found in [41] (see also [35,39]).

Asymptotic quasi-stability also implies the following result on determining functionals, which was proved in [41] (see also [27]).

Theorem 2.28 (On determining functionals of AQSDS)**.** *Let the Assumption 2.23 hold and DS (H, S_t) be dissipative and asymptotically quasi-stable on a bounded absorbing set $\mathcal{B}$. Denote by $\mathcal{L} = \{l_j : j = 1, \ldots, N\}$ a set of linearly independent functionals on X and let $\epsilon_{\mathcal{L}}(\mu_X)$ be its completeness defect with respect to the seminorm μ_X (see Definition 2.21). If there exists $\tau > 0$ such that*

$$\eta_\tau \equiv b(\tau) + \epsilon^2_{\mathcal{L}}(\mu_X)c(\tau) \sup_{s\in[0,\tau]} a(s) < 1,$$

then the relation

$$\lim_{t\to\infty} l_j(u^1(s) - u^2(s)) = 0, \quad j = 1, 2, \ldots, N,$$

implies $\lim_{t\to\infty} \|S_t y_1 - S_t y_2\|_H = 0$*. Here* $S_t y_i = (u^i(t); u^i_t(t); \theta^i(t))$*,* $i = 1, 2$*.*

The details of the proof can be found in [41].

3. Von Karman plates

Various versions of von Karman equations arise in applications to describe plate dynamics in the case of large deflections (see [58,59], in the case of stationary problems and [41,94] in the case of dynamical problems).

3.1. Von Karman equation with nonlinear internal damping. Let $\Omega \subset \mathbb{R}^2$ be a bounded set and $\alpha \in [0, 1]$. We use the notation $M_\alpha \equiv I - \alpha\Delta$ and consider the equation

$$M_\alpha w_{tt} + \Delta^2 w + \big[g(w_t) - \alpha \operatorname{div} G(\nabla w_t)\big] = [\mathcal{F}(w) + F_0, w] + p \tag{3.1}$$

in $\Omega \times (0, \infty)$ with clamped boundary conditions

$$w = \frac{\partial}{\partial n} w = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{3.2}$$

where the Airy function $\mathcal{F}(w)$ is a solution to the elliptic problem

$$\Delta^2 \mathcal{F}(w) = -[w, w] \quad \text{in } \Omega \quad \text{with} \quad \mathcal{F} = \frac{\partial}{\partial n}\mathcal{F} = 0 \quad \text{on } \Gamma,$$

and the von Karman bracket $[u, v]$ is defined as

$$[u, v] = \partial^2_{x_1} u \partial^2_{x_2} v + \partial^2_{x_2} u \partial^2_{x_1} v - 2\partial_{x_1x_2} u \partial_{x_1x_2} v.$$

A scalar function $g(s)$ is continuous, non-decreasing on $\mathbb{R}$ and $g(0) = 0$. In the case of $\alpha > 0$, the damping G has the form $G(s, \sigma) = (g_1(s); g_2(\sigma))$, where $g_i(s)$,

$i = 1, 2$, are continuous and monotone on $\mathbb{R}$ and $g_i(0) = 0$. Moreover, they have a polynomial growth: $|g_i(s)| \leq C(1 + |s|^{q-1})$ for some $q \geq 1$. We also assume that $F_0 \in H^{3+\delta}(\Omega) \cap H_0^1(\Omega)$, $\delta > 0$ and $p \in L_2(\Omega)$. The function F_0 represents the inner strain in the plate and p is a transverse load.

The case of $\alpha > 0$ is subcritical with respect to the von Karman nonlinearity $[\mathcal{F}(w), w]$.

In the case of $\alpha = 0$, the von Karman nonlinearity is critical. Problem (3.1), (3.2) possesses a unique global mild solution, i.e., generates a dynamical system (H_α, S_t), where $H_\alpha = H_0^2(\Omega) \times \mathcal{H}_\alpha(\Omega)$ and $\mathcal{H}_\alpha(\Omega)$ is defined by the formula

$$\mathcal{H}_\alpha(\Omega) = \begin{cases} L_2(\Omega), & \alpha = 0 \\ H_0^1(\Omega), & \alpha > 0 \end{cases}. \tag{3.3}$$

The dynamical system (H_α, S_t), generated by (3.1), (3.2), possesses a compact global attractor A in the space $H_\alpha \equiv H_0^2(\Omega) \times \mathcal{H}_\alpha(\Omega)$. Under certain additional conditions on the damping terms this dynamical system is gradient and the attractor has a regular structure described by Theorem 2.8. In the case of $\alpha > 0$, it follows from abstract results for second-order evolution equations with subcritical nonlinear source (see [39]). In the case of $\alpha = 0$, the von Karman nonlinearity becomes critical and the proof is based on Theorem 2.2 (the compensated compactness method). Let the following hold true.

Assumption 3.1. *Assume that*

(i) *in case of* $\alpha = 0$*: there exist* $m, M_0 > 0$ *such that*

$$0 < m \leq g'(s) \leq M_0[1 + sg(s)], \quad s \in \mathbb{R}, \tag{3.4}$$

(ii) *in case of* $\alpha > 0$*:* $g(s)$ *either has a polynomial growth at infinity, or satisfies* (3.4) *and there exists* $0 \leq \gamma < 1$ *such that* g_i *satisfies the inequality*

$$0 < m \leq g_i'(s) \leq M[1 + sg_i(s)]^\gamma, \quad s \in \mathbb{R},\ i = 1, 2.$$

Then the dynamical system, generated by problem (3.1), (3.2), is asymptotically quasi-stable, which yields the existence of a finite dimensional attractor, its regularity, and the existence of an exponential attractor. All the statements follow from abstract results in Section 2.4. See [41] for more details.

3.2. Von Karman equation with nonlinear boundary damping. We consider equation (3.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with appropriate initial conditions and dissipative hinged boundary conditions:

$$w = 0, \quad \Delta w = -g_0\left(\frac{\partial}{\partial n} w_t\right) \quad \text{on } \Gamma \times (0, T). \tag{3.5}$$

For problem (3.1), (3.5), we add an assumption on the nature of the boundary damping. A scalar function $g_0(s)$ is assumed to be globally Lipschitz and strictly increasing on $\mathbb{R}$ and equal to zero at zero. Then, under the same conditions as for problem (3.1), (3.2), there exists a unique global mild solution to problem (3.1), (3.5). To study asymptotic dynamics of problem (3.1), (3.5), we additionally assume that

$$g(s) = g_1 s + |s|^{m-1}s \quad \text{and} \quad G(s,\sigma) = G_1(s;\sigma) + (|s|^{m-1}s; \sigma^{m-1}\sigma) \tag{3.6}$$

and at the same time $g_1 > 0$.

The phase space of the dynamical system in the case of $\alpha > 0$ is the space $H \equiv \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times H_0^1(\Omega)$, and the corresponding dynamical system generated by problem (3.1), (3.5) is denoted by (H, S_t). The system (H, S_t) possesses a compact global attractor of regular structure described in Theorem 2.8.

The phase space in the case of $\alpha = 0$ is $H_0 \equiv \left[H^2(\Omega) \cap H_0^1(\Omega)\right] \times L_2(\Omega)$, and the regularizing effect of rotational inertia is absent. Under the above assumptions, the dynamical system (H_0, S_t) is gradient. To prove the asymptotic quasi-stability and, consequently, the existence of a finite dimensional attractor one needs to require that $g(s)$ is globally Lipschitz. Details can be found in [41].

One can also consider the problem with boundary dissipation, with hinged or simply supported boundary conditions (see [41]). As it is noted in [43], the following questions remain open for von Karman equations (3.1): general theory in the case of non-conservative forces destroying the gradient structure; asymptotic behavior in the case of localized (in a boundary layer) damping.

3.3. Thermoelastic von Karman plates. The problem has the form

$$\begin{cases} M_\alpha w_{tt} + \Delta^2 w - \beta \Delta\theta = B(w), & w|_{t=0} = w_0, \quad w_t|_{t=0} = w_1, \\ \theta_t + \eta\Delta\theta + \beta\Delta w_t = 0, & \theta|_{t=0} = \theta_0, \end{cases}$$

where w is the transverse displacement of the plate and θ is its temperature.

The system is subjected to clamped, hinged, or simply supported boundary conditions for the displacement and Dirichlet boundary conditions for the temperature. The state spaces for (w, w_t) and for θ are H_α and $L_2(\Omega)$ respectively. The system is gradient and asymptotically quasi-stable for $\alpha \geq 0$, $\beta \geq 0$. Moreover, the structure of the von Karman nonlinearity allows to prove asymptotic quasi-stability simultaneously in subcritical ($\alpha > 0$) and critical ($\alpha = 0$) cases deriving the estimates independent of the parameters of the problem. Thus, in addition to standard corollaries from quasi-stability (see Theorem 2.27), we obtain upper semicontinuity of the family of attractors with respect to the parameters (α, β) (see Theorem 2.9). Details can be found in [40] and [41, Chapter 11] (see also the references therein). The case of the Berger nonlinearity is considered in [7].

4. Kirchhoff–Boussinesq plate

4.1. Kirchhoff–Boussinesq system with internal damping. Using the notations introduced in Section 3, we consider the equation

$$M_\alpha w_{tt} + \Delta^2 w + a(x)\left[g(w_t) - \alpha \operatorname{div} G(\nabla w_t)\right] = \operatorname{div}\left[|\nabla w|^2 \nabla w\right] + P(w) \tag{4.1}$$

with clamped boundary conditions (3.2). The damping functions $g : \mathbb{R} \mapsto \mathbb{R}_+$, $G : \mathbb{R}^2 \mapsto \mathbb{R}^2_+$ satisfy condition (3.6), and the nonlinearity P has the form $\sigma\Delta[w^2] - \varrho|w|^{l-1}w$, where $\sigma > 0$, $\varrho \geq 0$, and $l \geq 1$.

Well-posedness for $\alpha > 0$ can be proved in the standard way, since the nonlinear term possesses the property $\operatorname{div} |\nabla w|^2\nabla w \in H^{-1}(\Omega)$ for solutions with finite energy w, i.e., is critical. The corresponding results can be found in [39]. The case of $\alpha = 0$ is more delicate. Its analysis requires non-standard arguments and relies essentially on the linearity of the damping function.

The main difficulty is created by the feedback force $\operatorname{div}\left[|\nabla w|^2\nabla w\right]$, which does not belong to $L_2(\Omega)$ for finite energy solutions. In the case of $\alpha = 0$, there exists a weak solution to (3.2), (4.1). If $g(s)$ is linear, the weak solution is unique and continuously depends on initial data (see [39, Chapter 7]). As it was mentioned in [43], the uniqueness of weak solutions to (3.2), (4.1) in the case of $\alpha = 0$ and nonlinear damping is an open question.

In the case of $\alpha > 0$, problem (3.2), (4.1) generates the dynamical system (H, S_t) on the phase space $H = H \equiv H^2_0(\Omega) \times H^1_0(\Omega)$. If also either the assumptions

$$m \leq 3, \quad \varrho > 0, \quad l \leq m, \quad \inf_{x\in\Omega}\{a(x)\} > 0$$

or the assumptions

$$m < 3, \quad \varrho = 0, \quad \inf_{x\in\Omega}\{a(x)\} \gg 1$$

hold true, the dynamical system (H, S_t) possesses a compact global attractor A. If the constant G_1 in (3.6) is positive, the dynamical system is asymptotically quasi-stable, and, consequently, the fractal dimension of the attractor A is finite and the dynamical system (H, S_t) possesses a generalized fractal exponential attractor (see Definition 2.16). The proofs can be found in [39, Chapter 7]), where the problem is considered under more general assumptions on the damping terms.

