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On the Sharpness of One Integral Inequality for Closed Curves in $\mathbb{R}^4$

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The sharpness of the integral inequality $\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds > 2\pi$ for closed curves with nowhere vanishing curvatures in $\mathbb{R}^4$ is discussed. We prove that an arbitrary closed curve of constant positive curvatures in $\mathbb{R}^4$ satisfies the inequality $\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds \geq 2\sqrt{5}\pi$.

Key words: closed curve, curvature, curves of constant curvatures.

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1. Introduction

The famous Fenchel–Borsuk theorem of the classical theory of curves states that the total curvature of an arbitrary smooth closed curve γ in $\mathbb{R}^n$ is greater than or equal to 2π:

$$\int_\gamma k_1\, ds \geq 2\pi, \tag{1.1}$$

and the equality holds if and only if γ is a convex closed curve in $\mathbb{R}^2$, see [1, Chap. 21], [4, 5].

In [6], the first author obtained a series of integral inequalities for curvatures of smooth closed curves in $\mathbb{R}^n$ which may be viewed as a direct generalization of the Fenchel–Borsuk inequality. Namely, let γ be an arbitrary smooth closed curve in $\mathbb{R}^n$, $n \geq 4$. Suppose that γ has nowhere vanishing curvatures $k_1, k_2, \ldots, k_j$ for some $2 \leq j \leq n-1$. Then the following inequality holds:

$$\int_\gamma \sqrt{k_{j-1}^2 + k_j^2 + k_{j+1}^2}\, ds > 2\pi, \tag{1.2}$$

where s stands for an arc-length of γ and k_n is taken to be identically zero.

Consequently, if all the curvatures $k_1, \ldots, k_{n-1}$ of $\gamma \subset \mathbb{R}^n$ are nowhere vanishing, then (1.2) holds true for each $2 \leq j \leq n-1$, and thus γ satisfies a sequence of $n-2$ different integral inequalities.

Inequality (1.2) is *sharp* in the case of an odd j, see [6]. Actually, for any fixed odd $2 \le j \le n-1$ one can construct a sequence of smooth closed curves $\{\gamma_m\}_{m=1}^{\infty}$ in $\mathbb{R}^n$ such that the values of

$$\int_{\gamma_m} \sqrt{k_{j-1}^2 + k_j^2 + k_{j+1}^2}\, ds$$

tend to 2π as $m \to \infty$. If n is even, then the desired sequence $\{\gamma_m\}_{m=1}^{\infty}$ may consist of closed curves of *constant* curvatures in $\mathbb{R}^n$; if n is odd, then $\{\gamma_m\}_{m=1}^{\infty}$ in $\mathbb{R}^n$ may be obtained by slight perturbations of curves of constant curvatures in $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. Thus, curves of constant curvatures provide the sharpness for (1.2) in the case of an odd j.

As for the case of an even j, the problem of the sharpness of (1.2) still remains quite challenging and interesting open problem, which motivated this research paper.

We start to discuss the problem by considering the simplest case $n = 4$. As stated above, an arbitrary smooth closed curve $\gamma \subset \mathbb{R}^4$ satisfies two inequalities:

$$\int_{\gamma} \sqrt{k_1^2 + k_2^2 + k_3^2} ds > 2\pi, \qquad \text{if} \qquad k_1 > 0, \qquad k_2 > 0, \tag{1.3}$$

$$\int_{\gamma} \sqrt{k_2^2 + k_3^2} ds > 2\pi, \qquad \text{if} \qquad k_1 > 0, \qquad k_2 > 0, \qquad k_3 > 0. \tag{1.4}$$

Inequality (1.4) is sharp since it corresponds to the odd value $j = 3$.

As for inequality (1.3), it looks rather trivial in view of (1.1), and hence one can expect that (1.3) is not sharp. The main result of the paper partially confirms this expectation.