In the case of $\alpha = 0$, the phase space is $H_0 \equiv H^2_0(\Omega) \times L_2(\Omega)$. If the damping is linear ($g(s) = s$) and

$$\sigma^2 < \frac{1}{4}k \min\{1, k\}, \quad k \equiv \inf_{\Omega} a(x) > 0,$$

then the dynamical system (H_0, S_t) possesses a compact global attractor $\mathfrak{A}$. To prove this result, one uses the Ball method (see Theorem 2.3), which requires the linearity of the damping term. In the case of $\sigma = 0$, it is easy to see that the energy is a strict Lapunov functional and the corresponding dynamical system is gradient. Since the nonlinearity is supercritical, the quasi-stability method cannot be applied without additional information about the attractor, however, quasi-stability plays an important role in investigation of asymptotic dynamics of strong solutions. In the phase space $H_{st} = \left[H^4(\Omega) \cap H^2_0(\Omega)\right] \times H^2_0(\Omega)$, the nonlinearity is only critical (for details, see [39]).

4.2. Kirchhoff–Boussinesq plates with boundary damping. Equation (4.1) in Subsection 4.1 can be considered with dissipative hinged boundary conditions (3.5). The functions of internal damping g and G satisfy (3.6). The boundary damping, as in the case of the von Karman equation, satisfies $g_0(s) \sim g_2 s + |s|^{q-1}s$, $q \geq 1$. The nonlinearity $P(w)$ has the same form as in (3.2), (4.1).

In the case of $\alpha > 0$, the proof of well-posedness is standard, while the case of $\alpha = 0$ is more subtle. Though the existence can be established for the damping of general form, the proof of the uniqueness requires linearity of the damping. Moreover, the continuous dependence on initial conditions requires the reversibility of the dynamics, which does not allow to consider boundary damping. From this point of view, most questions concerning the boundary damping remain open for the case of $\alpha = 0$.

To study the asymptotic dynamics for the case of $\alpha > 0$, one uses the same methods and obtains the same results as for the von Karman equation with boundary dissipation (see Subsection 3.2 and Section 10.3 in [41]). The asymptotic behavior of system (3.5), (4.1) in the case of $\alpha = 0$ is an open question. For details, see [37, 42].

5. Elastic plates models with structural damping depending on the state

We consider the problem

$$\begin{cases} \mu\partial_{tt}u + \sigma(u)\partial_t u + \Delta^2 u + \varphi(u) = f(x), & x \in \Omega, \quad t > 0, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1 \end{cases} \tag{5.1}$$

in a bounded domain Ω in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. The function $\sigma \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ is positive and $f \in L^2(\Omega)$, $\mu > 0$. Problem (5.1) describes nonlinear dynamics of elastic plates. Here u denotes the displacement of a point of the plate with respect to the equilibria, $\sigma(u)\partial_t u$ denotes friction in the system, the term $f - \varphi(u)$ represents nonlinear external forces.

Remark 5.1. Depending on the form of the nonlinearity $u \mapsto \varphi(u)$, the problem describes various plate models.

- *Kirchhoff model:* $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ satisfies the conditions

$$\liminf_{|s|\to\infty} \frac{\varphi(s)}{s} > -\lambda_1^2,$$

  where λ_1 is the first eigenvalue of the Laplace operator with the Dirichlet boundary conditions;

- *the von Karman model:* $\varphi(u) = [u, v(u) + F_0]$, see Section 3 for details;

- *the Berger model:*

$$\varphi(u) = \left[\kappa \int_\Omega |\nabla u|^2 dx - \Gamma\right] \Delta u,$$

  where $\kappa > 0$ and $\Gamma \in \mathbb{R}$.

Problem (5.1) can be rewritten as an abstract Cauchy problem

$$\mu\partial_{tt}u + K(u)\partial_t u + \mathcal{A}u + F(u) = 0, \quad u|_{t=0} = u_0, \;\; \partial_t u|_{t=0} = u_1. \tag{5.2}$$

The assumptions on the operators in (5.2) can be found in [33].

In the case of $\mu > 0$, problem (5.2) possesses a unique global weak solution and generates a dynamical system $(\mathcal{H}, S^\mu(t))$ in the space $\mathcal{H} = \mathcal{D}(\mathcal{A}^{1/2}) \times H$.

If $\mu = 0$, problem (5.2) generates a dynamical system $\big(\mathcal{D}(\mathcal{A}^{1/2}), S(t)\big)$.

To prove the existence of attractors for the dynamical systems $(\mathcal{H}, S^\mu(t))$ and $(\mathcal{D}(\mathcal{A}^{1/2}), S(t))$, one can use a version of the method from [89] (see Theorem 2.2). Under the conditions of additional regularity of the mapping F [33], the dynamical systems $(\mathcal{H}, S^\mu(t))$ and $(\mathcal{D}(\mathcal{A}^{1/2}), S(t))$ are quasi-stable (the first one is also asymptotically quasi-stable), which allows us to obtain additional properties of the attractor, described in Theorems 2.15, 2.26. Moreover, the attractors $\mathfrak{A}^\mu$ of (5.2) tend in some sense to the attractor of the singular limit as $\mu \to 0$ [33].

Model (5.2) can be rewritten in a separable Hilbert space H as a problem with strong nonlinear damping

$$\partial_{tt}u + D(u, \partial_t u) + \mathcal{A}u + F(u) = 0, \; t > 0; \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1. \tag{5.3}$$

This problem can have various applications in the plate dynamics theory. For instance, it can be assumed that $\mathcal{A} = (-\Delta_D)^2$, where Δ_D is the Laplace operator with the Dirichlet boundary conditions, the operator $D(u, u_t)$ can be as follows:

$$D(u, u_t) = \Delta\left[\sigma_0(u)\Delta u_t\right] - \operatorname{div}\left[\sigma_1(u, \nabla u)\nabla u_t\right] + g(u, u_t),$$

where $\sigma_0(s_1)$, $\sigma_1(s_1, s_2, s_3)$ and $g(s_1, s_2)$ are locally Lipschitz functions of the variables $s_i \in \mathbb{R}$, $i = 1, 2, 3$. The detailed description of assumptions on the parameters of the problem is given in [34]. The dynamical system $(\mathcal{H}, S(t))$ generated by (5.3) in the space $\mathcal{H} = \mathcal{D}(\mathcal{A}^{1/2}) \times H$ possesses a compact global attractor $\mathfrak{A}$, which has a finite fractal dimension. Besides, the system $(\mathcal{H}, S(t))$ possesses a (generalized) fractal exponential attractor $\mathfrak{A}_{exp}$ with finite fractal dimension in the space $\tilde{\mathcal{H}} = H_\theta \times H_{-1}$, where for $1/2 \le \theta \le 1$ $H_\theta = (H^{2\theta} \cap H_0^1)(\Omega)$, and for $0 < \theta < 1/2$ $H_\theta = H_0^{2\theta}(\Omega)$. These results are the corollary of the asymptotic quasi-stability of the dynamical system (see Theorem 2.26).

6. Problems of thermo- and viscoelasticity for the Berger model

In the framework of this model, it is assumed that the plate has the constant thickness and its mean surface, when in equilibrium, occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth or rectangle boundary $\partial\Omega$. We consider the system of integro-differential equations, which describes thermo-viscoelastic oscillations of a Berger plate:

$$u_{tt} + k_1(0)\Delta^2 u + \int_0^{+\infty} k_1'(s)\Delta^2 u(t-s)\, ds + \nu\Delta v$$

$$= p + M\left(\int_\Omega |\nabla u|^2\, d\mathbf{x}\right)\Delta u, \tag{6.1}$$

$$v_t - \omega\Delta v - \int_0^{+\infty} k_2(s)\Delta v(t-s)ds = \nu\Delta u_t, \tag{6.2}$$

$$u = k_1(0)\Delta u + \int_0^{+\infty} k_1'(s)\Delta u(t-s)\, ds = 0, \qquad \mathbf{x}\in\partial\Omega,\ t\geq 0, \tag{6.3}$$

$$v = 0, \qquad \mathbf{x}\in\partial\Omega,\ t\in\mathbb{R}, \tag{6.4}$$

$$u|_{t\leq 0} = u_0(-t,\mathbf{x}), \quad v|_{t\leq 0} = v_0(-t,\mathbf{x}), \qquad \mathbf{x}\in\Omega, \tag{6.5}$$

where $u(t,\mathbf{x})$ is the vertical displacement of a point on the mean surface, $v(t,\mathbf{x})$ is the temperature variation. To describe the process of heat conduction the equation of Gurtin–Pipkin ($\omega = 0$) [80] or Coleman–Gurtin type ($\omega\in(0,1)$) [62] is used instead of the classical Fourier model ($\omega = 1$). The parameter $\nu > 0$ describes a relation between the displacement and the temperature.

Having introduced the notations

$$\mu_1(s) = -k_1'(s), \quad \mu_2(s) = -(1-\omega)k_2'(s),$$

and additional variables of the Dafermos type [65],

$$\overline{\eta}^t(s) = u(t) - u(t-s), \quad \eta^t(s) = \int_0^s v(t-y)dy,$$

one can study the problem in the phase space

$$\mathcal{H} = H^2\cap H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times L^2_{\mu_1}(\mathbb{R}_+; H^2\cap H_0^1(\Omega))\times L^2_{\mu_2}(\mathbb{R}_+; H_0^1(\Omega))$$

(for more details about the method of the phase space extension and the definition of the component spaces, see Subsection 7.2).

It is assumed that

$$\mu_i(s)\in C^1(\mathbb{R}_+)\cap L^1(\mathbb{R}_+)\cap C[0,+\infty), \quad i = 1,2,\ \mu_i(s)\geq 0,$$

and there exist $\delta_i > 0$ such that

$$\mu_i'(s) + \delta_i\mu_i(s)\leq 0,$$

and

$$\begin{cases} \mathcal{M}(z) \equiv \displaystyle\int_0^z M(\xi)d\xi \geq -az - b, \quad a\in(0,\lambda_1),\ b\in\mathbb{R}, \\ M(z)\in C^2(\mathbb{R}_+). \end{cases}$$

Problem (6.1)–(6.5) is well-posed in the extended phase space in the sense of generalized solutions. The dynamical system $(\mathcal{H}, S_t)$ generated by (6.1)–(6.5) possesses a finite dimensional compact global attractor $\mathcal{A}$ [113], which follows from the asymptotic quasi-stability (see Theorem 2.26).

7. Mindlin–Timoshenko model

7.1. Pure elasticity. We consider a domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary Γ, which is occupied by the mean plane of the plate in the equilibrium. The Mindlin–Timoshenko system describing the dynamics of thin plates, if the effects of transverse shear (see [95, Chapter 1] and references therein) are taken into account, has the form

$$\alpha\,(v_{tt} + \alpha_1 v_t) - \mathcal{A}v + \mu(v + \nabla w) + v\left[\beta w + h(|v|^2)\right] = 0, \tag{7.1}$$

$$w_{tt} + \alpha_2 w_t - \mu\,\mathrm{div}(v + \nabla w) + h_0(w) = 0. \tag{7.2}$$

Here $v(x,t) = (v_1(x,t), v_2(x,t))$ is a vector function, whose components represent the angles of deflection of the cross-sections of the plate (these variables are responsible for the transverse shear effects), while the scalar function $w(x,t)$ denotes the transverse displacement of the mean plane of the plate. The terms $\alpha\alpha_1 v_t$ and $\alpha_2 w_t$ describe the structural damping (with intensities $\alpha_1 > 0$ and $\alpha_2 > 0$). The parameter $\alpha > 0$ is the rotational inertia of the cross-sections, $\mu > 0$ is the shear modulus. From the mechanical point of view, the limit case $\mu \to +0$ corresponds to in-plane oscillations, while $\mu \to +\infty$, to the absence of the shear modulus. The vector-function

$$f(v,w) = (v_1; v_2)\left[\beta w + h\left(|v_1|^2 + |v_2|^2\right)\right]$$

and the scalar function $h_0(w)$ denote external forces. The peculiarity of the model is in the presence of the non-conservative term $\beta v w$, i.e., the term which cannot be represented as a potential operator. This means that the energy of the system is not monotone. The operator A has the form

$$\mathcal{A} = \begin{bmatrix} \partial^2_{x_1} + \frac{1-\nu}{2}\partial^2_{x_2} & \frac{1+\nu}{2}\partial^2_{x_1 x_2} \\ \frac{1+\nu}{2}\partial^2_{x_1 x_2} & \frac{1-\nu}{2}\partial^2_{x_1} + \partial^2_{x_2} \end{bmatrix}, \tag{7.3}$$

where $0 < \nu < 1$ is the Poisson ratio. Equations (7.1), (7.2) are supplemented with the Dirichlet boundary conditions and appropriate initial conditions.

The Mindlin–Timoshenko hypotheses and the corresponding equations can be found in [94] and [95]. The asymptotic behavior of the solutions to the Mindlin–Timoshenko problems under various assumptions on the external forces, parameters, and the damping type was studied in [36] and [39].

Appropriate restrictions on the growth and properties of antiderivatives for the functions $h(s)$ and $h_0(s)$ from class C^1 are formulated in [36] and [39]. Under these assumptions, the result on the well-posedness of problem (7.1), (7.2) holds true in the energy space in the variational sense [36].

Under additional assumptions on $h(s)$ and $h_0(s)$ (see [39]) for problem (7.1), (7.2), the theorem on the existence of a compact global attractor in the energy space holds true.

In the limit case $\kappa \to \infty$, the attractors of system (7.1), (7.2) are close to the attractor of the Kirchhoff–Boussinesq system

$$(1 - \alpha\Delta)u_{tt} + (\alpha_2 - \alpha\alpha_1\Delta)u_t + \Delta^2 u - \mathrm{div}\left[h(|\nabla u|^2)\nabla u\right]$$

$$-\frac{\beta}{2}\Delta\left[u^2\right]+h_0(u)=0,$$

$$u(x,t)=0,\quad \nabla u(x,t)=0 \qquad \text{on } \Gamma\times\mathbb{R}_+$$

in the sense of singular limit.

The following problem with the weak damping is considered in [39]:

$$\alpha v_{tt}+\alpha_1 g(v_t)-\mathcal{A}v+\mu(v+\nabla w)=-f_1(v), \tag{7.4}$$

$$w_{tt}+\alpha_2 g_0(w_t)-\mu\operatorname{div}(v+\nabla w)=-f_2(w). \tag{7.5}$$

Under certain assumptions (see [39]), the dynamical system, generated by problem (7.4), (7.5), possesses a compact global attractor in the phase space. If, moreover, $sg_i(s)\ge m|s|^l$ for any $|s|\ge 1$ and some $l>p-1$, the fractal dimension of the attractor is finite.

7.2. Thermo- and viscoelasticity. The system of integro-differential equations for the nonlinear thermo-viscoelastic Mindlin–Timoshenko problem has the form (comp. (7.1)–(7.5)):

$$\alpha v_{tt}-[\lambda_\infty+a(0)]\mathcal{A}v-\int_0^\infty a'(s)\mathcal{A}v(t-s)ds+b(0)(v+\nabla w) \tag{7.6}$$

$$+\int_0^\infty b'(s)[v+\nabla w](t-s)ds+\beta\nabla\theta+\nabla_v\Phi(v)=0, \tag{7.7}$$

$$w_{tt}-b(0)\operatorname{div}(v+\nabla w)-\int_0^\infty b'(s)\operatorname{div}[v+\nabla w](t-s)ds+g(w)=0, \tag{7.8}$$

$$\gamma\theta_t-\int_0^\infty \kappa(s)\Delta\theta(t-s)ds+\beta\operatorname{div}v_t=0, \tag{7.9}$$

and is supplemented with the Dirichlet boundary conditions and appropriate initial conditions. Here the operator $\mathcal{A}$ is defined by formula (7.3), the vector function $\nabla_v\Phi(v)=(\partial_{v_1}\Phi(v_1,v_2),\partial_{v_2}\Phi(v_1,v_2))$ and the scalar function $g(w)$ describe external loads, $\lambda_\infty,\alpha,\beta,\gamma$ are positive constants.

In order to investigate the problem by the dynamical systems theory methods, system (7.6)–(7.8) is transformed into the form

$$\alpha v_{tt}-\lambda_\infty\mathcal{A}v-\frac{1}{\omega}\int_0^\infty \lambda_\omega(s)\mathcal{A}\phi ds+\frac{\mu_\infty}{\omega}(v+\nabla w)$$

$$+\frac{1}{\omega}\int_0^\infty \mu_\omega(s)[\phi+\nabla\xi]ds+\beta\nabla\theta+\nabla_v\Phi(v)=0, \tag{7.10}$$

$$w_{tt}-\frac{\mu_\infty}{\omega}\operatorname{div}(v+\nabla w)-\frac{1}{\omega}\int_0^\infty \mu_\omega(s)\operatorname{div}[\phi+\nabla\xi]ds+g(w)=0, \tag{7.11}$$

$$\gamma\theta_t-\frac{1}{\varepsilon}\int_0^\infty \eta_\varepsilon(s)\Delta\tau ds+\beta\operatorname{div}v_t=0, \tag{7.12}$$

$$\theta=\tau_t+\tau_s, \tag{7.13}$$

$$v_t=\phi_t+\phi_s, \tag{7.14}$$

$$w_t=\xi_t+\xi_s \tag{7.15}$$

by means of rescaling the kernels $a(s)$, $b(s)$, $\kappa(s)$ and introducing the notations

$$\lambda_\omega(s) = \frac{1}{\omega}\lambda\left(\frac{s}{\omega}\right) = -\frac{1}{\omega}a'\left(\frac{s}{\omega}\right),$$
$$\mu_\omega(s) = \frac{1}{\omega}\mu\left(\frac{s}{\omega}\right) = -\frac{1}{\omega}b'\left(\frac{s}{\omega}\right),$$
$$\eta_\varepsilon(s) = \frac{1}{\varepsilon}\eta\left(\frac{s}{\varepsilon}\right) = -\frac{1}{\varepsilon}\kappa'\left(\frac{s}{\varepsilon}\right),$$

as well as definition of new variables of Dafermos type τ, ϕ and ξ [65]:

$$\tau(x,t,s) = \int_{t-s}^{t}\theta(x,\xi)d\xi = \int_0^s \theta(x,t-\xi)d\xi, \qquad s \ge 0,\ t > 0, \tag{7.16}$$
$$\phi(x,t,s) = v(x,t) - v(x,t-s), \qquad s \ge 0,\ t > 0, \tag{7.17}$$
$$\xi(x,t,s) = w(x,t) - w(x,t-s), \qquad s \ge 0,\ t > 0, \tag{7.18}$$

for which we obtain from (7.16)–(7.18) three additional equations in system (7.10)–(7.15). Introduction of auxiliary variables and extension of the phase space allows us to overcome the problem that system (7.6)–(7.8) is non-autonomous.

The Hilbert spaces $L^2_\rho(\mathbb{R}^+, H^l(\Omega))$ are the spaces of functions on $\mathbb{R}^+$ with values in H^m ($m \in \mathbb{R}$), which are square integrable with the weight ρ.

The kernels η, λ, μ in system (7.10)–(7.15) are continuously differentiable, non-negative, exponentially decreasing, and μ can be estimated from above by the kernel λ (see [70] for details).

The phase space of the problem is as follows:

$$H_{\omega,\varepsilon} = \mathcal{D}(A_\omega^{1/2}) \times [L^2(\Omega)]^3 \times L^2(\Omega) \times \mathcal{B} \times L^2_\eta(\mathbb{R}^+, H^1_0(\Omega)),$$

where

$$\mathcal{B} = L^2_\lambda(\mathbb{R}^+, [H^1_0(\Omega)]^2) \times L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)).$$

Under appropriate assumptions on the growth of nonlinearities and the boundedness from below of their antiderivatives (see [70] for details), problem (7.10)–(7.15) possesses a unique generalized solution and generates a dynamical system on the phase space $H_{\omega,\varepsilon}$. This dynamical system possesses a compact global attractor $\mathfrak{A}_{\omega,\varepsilon}$ of finite fractal dimension. The asymptotic behavior of system (7.10)–(7.15) in the limit case $\omega \to 0$, $\varepsilon \to 0$ can be described by the system of Kirchhoff–Boussinesq type

$$(\alpha_2 - \alpha_1\Delta)w_{tt} + \lambda_1\Delta^2 w_t + \lambda_\infty\Delta^2 w + \beta\Delta\theta + \operatorname{div}[\nabla\Phi(-\nabla w)] + g(w) = 0,$$
$$\gamma\theta_t - \eta_1\Delta\theta + \beta\Delta w_t = 0$$

with clamped boundary conditions (3.2) and constant zero temperature on the ends.

Along with problem (7.10)–(7.15), the problem with viscous damping without memory is considered in [69]:

$$v_{tt} + \alpha_1 v_t - \mathcal{A}v + \mu(v + \nabla w) + \beta\nabla\theta + \nabla_v\Phi(v) = 0, \tag{7.19}$$

$$w_{tt} + \alpha_2 w_t - \mu \operatorname{div}(v + \nabla w) + g(w) = 0, \tag{7.20}$$

$$\gamma\theta_t - \frac{1}{\omega}\int_{-\infty}^{t} \kappa(\frac{t-s}{\omega})\Delta\theta(x,s)\,ds + \beta \operatorname{div} v_t = 0. \tag{7.21}$$

If the assumptions, analogous to those from the previous case, hold true, then for any $\omega > 0$ problem (7.19)–(7.21) generates a dynamical system possessing a compact global attractor $\mathfrak{A}_\omega$ of finite fractal dimension. In the case of $\omega \to 0$, the global attractor is close in an appropriate sense to the attractor of the Mindlin–Timoshenko system

$$\begin{aligned} &v_{tt} + \alpha_1 v_t - \mathcal{A}v + \mu(v + \nabla w) + \beta\nabla\theta + \nabla_v \Phi(v) = 0,\\ &w_{tt} + \alpha_2 w_t - \mu \operatorname{div}(v + \nabla w) + g(w) = 0,\\ &\gamma\theta_t - \eta_1 \Delta\theta + \beta \operatorname{div} v_t = 0 \end{aligned}$$

with clamped boundary conditions and zero initial conditions (see [69]).

8. Fluid-plate/shell interaction models

8.1. Models description. In this section, we consider a number of models arising from interaction of 3D fluid and 2D elastic body. A fluid motion is described by the linearized equations (e.g., Navier–Stokes, Euler), and for the plate components we use several models that can be nonlinear. Thus, we assume that "large" deflections of the plate produce a "small" effect on the fluid. This corresponds to a system, in which a container with fluid is large enough with respect to the size of the plate.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\mathcal{O}$. We assume that $\partial\mathcal{O} = \Omega \cup S$, where $\Omega \subset \{x = (x_1; x_2; 0) : x' \equiv (x_1; x_2) \in \mathbb{R}^2\}$ is a flat domain with a smooth contour $\Gamma = \partial\Omega$ and S is a surface in $\mathbb{R}^3_- = \{x_3 \le 0\}$. We denote by n the outer normal to $\partial\mathcal{O}$. Note that $n = (0; 0; 1)$ on Ω. The surface S corresponds to the rigid walls of the fluid container and Ω is occupied by an elastic plate or a shell placed over the fluid.

To describe the fluid motion, we use the linearized Navier–Stokes equations in the domain $\mathcal{O}$ for the fluid velocity field $v = v(x,t) = (v^1(x,t); v^2(x,t); v^3(x,t))$ and the pressure $p(x,t)$:

$$v_t - \nu\Delta v + \nabla p = G_f \qquad \text{in } \mathcal{O} \times (0, +\infty)\,, \tag{8.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \mathcal{O} \times (0, +\infty), \tag{8.2}$$

where $\nu > 0$ is the dynamical viscosity of the fluid and G_f is a volume force.

We denote by $T_f(v)$ the surface force acting on the plate from the fluid; it equals $Tn|_\Omega$, where $T = \{T_{ij}\}_{i,j=1}^3$ is a stress tensor of the fluid,

$$T_{ij} \equiv T_{ij}(v) = \nu\left(v^i_{x_j} + v^j_{x_i}\right) - p\delta_{ij}, \quad i,j = 1,2,3.$$

Since $n = (0; 0; 1)$ on Ω, we have

$$T_f(v) = (\nu(v^1_{x_3} + v^3_{x_1}), \nu(v^2_{x_3} + v^3_{x_2}), 2\nu\partial_{x_3}v^3 - p).$$

The exact form of this force, as well as the boundary conditions on a part of the boundary Ω for the Navier–Stokes equations, depends on a plate/shell model chosen.