Theorem 1.1. *Let γ be a smooth closed curve in $\mathbb{R}^4$ with nowhere vanishing constant curvatures k_1, k_2, and k_3. Then the following sharp lower bound holds:*

$$\int_{\gamma} \sqrt{k_1^2 + k_2^2 + k_3^2} ds \ge 2\sqrt{5}\pi. \tag{1.5}$$

Moreover, the equality in (1.5) is attained if and only if γ is represented in $\mathbb{R}^4$ as $x^1 = a^1 \cos t$, $x^2 = a^1 \sin t$, $x^3 = a^2 \cos 2t$, $x^4 = a^2 \sin 2t$, $t \in [0, 2\pi]$, where a^1 and a^2 are arbitrary non-zero constants.

A computer-aided numerical analysis demonstrates that (1.5) holds true for some closed curves with nonconstant curvatures too. This, together with Theorem 1.1, allows us to think that inequality (1.5) remains true for any closed curve with nowhere vanishing curvatures in $\mathbb{R}^4$.

Let us recall the idea of the proof of inequality (1.3). For an arbitrary smooth curve $\gamma \subset \mathbb{R}^4$ with nowhere vanishing curvatures k_1 and k_2, one can consider a well-defined family of two-dimensional osculating planes of γ, which are spanned by the first and second vectors of the Frenet frame of γ. This family of planes may be interpreted as a smooth closed curve in the Grassmann manifold $G_{2,4}$; it is called *the osculating indicatrix* of γ and denoted by $\tilde{\gamma}$. The Grassmann manifold $G_{2,4}$ can be embedded into $\mathbb{R}^6$ via the Plücker embedding, see [2, Chap. 8.2], [3], and therefore $\tilde{\gamma}$ can be viewed as a smooth closed curve in $\mathbb{R}^6$. It turns out that the left-hand side of (1.3) is the total curvature of $\tilde{\gamma}$, and thus inequality (1.3) for γ is just the Fenchel–Borsuk inequality for $\tilde{\gamma}$, c.f., [6].

If γ has constant curvatures, then the the following stronger result holds.

Theorem 1.2. *Let γ be a smooth closed curve in $\mathbb{R}^4$ with nowhere vanishing constant curvatures k_1, k_2 and k_3. Let $\tilde{\gamma} \subset \mathbb{R}^6$ be the osculating indicatrix of γ. Then the following holds:*

1) *$\tilde{\gamma}$ is a smooth closed curve;*

2) *$\tilde{\gamma}$ lies in a four-dimensional affine subspace $\mathbb{R}^4 \subset \mathbb{R}^6$;*

3) *the curvatures $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$ of $\tilde{\gamma}$ are non-zero constants;*

4) $\displaystyle\int_{\tilde{\gamma}} \tilde{k}_1 d\tilde{s} = \int_{\gamma} \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds;$

5) $\displaystyle\int_{\tilde{\gamma}} \sqrt{\tilde{k}_1^2 + \tilde{k}_2^2 + \tilde{k}_3^2}\, d\tilde{s} = \sqrt{2} \int_{\gamma} \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds$, *where $\tilde{s}$ is an arc-length of $\tilde{\gamma}$.*

We were very surprised by the relationship 5) which looks quite elegant although individual expressions for $\tilde{k}_j$ in terms of k_i are rather cumbersome.

In view of Theorem 1.2, the procedure of constructing the osculating indicatrix described above can be viewed as a particular transformation of closed curves of constant curvatures in $\mathbb{R}^4$. The transformation can be iterated, and then at every step we obtain a new curve of constant curvatures in $\mathbb{R}^4$. This results in a specific sequence of curves of constant curvatures in $\mathbb{R}^4$ which is generated by the initial curve γ of constant curvature. Notice that the value of

$$\int_{\gamma} \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds$$

is multiplied by $\sqrt{2}$ at each step of iteration.

It would be interesting to extend Theorem 1.1 and Theorem 1.2 to more general families of closed curves with *nonconstant* curvatures in $\mathbb{R}^4$.

2. Closed curves of constant curvatures in $\mathbb{R}^4$

Let γ be a smooth curve in $\mathbb{R}^4$, whose curvatures k_1, k_2, k_3 are non-zero constant. Then γ is parameterized as follows:

$$x_1 = a_1 \cos \alpha_1 t, \quad x_2 = a_1 \sin \alpha_1 t, \quad x_3 = a_2 \cos \alpha_2 t, \quad x_4 = a_2 \cos \alpha_2 t, \tag{2.1}$$

where a_1, a_2, α_1 and α_2 are constants, see [1, Chap. 33].