To describe a velocity field of incompressible fluid, we use the space

$$X = \left\{ v = (v^1; v^2; v^3) \in [L_2(\mathcal{O})]^3 : \operatorname{div} v = 0, \gamma_n v \equiv (v, n) = 0 \text{ on } S \right\}$$

endowed with L_2-norm. Details on the Navier–Stokes equations can be found in [91, 127].

8.1.1. Full von Karman (Marguerre–Vlasov) equations. The full von Karman (Marguerre–Vlasov) equations describe a shallow shell, which moves both in longitudinal and transversal directions [94]. To model interaction of the shell with the fluid, we supplement (8.1) and (8.2) with the non-slip boundary conditions on the fluid velocity field $v = v(x, t)$:

$$v = 0 \quad \text{on } S, \qquad v \equiv (v^1; v^2; v^3) = (u^1_t; u^2_t; w_t) \quad \text{on } \Omega, \tag{8.3}$$

where $u = u(x, t) \equiv (u^1; u^2; w)(x, t)$ is the displacement of the shell occupying Ω: w states for the transversal displacement, $\bar{u} = (u^1; u^2)$ denotes in-plane displacement.

In this model, rotational inertia of the shell filaments and in-plane accelerations are accounted for:

$$M_\alpha(w_{tt} + \gamma w_t) + \Delta^2 w + \operatorname{trace}\{K\mathcal{N}(u)\} - \operatorname{div}\{\mathcal{N}(u)\nabla w\} = G_3 - 2\nu\partial_{x_3} v^3 + p, \tag{8.4}$$

$$\bar{u}_{tt} = \operatorname{div}\{\mathcal{N}(u)\} + \left(G_1 - \nu(v^1_{x_3} + v^3_{x_1}); G_2 - \nu(v^2_{x_3} + v^3_{x_2})\right), \tag{8.5}$$

where $M_\alpha = 1 - \alpha\Delta$, $K = \operatorname{diag}(k_1, k_2)$, and

$$\mathcal{N}(u) \equiv \begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{pmatrix} = \mathcal{C}(\epsilon_0(\bar{u}) + wK + f(\nabla w))$$

with $\bar{u} = (u_1; u_2)$, $\mathcal{C}(\epsilon) = D\,[\mu \operatorname{trace} \epsilon I + (1 - \mu)\epsilon]$ and

$$\epsilon_0(\bar{u}) = \frac{1}{2}(\nabla\bar{u} + \nabla^T\bar{u}), \quad f(s) = \frac{1}{2} s \otimes s, \qquad s \in \mathbb{R}^2.$$

All the constants are assumed to be positive, except of $\gamma \geq 0$, which is the intensity of viscose damping of the shell material. We also denote by $G_{sh} \equiv (G_1; G_2; G_3)$ a (known) external force. The equations are supplemented with the clamped boundary conditions:

$$u^1\big|_{\partial\Omega} = u^2\big|_{\partial\Omega} = 0 \tag{8.6}$$

and

$$w|_{\partial\Omega} = \frac{\partial w}{\partial n}\bigg|_{\partial\Omega} = 0. \tag{8.7}$$

We complement (8.1)–(8.7) with the initial data of the form

$$v|_{t=0} = v_0, \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \tag{8.8}$$

where $\bar{u} = (u^1; u^2)$. Here $v_0 = (v_0^1; v_0^2; v_0^3)$, $u_i = (u_i^1; u_i^2; w_i)$.

Equations (8.2) and (8.3) imply the compatibility condition

$$\int_\Omega w(x', t)\, dx' = 0 \quad \text{for all } t \geq 0, \tag{8.9}$$

which can be interpreted as a fluid volume conservation.

For this model, we use the following space as a phase space:

$$\mathcal{H} = \{(v_0; u_0; u_1) \in X \times W \times Y : \; v_0 = u_1 \text{ on } \Omega\},$$
$$W = H_0^1(\Omega) \times H_0^1(\Omega) \times \widehat{H}_0^2(\Omega), \quad Y = L_2(\Omega) \times L_2(\Omega) \times \widehat{H}_0^1(\Omega).$$

There exists a unique weak solution to problem (8.1)–(8.9) (provided the right-hand side is smooth enough). Smoothing of initial velocities of longitudinal component $\bar{u}_t \in L_2(0, T; H^{1/2}(\Omega))$ takes place. The proof of the existence uses an idea from [12]. The strong super-criticality of full von Karman equations requires an application of the Sedenko method [121] and Bresis–Gallouet type inequalities to prove the uniqueness. The energy equality is proved by means of finite difference relations in the same way as in [90]. In the case of $\alpha = 0$, one can prove the existence of a weak solution, however, for proving the uniqueness, the Sedenko method fails because of strong super-criticality of nonlinearity. It is an open question by now. The model was studied in [53].

8.1.2. Simplified model for pure in-plane deformations. Models of this type arise in problems of blood flows in large arteries [112]. The transversal displacement of the plate w is assumed to be small with respect to its in-plane displacement $(u^1; u^2)$. Thus, we obtain the following boundary conditions on the fluid velocity field $v = v(x, t)$:

$$v = 0 \quad \text{on } S, \qquad v \equiv (v^1; v^2; v^3) = (\bar{u}_t; 0) \equiv (u_t^1; u_t^2; 0) \quad \text{on } \Omega, \tag{8.10}$$

where $\bar{u} = \bar{u}(x, t) \equiv (u^1(x, t); u^2(x, t))$ is the in-plane displacement of the plate placed on Ω. Since $v^3(x_1; x_2; 0) = 0$ for $(x_1; x_2) \in \Omega$ because of (8.10), we have $v^3_{x_i} = 0$ on Ω, $i = 1, 2$. Thus, after rescaling we obtain the following equations for the longitudinal displacement $\bar{u} = (u^1; u^2)$:

$$u_{tt}^i - \Delta u^i - \lambda \partial_{x_i} [\operatorname{div} \bar{u}] + \nu v^i_{x_3}|_{x_3=0} + f^i(\bar{u}) = 0, \quad i = 1, 2, \tag{8.11}$$

where λ is a nonnegative parameter and a nonlinear feedback force is potential. We impose clamped boundary conditions (8.6) on the displacement $\bar{u} = (u^1; u^2)$ on $\Gamma = \partial\Omega$. The problem with $\lambda = 0$, $f^i(\bar{u}) \equiv 0$ and strong (Kelvin–Vogt) internal damping present in the plate equation was considered in [79]. In contrast to [79], any type of mechanical or thermal damping is absent in the plate equation for this model.

If nonlinearity is potential and the potential $\Phi(\bar{u})$ satisfies a number of assumptions (see [21]), then there exists a unique weak solution to this problem in the space

$$\mathcal{H} = \{v \in X : \ (v, n) = 0 \text{ on } \Omega\} \times \left[H_0^1(\Omega)\right]^2 \times [L_2(\Omega)]^2 .$$

In the linear case ($\Phi(\bar{u}) \equiv 0$), the problem generates an exponentially stable C_0-semigroup of contractions in $\mathcal{H}$. Note that this property improves the result obtained in [79], where only strong stability (for individual trajectories) is proved. The model described above was studied in [21]. Recently the stochastic version of this problem was studied in [55].

8.1.3. Simplified model for pure transversal displacements. For this model, a special structure of the longitudinal displacement $\bar{u} = \bar{u}(x,t) \equiv (u^1(x,t); u^2(x,t))$ is assumed in (8.4). It is considered as a function of the transversal displacement w [94, 128]. Here it is also assumed that $\alpha = 0$. If we formally discard equation (8.5), we arrive to the following boundary conditions for the fluid velocity field $v = v(x,t)$:

$$v = 0 \quad \text{on } S; \qquad v \equiv (v^1; v^2; v^3) = (0; 0; w_t) \quad \text{on } \Omega. \tag{8.12}$$

Equation (8.2) implies $\partial_{x_3} v^3 = 0$, thus the third (transversal) component of the fluid stress tensor $T_f(v)$ exactly equals the fluid pressure p on Ω. Finally, the transversal displacement of the plate $w = w(x,t)$ satisfies the equation

$$w_{tt} + \Delta^2 w + \mathcal{F}(w) = G_{pl} + p|_\Omega \quad \text{in } \Omega \times (0, \infty),$$

where G_{pl} is a prescribed force acting on the plate, $\mathcal{F}(w)$ is a nonlinear feedback force, which assumed to be potential and locally Lipschitz from $H_0^{2-\epsilon}(\Omega)$ to $H^{-1/2}(\Omega)$ with some $\epsilon > 0$. These assumptions are satisfied for a number of nonlinearities important in elasticity such as Kirchhoff's, von Karman's and Berger's nonlinearities. Details can be found in [52].

The phase space of the system has the form

$$\mathcal{H} = \left\{ (v_0; w_0; w_1) \in X \times H_0^2(\Omega) \times L^2 : \ v_0^3 = w_1 \text{ on } \Omega, \ \int_\Omega w_0 = \int_\Omega w_1 = 0 \right\}.$$

Provided the right-hand side is smooth enough, there exists a unique weak solution to the problem. In the linear case ($G_f \equiv 0$, $G_{pl} \equiv 0$, $\mathcal{F}(w) \equiv 0$), the problem generates a C_0 exponentially stable semigroup T_t on $\mathcal{H}$. Smoothing of the plate velocity takes place and $w_t \in L_2(0, T; H^{1/2}(\Omega))$.

We emphasize that even in the linear case we cannot decompose system (8.1)–(8.8) into two equation sets, which describe the longitudinal and the transversal displacements separately, because of the structure of the surface fluid stress $T_f(v)$. The point is that the equation for in-plane oscillations of the plate (8.11) (Section 8.1.2) does not contain terms $v^3_{x_i}$ and the model does not require any compatibility conditions of the form (8.9) since the volume of the fluid is constant. In the case of purely transversal displacements, the force exerted on the plate by the fluid contains the pressure only. For more details, see [53].

8.1.4. Inviscid incompressible fluid. The model considered in Section 8.1.3 was studied in [26] for the case of inviscid incompressible fluid. In this case, the fluid motion is described by the Euler equations:

$$\begin{aligned} & v_t + \alpha v + \nabla p = G_f(t) && \text{in} \quad \mathcal{O}\times(0,+\infty), \\ & \operatorname{div} v = 0 && \text{in } \mathcal{O}\times(0,+\infty), \\ & (v,n) = 0 && \text{on } S; \\ & (v,n) = u_t && \text{on } \Omega. \end{aligned}$$

Here the fluid resistance is modeled by αv (with $\alpha \geq 0$) (see, e.g., [104]). To obtain the result on the existence of an attractor, we need a viscose damping in the plate equation,

$$w_{tt} + \gamma w_t + \Delta^2 w + \mathcal{F}(w) = G_{pl}(t) + p|_\Omega \quad \text{in } \Omega\times(0,\infty).$$

The phase space and the conditions on the nonlinearity are the same as in Subsection 8.1.3.

8.1.5. Viscose compressible fluid. The model from Section 8.1.3 was considered in [25] for the case of viscose compressible fluid. Thus, the fluid is described by two quantities: the density ϱ and the velocity field v. Provided the fluid is isothermal, we arrive at the following equations [25]:

$$\begin{aligned} & \varrho_t + \operatorname{div} v = G_d && \text{in } \mathcal{O}\times\mathbb{R}_+, && (8.13) \\ & v_t - \nu\Delta v - (\nu+\lambda)\nabla \operatorname{div} v + \gamma v + \nabla\varrho = G_f && \text{in } \mathcal{O}\times\mathbb{R}_+, && (8.14) \\ & (Tn,\tau) = 0 \ \text{ on } \partial\mathcal{O}, \ (v,n) = 0 \ \text{ on } \partial\mathcal{O}\setminus\Omega, \ (v,n) = w_t && \text{on } \Omega, && (8.15) \\ & w_{tt} + \Delta^2 w + \mathcal{F}(w) + T^{33}(v,\varrho)|_\Omega = G_{pl} && \text{in } \Omega\times(0,\infty), && (8.16) \end{aligned}$$

where T is a fluid stress tensor defined by

$$T^{ij} \equiv T^{ij}(v,p) = \nu\left(v^i_{x_j} + v^j_{x_i}\right) + \left[\lambda \operatorname{div} v - p\right]\delta_{ij}, \quad i,j = 1,2,3.$$

The main boundary conditions on $\partial\mathcal{O}$ are non-penetrating ones (8.15). However, one can consider the problem above with non-slip boundary conditions (8.12) as well.

We use the phase space

$$\mathcal{H} = \left\{(\varrho_0; v_0; u_0; u_1) \in L_2(\mathcal{O}) \times [L_2(\mathcal{O})]^3 \times H_0^2(\Omega) \times L_2(\Omega)\right\}.$$

Under the same conditions on the nonlinearity as in Subsection 8.1.3 and provided $\nu > 0$, $\lambda \geq 0$, $\gamma \geq 0$, the problem possesses a unique weak solution. Smoothing of the plate velocity takes place.

8.1.6. Unbounded container. Problems in Subsections 8.1.3–8.1.5 can be considered for the case when the fluid occupies an unbounded domain satisfying the Fridrichs–Poincare conditions (see [54]). A typical example of this domain is a tube, so we can generalize our considerations for a fluid flow. In this case, the fluid motion is described by the 3D Navier–Stokes equations linearized near a Poiselle (or Ossen) flow $a_0(x)$. Then equation (8.1) in Subsection 8.1.3 is replaced by

$$v_t - \nu\Delta v + L_0 v + \nabla p = G_f \quad \text{in } \mathcal{O} \times (0, +\infty),$$

where L_0 is a first-order linear differential operator of the form $L_0 v = (a_0, \nabla)v + Av$. Here $a_0(x)$ is a smooth divergence-free field on $\overline{\mathcal{O}}$ such that $(n, a_0) = 0$ on $\partial\mathcal{O}$, and $A = A(x)$ is a bounded 3×3 matrix, $x \in \overline{\mathcal{O}}$. This problem was considered in [54]. One can modify the models in Subsections 8.1.4, 8.1.5 in the same way. For these models, the cases of unbounded domains were considered together with the cases of bounded domains in [25, 26].

8.2. Asymptotic behavior. An asymptotic behavior of the models with viscose fluid from Subsections 8.1.1–8.1.3, 8.1.5 has many common features, therefore, we consider them together. The dynamical systems generated by the above mentioned problems are dissipative under some standard assumptions on the nonlinearity. Dynamical systems in Subsections 8.1.2, 8.1.3, 8.1.5 are asymptotically quasi-stable because of a special structure of the (critical) von Karman nonlinearity, or because of subcriticality of the nonlinearity. Provided $\epsilon > 0$, the DS from Subsection 8.1.1 is asymptotically smooth. The full von Karman equation maps the phase space in a space of lower smoothness (that is, is supercritical), thus we can not prove the stabilization inequality. Since the nonlinearity is potential, one can prove the asymptotic smoothness by means of the Ball method.

Thus, Theorems 2.8, 2.26, 2.27 imply that the dynamical systems in Subsections 8.1.2, 8.1.3, 8.1.5

(1) have compact global attractors of finite fractal dimension;

(2) the attractors have the regular structure described in Theorem 2.8;

(3) the attractors are smooth sets in the phase space;

(4) have generalized fractal exponential attractors;

(5) have finite sets of determining functionals.

The dynamical system from Subsection 8.1.1 has a compact global attractor of regular structure provided $\epsilon > 0$ and external forces have the form $G_f \equiv 0$, $G_1 = G_2 \equiv 0$, $G_3 \equiv g \in H^{-1}(\Omega)$.

One can use the same methods for the problem in Subsection 8.1.4, but then the result on the existence of an attractor can be obtained for subcritical nonlinearities only because of the lack of smoothing of the plate velocity. A reduction of the entire problem to the damped plate equation is used to treat critical nonlinearities in this problem. We have the following result [26]: if $G_{pl} \equiv 0$, $G_f \equiv 0$,

$\alpha > 0$, $\gamma > 0$, then the dynamical system in Subsection 8.1.4 is gradient, dissipative and possesses a weak attractor (that is, the uniform attraction takes place in a weak topology, not with respect to the norm). If the potential of nonlinearity is continuous on $H_0^{2-\epsilon}(\Omega)$ for some $\varepsilon > 0$, the weak attractor $\mathfrak{A}$ is also a compact global attractor and it has a regular structure described in Theorem 2.8.

9. Interaction of elastic plates with gas

In this section, we consider problems of interaction of a 2D plate with a gas occupying a finite or infinite domain. In the framework of this section, the transverse oscillations of the plate are modeled by the Kirchhoff equation with von Karman's nonlinearity (or general critical nonlinearity). The field of gas velocities is assumed to be potential. The survey of recent results on the interaction of plates with gas can be found in [30, 31].

9.1. Structural acoustic model. The mathematical formulation consists of a semilinear wave equation in a bounded domain $\mathcal{O}$ strongly coupled with a nonlinear equation of dynamics of an elastic Berger's or von Karman's plate (possibly, under presence of thermal effects) on the plane part of the boundary $\mathcal{O}$. This class of problems, known as structural acoustic models, appears in modeling the gas pressure in acoustic cameras surrounded by a combination of rigid and elastic walls.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial\mathcal{O}$. We assume that $\partial\mathcal{O} = \overline{\Omega} \cup \overline{S}$, where $\Omega \cap S = \emptyset$, $\Omega \subset \{x = (x_1; x_2; 0) : x' \equiv (x_1; x_2) \in \mathbb{R}^2\}$ is bounded by a smooth contour $\Gamma = \partial\Omega$, and S is a surface lying in the half-space $\mathbb{R}^3_- = \{x_3 \le 0\}$. The outward normal to $\partial\mathcal{O}$ is denoted by n. The variable v stands for the transverse displacement of the plate and z is the velocity potential of the gas medium. They satisfy the coupled system

$$
\begin{aligned}
& z_{tt} + g(z_t) - \Delta z + f(z) = 0 && \text{in } \mathcal{O} \times (0, T), \\
& \frac{\partial z}{\partial n} = 0 && \text{on } S \times (0, T), \\
& \frac{\partial z}{\partial n} = \alpha v_t && \text{in } \Omega \times (0, T), \\
& v_{tt} + b(v_t) + \Delta^2 v + B(v) + \beta z_t|_\Omega = 0 && \text{in } \Omega \times (0, T), \\
& v = \Delta v = 0 && \text{on } \partial\Omega \times (0, T)
\end{aligned}
$$

and the initial conditions. Here $g(s)$ and $b(s)$ are non-decreasing functions, describing dissipative effects of the model, and the term $f(z)$ is a nonlinear feedback acting on the wave component, $B(v)$ is von Karman's or Berger's nonlinearity, $\alpha,\ \beta > 0$.

The well-posedness of the problem follows from the abstract theory for interacting systems of the second order [96, Section 2.6] (see also [41, Chapter 6]). Asymptotic dynamics of the problem was investigated in [41, Section 12] from the point of view of quasi-stable systems for the von Karman plates and in [8], for the Berger plates. Various cases, when the flexible wall is a thermoelastic plate, are studied in [7] and [41, Section 12].

9.2. Interaction of plates with gas flow. The objects considered occupy the domain $\mathbb{R}^3_+ = \{(x, y, z) : z \geq 0\}$. The plate occupies a bounded domain $\Omega \subset \mathbb{R}^2_{\{(x,y)\}} = \{(x, y, z) : z = 0\}$ with a smooth boundary $\partial\Omega = \Gamma$. It is fastened in a " large" rigid body occupying the domain $\mathbb{R}^2 \setminus \Omega$. A flow of gas moves over the body in the opposite to the axis x direction with the speed $U \neq 1$. We consider the equations normalized in such a way that U is the Mach number, i.e., $0 \leq U < 1$ corresponds to subsonic speeds and $U > 1$, to supersonic speeds.

In most works, a nonlinear Kirchhoff plate with the clamped boundary conditions is considered. However, all the results obtained are true for hinged boundary conditions without essential changes in the proof. Far more complicated is the case of combined free-clamped boundary conditions. Some results for this case can be found in [47].

We denote the transverse displacement of the plate (in the direction of the z-axis) by u, and the potential of velocities of perturbed flow by ϕ. These variables satisfy the system

$$(1 - \alpha\Delta)u_{tt} + \Delta^2 u + f(u) = \big(\partial_t + U\partial_x\big)\gamma[\phi] \quad \text{on} \quad \Omega \times (0, T), \tag{9.1}$$

$$u(0) = u_0; \quad u_t(0) = u_1, \tag{9.2}$$

$$BC(u) \quad \text{on} \quad \partial\Omega \times (0, T), \tag{9.3}$$

$$(\partial_t + U\partial_x)^2\phi = \Delta\phi \quad \text{in} \quad \mathbb{R}_+ \times (0, T), \tag{9.4}$$

$$\phi(0) = \phi_0; \quad \phi_t(0) = \phi_1, \tag{9.5}$$

$$\frac{\partial}{\partial n}\phi = -\big[(\partial_t + U\partial_x)u(\mathbf{x})\big]\mathbf{1}_\Omega(\mathbf{x}) \quad \text{on} \quad \mathbb{R}^2_{\{(x,y)\}} \times (0, T), \tag{9.6}$$

where $\alpha \geq 0$ is the rotational inertia of the elements of the plate, $f(u)$ is von Karman's or Berger's nonlinearity, while $BC(u)$ are boundary conditions. Problem (9.1)–(9.6) was also considered with temperature and mechanical damping in the plate equation.

One of the key parameters, which influences the qualitative properties of the system, is the rotational inertia $\alpha \geq 0$. In the cases of $\alpha = 0$ and $\alpha > 0$, the functional spaces for weak solutions to problem (9.1)–(9.6) are different: $u_t \in L_2(\Omega)$ for the case of $\alpha = 0$ and $u_t \in H^1_0(\Omega)$ for the case of $\alpha > 0$. This makes different the arguments for proving the well-posedness and investigation of asymptotic behavior of problem (9.1)–(9.6).

For the case of $\alpha > 0$, the proof of well-posedness for the initial conditions $(u_0; u_1; \phi_0; \phi_1) \in H^2_0(\Omega) \times H^1_0(\Omega) \times H^1(\mathbb{R}^3_+) \times L_2(\mathbb{R}^3_+)$ (both for subsonic and supersonic flows) is given in [5].

The case of $\alpha = 0$ is more complicated and was solved for the supersonic flow not long ago in [46]. The phase space of the dynamical system in this case is $(u_0; u_1; \phi_0; \phi_1) \in H^2_0(\Omega) \times L_2(\Omega) \times H^1(\mathbb{R}^3_+) \times L_2(\mathbb{R}^3_+)$.

The main difficulty in studying the asymptotic behavior of a system of the type "plate+gas flow" is the unboundedness of the domain occupied by gas.

There are two methods for solving this problem. The first one is to study the asymptotic behavior of the gas components not in the whole space but in an arbitrary ball. The second one is to reduce the influence of a gas on the plate to a term with delay in the equation for the plate.

The first approach was used for the systems of the type (9.1)–(9.6) with structural [17,41,97,98] and thermal [118,120] damping. In the above-mentioned works, the asymptotic behavior was studied in the case of subsonic gas flow ($U < 1$). The following results on stabilization were obtained: in general position, for any solution to system (9.1)–(9.6) there exists a stationary point $(\overline{u}, 0, \overline{\phi}, 0)$ such that

$$\lim_{t\to+\infty} \Big\{ \|u(t) - \overline{u}\|_{2,\Omega}^2 + \|u_t(t)\|_\Omega^2 + \alpha\|\nabla u_t(t)\|_\Omega^2 + \|\nabla(\phi(t) - \overline{\phi})\|_{B_R^+}^2 + \|\phi_t(t)\|_{B_R^+}^2 \Big\} = 0 \quad (9.7)$$

for any $R > 0$, where $B_R^+ = \{x \in \mathbb{R}^3_+ : |x| < R\}$. In the case of thermoelastic plate, in addition to the convergence (9.7), $\|\theta(t)\| \to 0$ holds true when $t \to +\infty$ (here θ stands for the temperature of the plate).

The proof is based on the idea of splitting of an individual trajectory of (9.1)–(9.6) into converging to 0 and compact components (in an arbitrary half of a ball B_R^+). If $u_t \in H_0^1(\Omega)$ (either in view of the structure of the phase space in the case of $\alpha > 0$, or because of regularizing effects of structural $(-\Delta u_t)$ or thermal damping ($\alpha = 0$)), the compactness (in B_R^+) of the gas component follows from the compactness of the plate components. In the case of $\alpha = 0$, there is no regularization (see [97] for von Karman equations with viscous and static damping and [98] for the Berger equation with viscous damping). The result is established for smooth solutions and then extended on weak solutions by means of the limit transition.

The second approach was fully developed in [5], however, equations with memory were applied for description of motion of plates in a flow of gas up to then. The asymptotic behavior of solutions to the Berger models was studied in [6, 49]. Abstract methods of analysis of asymptotic behavior of solutions to the retarded nonlinear PDEs were developed in [14] (see also [41]). However, the models considered included either rotational inertia ($\alpha > 0$) or structural damping (the term of the form $-\varepsilon\Delta u_t$), which assures higher regularity of u_t and subcritical nonlinearity and delay.

The reduction principle for the full system "plate+gas flow" is formulated as follows [5].

Theorem 9.1. *Let there exist R such that $\phi_0(\mathbf{x}) = \phi_1(\mathbf{x}) = 0$ for $|\mathbf{x}| > R$. Then there exists $t^\#(R, U, \Omega) > 0$ such that for all $t > t^\#$ a weak solution $u(t)$ to (9.1)–(9.6) satisfies the equation*

$$(1 - \alpha\Delta)u_{tt} + \Delta^2 u + f(u) = p_0 - (\partial_t + U\partial_x)u - q^u(t), \quad (9.8)$$

where

$$q^u(t) = \frac{1}{2\pi}\int_0^{t^*} ds \int_0^{2\pi} d\theta\, [M_\theta^2 \widehat{u}](x - (U + \sin\theta)s, y - s\cos\theta, t - s).$$

Here $\widehat{u}$ *is the extension of* u *by zero outside* Ω; $M_\theta = \sin\theta\partial_x + \cos\theta\partial_y$, *and*

$$t^* = \inf\{t:\ \mathbf{x}(U,\theta,s) \notin \Omega\ \forall \mathbf{x} \in \Omega,\ \theta \in [0,2\pi],\ \textit{and}\ s > t\}$$

with $\mathbf{x}(U,\theta,s) = (x - (U + \sin\theta)s, y - s\cos\theta) \subset \mathbb{R}^2$.

Since the trajectories of the entire system with initial data with compact supports coincide in some time with the trajectories of system (9.8), investigation of the limit regimes of (9.8) allows to gain insight into the asymptotic behavior of (9.1)–(9.6).

In what follows, we will use a conventional for systems with delay (see, e.g., [66]) notation $u^t(\cdot)$ for the functions on $s \in [-t^*, 0]$ of the form $s \mapsto u(t+s)$, where $0 < t^* < +\infty$ is the delay time. We consider the system with delay (which somewhat generalizes (9.8)):

$$\begin{aligned}
&(1-\alpha\Delta)u_{tt} + \Delta^2 u + (\varepsilon_1 - \varepsilon_2\Delta)u_t + f(u) + Lu \\
&\qquad\qquad = p_0 + q(u^t, t) \quad \text{in } \Omega \times (0,T), && (9.9)\\
&BC(u) \qquad \text{on } \partial\Omega \times (0,T), && (9.10)\\
&u(0) = u_0, \quad u_t(0) = u_1, && (9.11)\\
&u|_{t\in(-t^*,0)} = \eta \in L_2(-t^*, 0; H_0^2(\Omega)). && (9.12)
\end{aligned}$$

Here $f(u)$ is von Karman's nonlinearity (or any other nonlinearity possessing certain properties, for instance, subcritical), $q(u^t,t)$ is a delay term, the operator L includes lower order derivatives not having gradient structure (for instance, $-Uu_x$ in (9.8)).

Problem (9.9)–(9.12) generates a dynamical system on the space

$$(u; u_t; u^t) \in H_0^2(\Omega) \times \mathcal{H}_\alpha(\Omega)(\Omega)) \times L_2(-t^*, 0; H_0^2(\Omega)),$$

where $\mathcal{H}_\alpha(\Omega)$ is defined in (3.3).

In the case of the rotational inertia ($\alpha > 0$) $u_t \in H_0^1(\Omega)$, von Karman's nonlinearity and the delay are subcritical. If, in addition, $\varepsilon_2 > 0$, it is easy to show that the dynamical system generated by (9.9)–(9.12) is dissipative and the proof of the existence of a global attractor is standard (see, e.g., [14]).

For $\alpha > 0$ and $\varepsilon_2 = 0$, the uniform dissipativity is not established and, probably, it does not hold true. Recent results obtained for a plate with rotational inertia interacting with the fluid [1] allow us to predict a pointwise dissipativity. However, the method used in this work is based on the spectral properties of the generator and can not be applied to nonlinear problems.

If the rotational inertia is neglected ($\alpha = 0$) and the viscous damping (inherited from interaction with a flow of gas) $u_t \in L_2(\Omega)$, the nonlinearity and the delay become critical. The dynamical system is asymptotically smooth and, being dissipative, possesses a compact global attractor $\mathbb{A}$ [47, 48]. The delay term, as well as von Karman's nonlinearity, possesses "the compensated compactness property"; therefore, the quasi-stability method (see Section 2.4.3) was successfully applied for studying the properties of the attractor of problem (9.9)–(9.12) in [47, 48].

The results obtained mean that despite of the conservativity of system (9.1)–(9.6) without damping there is the stabilization of the system to equilibria on any bounded domain (for subsonic flow) since the energy of the system is dissipated by an unbounded volume of gas at infinity. However, this phenomena does not take place for an arbitrary unbounded domain, there may exist periodic solutions for some configurations [29].

10. Wave equation with damping

This type of models was studied by many authors (see, e.g., [84] and references in [38, 39, 44, 45]). We consider two situations as examples.

10.1. Wave equation with nonlinear damping. We consider the following wave equation in a bounded domain $\Omega \subset \mathbb{R}^3$ with Dirichlet boundary conditions and appropriate initial conditions:

$$w_{tt} - \Delta w + a(x)g(w_t) = f(w) \quad \text{in } \Omega \times (0,T); \qquad w = 0 \quad \text{on } \Gamma \times (0,T), \tag{10.1}$$

where $T > 0$ may be finite or infinite. Let the damping have the structure $g(s) = g_1 s + |s|^{m-1}s$ for some $m \geq 1$, and also $f(w) \sim -|w|^{p-1}w$, where either $1 \leq p \leq 3$, or $3 < p \leq \min\left\{5, \frac{6m}{m+1}\right\}$. A nonnegative function $a \in C^1(\Omega)$ denotes the intensity of the damping (for more details on the assumptions on the parameters of problem (10.1), see [38]).

The well-posedness result holds if $p \leq m$, or under the dissipation condition

$$\liminf_{|s|\to\infty} \frac{-f(s)}{s} > -\lambda_1,$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary conditions. If $p > 3$, $p > m$ and $f(s) = |s|^{p-1}s$, then the local solution "blows up" at a finite time for the initial conditions with negative energy [38].

For strictly positive damping intensity in the critical case, the dynamical system $(H_0^1(\Omega) \times L_2(\Omega), S_t)$ is gradient and possesses a compact global attractor of finite fractal dimension. The result follows from asymptotic quasi-stability of the system. The same property allows to get the result on the regularity of elements of the attractor. It turns out that the result on the existence of a strong (corresponding to the strong topology) attractor does not follow from the smoothness of attractors except the case of subcritical damping. The first question arising in this connection is the existence of an attractor for strong solutions, i.e., of an attractor for solutions with smooth initial data, that asymptotically tend to the attractor $\mathcal{A}_1$, which is strongly included in $\mathcal{A}$. The first step is the proof of dissipativity of strong solutions which can be inferred from quasi-stability. Using the dissipativity of strong solutions and interpolation, one can show the existence of an attractor for strong solutions in a stronger topology [38].

10.2. Wave equation with boundary damping. We consider wave equation (10.1) in a bounded domain $\Omega \subset \mathbb{R}^3$ with nonlinear boundary conditions of Neumann type:

$$\frac{\partial}{\partial n} w + g_0(w_t) = h(w) \quad \text{in } \Gamma \times (0, T). \tag{10.2}$$

The functions g and f satisfy the same assumptions as in Subsection 10.1. The boundary damping has the form $g_0(s) = g_2 s + |s|^{q-1} s$, and the boundary source has the form $h(w) \sim -|w|^{k-1} w$, where $1 \le k \le \max\left\{3, \frac{4q}{q+1}\right\}$ (for more details on the parameters of the problem, see [38]).

The fundamental difference of the boundary and the inner damping is the essential condition of the linear behavior at infinity. Asymptotic quasi-stability of dynamical systems generated by problems (10.1), (10.2) was established in [32, 39, 44, 45].

10.3. Kirchhoff wave models. We consider the problem in a bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary:

$$u_{tt} - \sigma(\|\nabla u\|^2)\Delta u_t - \phi(\|\nabla u\|^2)\Delta u + f(u) = h(x), \qquad x \in \Omega,\ t > 0, \tag{10.3}$$

$$u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \tag{10.4}$$

Here $\sigma, \phi \in C^1(\mathbb{R}_+)$ are scalar positive functions, $f(u)$ is a source function, on which one imposes appropriate growth assumptions depending on the space dimension, $h \in L^2(\Omega)$ is a known function (for more details on the parameters of the problem, see [22]).

This model was introduced by G. Kirchhoff (for the case of $d = 1$, $\phi(s) = \phi_0 + \phi_1 s$, $\sigma(s) \equiv 0$, $f(u) \equiv 0$) and was studied by many authors under various assumptions on the parameters of the problem (see, e.g., [22, 101] and references therein). Model (10.3), (10.4) is characterized by the presence of three nonlinear terms: the source, the damping and the rigidity.

Well-posedness of problem (10.3), (10.4) is proved in the space $\mathcal{H} = [H_0^1(\Omega) \cap L_{p+1}(\Omega)] \times L_2(\Omega)$ with partially strong topology. A sequence $\{(u_0^n; u_1^n)\} \subset \mathcal{H}$ is called *partially strongly convergent* to $(u_0; u_1) \in \mathcal{H}$ if $u_0^n \to u_0$ strongly in $H_0^1(\Omega)$, $u_0^n \to u_0$ weakly in $L_{p+1}(\Omega)$, and $u_1^n \to u_1$ strongly in $L_2(\Omega)$, when $n \to \infty$ (in the case of $d \le 2$, $1 < p < \infty$ can be chosen arbitrarily). Obviously, partially strong convergence becomes strong under the supercritical level since $H_0^1(\Omega) \subset L_{p+1}(\Omega)$.

To describe the asymptotic properties of the corresponding dynamical system, we introduce the notion of a global partially strong attractor, which is defined as a bounded set possessing the properties:

(i) $\mathfrak{A}$ is a closed set in the partially strong topology,

(ii) $\mathfrak{A}$ is strictly invariant ($S_t\mathfrak{A} = \mathfrak{A}$ for all $t > 0$),

(iii) $\mathfrak{A}$ attracts uniformly all bounded sets in the partially strong topology: for any vicinity (partially strong) $\mathcal{O}$ of the set $\mathfrak{A}$ and any bounded set B in $\mathcal{H}$ there exists $t_* = t_*(\mathcal{O}, B)$ such that $S_t B \subset \mathcal{O}$ for all $t \ge t_*$.

The dynamical system $(S_t, \mathcal{H})$ generated by (10.3), (10.4) possesses a global partially strong attractor $\mathfrak{A}$ in the space $\mathcal{H}$. Moreover, $\mathfrak{A} \subset \mathcal{H}_1 = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$. The attractor $\mathfrak{A}$ has a finite fractal dimension as a compact set in $[H^{1+r}(\Omega) \cap H_0^1(\Omega)] \times H^r(\Omega)$ for any $r < 1$. The proof is based on the quasi-stability method (for details, see [22]).

11. Plasma dynamics equations

11.1. Quantum Zakharov system. We consider the following system in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \le 3$:

$$n_{tt} - \Delta\left(n + |E|^2\right) + h^2\Delta^2 n + \alpha n_t = f(x), \qquad x \in \Omega,\ t > 0, \tag{11.1}$$

$$iE_t + \Delta E - h^2\Delta^2 E + i\gamma E - nE = g(x), \qquad x \in \Omega,\ t > 0, \tag{11.2}$$

$$n|_{\partial\Omega} = \Delta n|_{\partial\Omega} = 0, \quad E|_{\partial\Omega} = \Delta E|_{\partial\Omega} = 0. \tag{11.3}$$

Here $E(x,t)$ is a complex-valued function and $n(x,t)$ is a real-valued function; $h > 0$, $\alpha \ge 0$ and $\gamma \ge 0$ are parameters, and $f(x)$, $g(x)$ are known functions.

In the dimension $d = 1$, this system was obtained [76] by means of quantum fluids approach to model the interaction between the Langmuir quantum waves and quantum ion acoustic waves in the electron-ion plasma. A 3D version of equations (11.1)–(11.3) was suggested later in [81].

Problem (11.1)–(11.3) with the Dirichlet boundary conditions is well-posed in the phase space $L_2(\Omega) \times H_2 \times \overline{H}_2$, where $H_2 = H^2(\Omega) \bigcap H_0^1(\Omega)$, and $\overline{H}_2$ is the complexification of H_2 (see [23]). One can show the existence of a finite dimensional compact global attractor using quasi-stability approach (see [23]).

If $h = 0$, we obtain the classical Zakharov system [131]. The global attractors of the system were studied in [71, 77] in the one-dimensional case and in [56], in the two-dimensional case. In the latter case, the phase space is less smooth and the corresponding nonlinearity is supercritical, therefore, the Sedenko method is used in order to prove the uniqueness, and Ball's method is used to show the existence of an attractor.

11.2. Schrödinger–Boussinesq equations. The methods, analogous to those described above, may be used to study the qualitative behavior of the system consisting of the Schrödinger equation and the Boussinesq equation interacting in a smooth two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. The system has the form:

$$\begin{aligned} w_{tt} + \gamma_1 w_t + \Delta^2 w - \Delta\left(f(w) + |E|^2\right) &= g_1(x), \\ iE_t + \Delta E - wE + i\gamma_2 E &= g_2(x), \qquad x \in \Omega,\ t > 0, \end{aligned}$$

where $E(x,t)$ and $w(x,t)$ are unknown functions, $E(x,t)$ is a complex-valued function, $w(x,t)$ is a real-valued function. Here γ_1 and γ_2 are non-negative parameters, $g_1(x)$ and $g_2(x)$ are known L_2-functions.

The asymptotic dynamics of the problem was studied in [57] using the methods described above under the assumptions that

$$
\begin{aligned}
&f \in C^1(\mathbb{R}), && f(0)=0,\\
&\exists c_1, c_2 \geq 0: && F(r)=\int_0^r f(\xi)d\xi \geq -c_1 r^2 - c_2,\ |r| \geq r_0,\\
&\exists M \geq 0, p \geq 1: && |f'(s)| \leq M(1+|s|^{p-1}),\ s \in \mathbb{R}.
\end{aligned}
$$

12. Equations with delay depending on the state

12.1. Hyperbolic equations. We study a class of nonlinear second-order evolution equations with delay depending on the state. The main goal is to investigate the well-posedness and asymptotic dynamics of equations of the form

$$
u_{tt}(t) + ku_t(t) + Au(t) + F(u(t)) + M(u^t) = 0, \quad t > 0, \tag{12.1}
$$

in a Hilbert space H. Here A is a positive linear operator with discrete spectrum, $F(\cdot)$ is a nonlinear operator, $M(u^t)$ describes (nonlinear) delay effects. We use the conventional for systems with delay (see, e.g., [83, 130]) notation $u^t(\cdot)$ for a function on $s \in [-h, 0]$ of the form $s \mapsto u(t+s)$, where $0 < h < +\infty$ is the delay time. Details on the assumptions on the nonlinearities can be found in [50].

The main example of this model is the nonlinear plate equation of the form

$$
u_{tt}(t,x) + ku_t(t,x) + \Delta^2 u(t,x) + F(u(t,x)) + au(t-\tau[u(t)], x) = 0
$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with appropriate boundary conditions. Here τ is a mapping defined on the solutions on an interval $[0,h]$, k and a are constants. The plate is located on a basement; the term $au(t-\tau[u(t)],x)$ models the effect of the Winkler type basement [123] with the resistance with delay. Nonlinear forces F may be of Kirchhoff, Berger, or von Karman types.

Equation (12.1) is considered under the initial conditions

$$
u_0 = u_0(\theta) \equiv u(\theta) = \varphi(\theta), \quad \text{for } \theta \in [-h, 0],\ \varphi \in W, \tag{12.2}
$$

where $W = C([-h,0], D(A^{1/2})) \bigcap C^1([-h,0], H)$ with A denoting the biharmonic operator with appropriate boundary conditions. For any $\varphi \in W$, there exists a unique global generalized solution $U(t) \equiv (u(t); \dot{u}(t))$ to problem (12.1), (12.2) on the interval $[0, +\infty)$.

By using the quasi-stability method (see Section 2.4.3), it was proved that under certain assumptions on the nonlinearity F, for globally Lipschitz functional G and locally Lipschitz nonlinearity $M : W \mapsto H$ of the form

$$
M(u^t) = G(u(t-\tau(u^t))) \equiv G\left(u(t) - \int_{t-\tau(u^t)}^{t} \dot{u}(s)\, ds\right),
$$

where τ maps W into the interval $[0,h]$ (for more details, see [50]), the dynamical system (S_t, W) generated by the generalized solutions of (12.1) possesses a finite dimensional compact global attractor $\mathfrak{A}$.

12.2. Parabolic equations. We study the well-posedness and asymptotic properties of solutions to the abstract first-order in time evolution equations with delay of the form

$$u_t(t) + Au(t) + M(u^t) + F(u(t)) = g, \quad t > 0, \tag{12.3}$$

in a Hilbert space H. All notation are analogous to those used in the previous subsection.

An example of such equations is the following system with discrete delay:

$$u_t(t,x) - \Delta u(t,x) + b\left(M[u(t-\eta(u^t),\cdot)](x)\right) + f(u(t,x)) = g(x)$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $M : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator, $b : \mathbb{R} \to \mathbb{R}$ is a Lipschitz mapping. The function $\eta : C([-h,0]; L^2(\Omega)) \to [0,h] \subset \mathbb{R}_+$ denotes a discrete delay. The Nemytskii operator $u \mapsto f(u)$, where f is a function of class C^1 which describes a nonlinear resistance without delay, and $g(x)$ denotes an external source. The form of the delay term is motivated by some population dynamics models (see [78] and [116]).

Equation (12.3) is supplemented with the initial conditions

$$u(\theta) = \varphi(\theta), \quad \theta \in [-h,0]. \tag{12.4}$$

Under certain assumptions, it is possible to show the well-posedness of problem (12.3), (12.4) and to use the idea of the quasi-stability method [51] to prove the existence of a compact global attractor with the properties described in Theorems 2.15, 2.17.

13. Quasi-stability and synchronization

In this section, we discuss the application of quasi-stability method to the analysis of synchronization phenomena at the level of attractors. This means that in the synchronization regime an attractor of the interacting system becomes "diagonal" in some sense. It should be mentioned that the question of synchronization of interacting equations has recently attracted an ample attention. There are several monographs on this topic [3,100,107,111,125,129], including extensive literature. The problems of synchronization of infinite dimensional systems were studied in [9–11,82,117] for parabolic systems. Syncronization of Berger plates (as examples of abstract models considered below) was investigated in [108–110]. Master-slave synchronization of coupled parabolic and hyperbolic equations was studied in [19,20].

13.1. The main model. The abstract form of the synchronization problem for second-order systems is stated in the following way. We consider the system in a Hilbert space H,

$$u_{tt} + Au + D_1 u_t + \alpha K(u_t - v_t) + \varkappa K(u - v) + B_1(u) = 0, \tag{13.1}$$

$$v_{tt} + Av + D_2 v_t + \alpha K(v_t - u_t) + \varkappa K(v - u) + B_2(v) = 0, \tag{13.2}$$

where α and $\varkappa$ are non-negative parameters with the initial conditions

$$u(0) = u_0, \quad u_t(0) = u_1, \quad v(0) = v_0, \quad v_t(0) = v_1. \tag{13.3}$$

We assume that

(i) A is a self-adjoint positive densely defined operator (with the domain $\mathcal{D}(A)$) in a separable Hilbert space H. The resolvent of A is compact in H.

(ii) The damping operators $D_i : D(A^{1/2}) \mapsto H$ are non-negative.

(iii) The coupling operator K is strictly positive in H with the domain $\mathcal{D}(K) \supseteq D(A^{1/2})$.

(iv) The nonlinear operators $B_i : D(A^{1/2}) \to H$ are locally Lipschitz.

These hypotheses are mainly motivated by the systems of wave and plate equations.

Using the notations

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} A, \quad \mathcal{B}(U) = \begin{pmatrix} B_1(u) \\ B_2(v) \end{pmatrix},$$

one can rewrite problem (13.1)–(13.3) in the form

$$U_{tt} + \mathcal{A}U + (\mathcal{D}_0 + \alpha\mathcal{K})U_t + \varkappa\mathcal{K}U + \mathcal{B}(U) = 0, \quad U(0) = U_0, \quad U_t(0) = U_1, \tag{13.4}$$

where

$$\mathcal{K} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} K, \quad \mathcal{D}_0 = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Under some additional conditions, problem (13.1)–(13.3) has a unique weak solution and generates a dynamical system $(\mathcal{H}, S_t)$ in the space

$$\mathcal{H} \equiv H^{1/2} \times H^{1/2} \times H \times H, \quad H^{1/2} = D(A^{1/2}).$$

Synchronization problem consists of analysis of asymptotic closeness of components of solutions $u(t)$ and $v(t)$ in some sense. Moreover, the question on synchronization at the global attractor level in the limit case $\alpha, \kappa \to \infty$ plays an important role. This requires uniform estimates for global attractors of problem (13.1)–(13.3). To study problem (13.4), we split the research into two steps: the uniform (with respect to α, κ) dissipativity and the uniform asymptotic quasi-stability.

Under some assumptions on the operators of the system, it is possible to answer these questions for a wide class of models of type (13.1)–(13.3). Restrictions on the nonlinear feedback forces $B_i(u)$ were introduced earlier for the systems with nonlinear damping (see [39, p. 98] and [33]) in order to consider a number of critical nonlinearities. This assumptions are satisfied for Berger's and von Karman's nonlinearities (see [39] p. 156 and p. 160, respectively). Moreover, it is shown in [39, p. 137] that the assumption is satisfied for the coupled 3D wave equations in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$u_{tt} + \sigma_1 u_t - \Delta u + k_1(u - v) + \varphi_1(u) = f_1(x), \qquad u|_{\partial\Omega} = 0, \tag{13.5}$$
$$v_{tt} + \sigma_2 v_t - \Delta v + k_2(u - v) + \varphi_2(u) = f_2(x), \qquad v|_{\partial\Omega} = 0, \tag{13.6}$$

if $\varphi_i \in C^2(\mathbb{R})$ satisfy $|\varphi_i''(s)| \le C(1 + |s|)$ for all $s \in \mathbb{R}$. The parameters σ_i and k_i are non-negative. Consequently, the abstract model covers 3D wave dynamics with critical nonlinear terms.

To study the synchronization phenomena, it is important to obtain the uniform with respect to the parameters α, κ quasi-stability estimates. Under certain assumptions, it is possible to prove the following statement (see [28]).

Proposition 13.1 (Uniform quasi-stability). *Let $\mathcal{M} \subset \mathcal{H}$ be a bounded positively invariant set with respect to S_t, and $Y^i = (U^i(t), U^i_t(t)) = S_t Y^i_0$, $i = 1, 2$, be two solutions to (13.4) with different initial data $Y^i_0 \in \mathcal{M}$. Then there exist $C, \gamma > 0$ such that*

$$E_Z(t) \le C E_Z(0) e^{-\gamma t} + C \max_{[0,t]} \|\mathcal{A}^\sigma Z(\tau)\|^2, \quad \forall t > 0,$$

where $0 \le \sigma < 1/2$, $Z = U^1 - U^2$, and

$$E_Z(t) = \frac{1}{2}\left(\|Z(t)\|^2 + \|\mathcal{A}^{1/2} Z(t)\|^2 + \varkappa \|\mathcal{K}^{1/2} Z\|^2\right).$$

If $\mathcal{M}$ is bounded in $\mathcal{H}$ uniformly with respect to $(\alpha; \varkappa) \in \Lambda$ and

$$(\alpha; \varkappa) \in \Lambda_\beta \equiv \{(\alpha; \varkappa) \in \Lambda : \ \alpha \le \beta(1 + \varkappa)\}$$

for some $\beta > 0$, then the constants C, γ do not depend on $(\alpha; \varkappa)$, though they may depend on β.

13.2. Asymptotic synchronization. The study of asymptotic synchronization phenomena includes investigation of qualitative behavior of systems in the case of large coupling parameters $\varkappa \to \infty$ (and/or $\alpha \to \infty$). It follows from the uniform estimates for the attractor that $u = v$ in this limit case. Therefore, it is natural to consider the limit problem of the form

$$w_{tt} + Aw + Dw_t + B(w) = 0, \quad w(0) = w_0, \quad w_t(0) = w_1, \tag{13.7}$$

where

$$D = \frac{1}{2}(D_1 + D_2), \quad B(w) = \frac{1}{2}(B_1(w) + B_2(w)).$$

It can be shown that problem (13.7) possesses a compact global attractor $\mathfrak{A}$ in the space $H^{1/2} \times H$, and in the limit case $\varkappa \to \infty$ we have

$$\lim_{\varkappa \to \infty} \left[\sup \left\{ \operatorname{dist}_{\mathcal{H}_\varepsilon}(Y, \widetilde{\mathfrak{A}}) : \ Y \in \mathfrak{A}^{\alpha,\varkappa} \right\} \right] = 0, \tag{13.8}$$

where $\mathcal{H}_\varepsilon = [H^{1/2-\varepsilon}]^4$ and $\widetilde{\mathfrak{A}} = \{(u_0; u_0; u_1; u_1) : (u_0; u_1) \in \mathfrak{A}\}$. Here $\mathfrak{A}$ is a global attractor of the dynamical system generated by (13.7) and $\mathfrak{A}^{\alpha,\kappa}$ is a global attractor of the dynamical system generated by (13.1). Moreover, if the operator K commutes (one can choose $K = A^\sigma$ for some $0 \le \sigma \le 1/2$) with A, then convergence (13.8) takes place in the space $H^{1/2} \times H^{1/2} \times H^{1/2-\varepsilon} \times H^{1/2-\varepsilon} \subset \mathcal{H}$.

Result (13.8) means that the attractor $\mathfrak{A}^{\alpha,\varkappa}$ becomes "diagonal" for large intensity $\varkappa$ with a fixed or even absent interaction between the speed components. Therefore, the components of the system synchronize in the limit case at the level of attractors. In particular, this means that any solution $U(t) = (u(t); v(t))$ to equation (13.1)–(13.3) shows the following synchronization:

$$\forall \varepsilon > 0\ \exists \varkappa_*\ \forall \varkappa \ge \varkappa_* \quad \limsup_{t\to+\infty} \left[\|u_t(t) - v_t(t)\|^2 + \|A^{1/2}(u(t) - v(t))\|^2\right] \le \varepsilon.$$

Let us note that there is a problem in proving synchronization for small fixed $\varkappa$ and large α. The point is that in the case of $\varkappa = 0$, under certain conditions on nonlinear forces B_i, the existence of two stationary solutions is possible, which demonstrates the absence of asymptotic synchronization.

In the case of identical interacting subsystems, i.e., if $D_1 = D_2 \equiv D$, $B_1(w) = B_2(w) \equiv B(w)$, the asymptotic synchronization is observed for finite $\varkappa$. Namely, if the parameter

$$s_\varkappa = \inf\left\{\nu(Aw, w) + \varkappa(Kw, w) : \ w \in H^{1/2},\ \|w\| = 1\right\}$$

is large enough (it can be shown that $s_\varkappa \ge \varkappa \inf \operatorname{spec}(K)$, i.e., if K is non-singular, $s_\varkappa \to +\infty$ if $\varkappa \to +\infty$), then the property of exponential synchronization holds. This means that there exists $\kappa_0 > 0$ such that for any $\kappa > \kappa_0$ there exists $\omega > 0$ such that

$$\lim_{t\to\infty} \left\{e^{\omega t}\left[\|u_t(t) - v_t(t)\|^2 + \|A^{1/2}(u(t) - v(t))\|^2\right]\right\} = 0$$

for any solution $U(t) = (u(t); v(t))$ to problem (13.1)–(13.3). In this case, $\mathfrak{A}^{\alpha,\varkappa} \equiv \widetilde{\mathfrak{A}}$ for all $\varkappa$ such that $s_\varkappa \ge s_*$.

Similar results can be established for N interacting second-order in time equations

$$\begin{aligned}
&u^1_{tt} + Au^1 + D_1 u^1_t + \alpha K(u^1_t - u^2_t) + \varkappa K(u^1 - u^2) + B_1(u^1) = 0,\\
&u^j_{tt} + Au^j + D_i u^j_t - \alpha K(u^{j+1}_t - 2u^j_t + u^{j-1}_t)\\
&\qquad\qquad\qquad - \varkappa K(u^{j+1} - 2u^j + u^{j-1}) + B_j(u^j) = 0,\\
&\qquad\qquad\qquad \dots\\
&u^N_{tt} + Au^N + D_N u^N_t + \alpha K(u^N_t - u^{N-1}_t) + \varkappa K(u^N - u^{N-1}) + B_N(u^N) = 0.
\end{aligned}$$

Let us note that in the case of ODEs ($\nu_i \equiv 0$, $K = \mathrm{id}$) synchronization for this model was studied in [82] under the assumption that *both parameters* α and $\varkappa$ become large and even tend to infinity. Our approach allows us to establish asymptotic synchronization for fixed α and $\varkappa \to +\infty$ (for coincident systems, it is sufficient to assume that $\varkappa$ is large enough). A similar result was obtained in [109] for the plates with Berger's nonlinearity in the case of $D_{0j} = d_j\mathrm{id}$, $\alpha = 0$ and $K = \mathrm{id}$.

13.3. Applications. Finally, we briefly describe the applications of general results stated above.

13.3.1. Plates. We consider the plate equations coupled by elastic coupling, namely, the system:

$$\begin{aligned} &u_{tt} + \gamma_1 u_t + \Delta^2 u + \varkappa K(u - v) + \varphi_1(u) = f_1 && \text{in } \Omega \subset \mathbb{R}^2, \\ &v_{tt} + \gamma_2 v_t + \Delta^2 v + \varkappa K(v - u) + \varphi_2(u) = f_2 && \text{in } \Omega \subset \mathbb{R}^2, \end{aligned}$$

with the hinged boundary conditions

$$u = \Delta u = 0, \quad v = \Delta v = 0 \quad \text{on } \partial\Omega.$$

The nonlinear term $\varphi_i(u)$ may describe Kirchhoff's, von Karman's, or Berger's models (see Section 5).

The coupling operator K may have the form: $K = id$, $K = -\Delta$, and $K = A^\sigma$ with $0 < \sigma < 1/2$. In the case of the Kirchhoff plate with globally Lipschitz functions φ_i, one can show that two plates can be synchronized by means of finite number of point couplings. Let us note that in the case when both φ_1 and φ_2 are Berger's nonlinearities (possibly with different parameters) the results on synchronization with the coupling operator $K = id$ can be found in [108], see also [109, 110].

The abstract results stated above can be also applied to other plate models (see [20, 28]).

13.3.2. Coupled wave equations. For coupled wave equations (13.5), (13.6), the standard (critical) hypothesis on the nonlinear source $\varphi \in C^2(\mathbb{R})$ in the 3D case is as follows:

$$\liminf_{|s|\to\infty} \left\{\varphi_i(s)s^{-1}\right\} > -\lambda_1, \quad |\varphi_i''(s)| \le C(1 + |s|), \ s \in \mathbb{R},$$

where λ_1 is the first eigenvalue of the Laplace operator with the Dirichlet boundary conditions, see, e.g., [39, Chapter 5].

One can also consider various forms of dissipative sine-Gordon equations. They are used to model dynamics of the superconductive Josephson tunneling controlled by a current source (see, e.g., [126]). For example, one can consider the system

$$\begin{aligned} u_{tt} + \gamma u_t - \Delta u + \beta u + \varkappa(u - v) + \lambda \sin u = f(x), \\ v_{tt} + \gamma v_t - \Delta v + \beta u + \varkappa(v - u) + \lambda \sin v = f(x) \end{aligned}$$

in a smooth domain $\Omega \subset \mathbb{R}^d$ with the Neumann boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0.$$

It is easy to see that in the case of the Dirichlet boundary conditions one can apply the above mentioned theory. The same is true for $\beta > 0$. In the case of $\beta = 0$, the situation is more complicated (see [28]).

Another interesting system of the coupled sine-Gordon equations is as follows:

$$u_{tt} + \gamma u_t - \Delta u + \lambda \sin(u - v) = f_1(x), \tag{13.9}$$

$$v_{tt} + \gamma v_t - \Delta v + \lambda \sin(v - u) = f_2(x), \tag{13.10}$$

$$u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0. \tag{13.11}$$

Formally, this model is out of the scope of the developed theory. However, using the ideas introduced above, it is possible to answer some questions about synchronization regimes of the system. For ODEs, the synchronization phenomena for (13.9)–(13.11) were studied in [99, 100]. For this system, the shifted in some sense synchronization in the antiphase is observed. For details, we refer to [28].

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Метод квазістійкості в дослідженні асимптотичного поводження динамічних систем

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В огляді здійснено спробу представити сучасні ідеї та методи дослідження якісної динаміки нескінченновимірних дисипативних систем. Представлено такі основні поняття, як дисипативність та асимптотична гладкість динамічних систем, глобальний та фрактальний атрактори, визначальні функціонали, регулярність асимптотичної динаміки. Акцент зроблено на методі квазістійкості, розробленому І. Чуєшовим та І. Лашецькою. Цей метод базується на відповідному розкладі різниці траєкторій на стійку та компактну частини. Існування такого розкладу має багато важливих наслідків: асимптотичну гладкість, існування та скінченновимірність атракторів, існування скінченної множини визначальних функціоналів та існування (за деяких додаткових умов) фрактального експоненціального атрактора. Решта статті ілюструє застосування абстрактної теорії до конкретних проблем. Основну увагу приділено демонстрації області застосування методу квазістійкості.

Ключові слова: нескінченновимірні динамічні системи, асимптотичне поводження, глобальні атрактори, фрактальні експоненціальні атрактори, детермінуючи функціонали, фінітний фрактальний вимір, квазістійкість, стійкість, диференціальні рівняння з частинними похідними.