For an arc-length s of γ, one has

$$\frac{ds}{dt} = \sqrt{a_1^2 \alpha_1^2 + a_2^2 \alpha_2^2}. \tag{2.2}$$

Hence one needs to assume $a_1^2\alpha_1^2 + a_2^2\alpha_2^2 \neq 0$ to guarantee the smoothness of γ.

For calculating the curvatures of γ, one can apply standard formulae of the classical theory of curves, see [1, Chap. 32]. Elementary differential-geometric calculations result in the following statement.

Proposition 2.1. *The curvatures of γ are expressed as follows:*

$$k_1 = \frac{\sqrt{a_1^2\alpha_1^4 + a_2^2\alpha_2^4}}{a_1^2\alpha_1^2 + a_2^2\alpha_2^2}, \tag{2.3}$$

$$k_2 = \frac{a_1 a_2 \alpha_1 \alpha_2 |\alpha_1^2 - \alpha_2^2|}{(a_1^2\alpha_1^2 + a_2^2\alpha_2^2)\sqrt{a_1^2\alpha_1^4 + a_2^2\alpha_2^4}}, \tag{2.4}$$

$$k_3 = \frac{\alpha_1\alpha_2}{\sqrt{a_1^2\alpha_1^4 + a_2^2\alpha_2^4}}. \tag{2.5}$$

Therefore, in order to guarantee the smoothness of γ and the nowhere vanishing of its curvatures, one needs to assume that no one of the four constants a_1, a_2, α_1, α_2 is zero and, moreover, $\alpha_1^2 \neq \alpha_2^2$. If some of these constants are negative, then one can apply a symmetry transformation in $\mathbb{R}^4$ to make them positive. Thus, from now on we will assume that a_1, a_2, α_1, α_2 are positive and $\alpha_1^2 \neq \alpha_2^2$.

Clearly, the curve γ represented by (2.1) lies on the Clifford torus $T^2 \subset \mathbb{R}^4$ given implicitly by $x_1^2 + x_2^2 = a_1^2$, $x_3^2 + x_4^2 = a_2^2$. The curve is closed if and only if $\frac{\alpha_1}{\alpha_2} \in \mathbb{Q}$, i.e., $\frac{\alpha_1}{\alpha_2} = \frac{m_1}{m_2}$, where m_1 and m_2 are coprime integers. The minimal period T for the parameter t is expressed by the obvious formulae

$$T = \frac{2\pi m_1}{\alpha_1} = \frac{2\pi m_2}{\alpha_2}. \tag{2.6}$$

Our aim is to analyze the value of $\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2} ds$. Applying (2.2)–(2.5) and taking into account (2.6), one gets the following.

Proposition 2.2. *We have*

$$\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2} ds = 2\pi\sqrt{m_1^2 + m_2^2}. \tag{2.7}$$

We would like to emphasize that the arc-length and the curvatures of γ depend on all the constants a_1, a_2, α_1, α_2. However, the value of

$$\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds$$

depends only on the coprime integers m_1 and m_2. Consequently, the range of possible values of the integral in question is countable.

The minimal possible value is equal to $2\sqrt{5}\pi$, and it is achieved if either $m_1 = 1$, $m_2 = 2$ or $m_1 = 2$, $m_2 = 1$. The cases of (m_1, m_2) being equal to $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, which give to $\sqrt{m_1^2 + m_2^2}$ values less than $\sqrt{5}$, are prohibited because

α_1 and α_2 are assumed to be positive and different. This completes the proof of Theorem 1.1.

Notice that γ represents the class (m_1, m_2) in the fundamental group of the torus T^2. Therefore, in the general case of curves with nonconstant curvatures, one may conjecture that the values of

$$\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds$$

have to obey some topological (homotopical) restrictions.

3. Osculating indicatrices of closed curves of constant curvatures in $\mathbb{R}^4$

Now let us construct the osculating indicatrix of the curve γ, c.f., [6]. By definition, the osculating plane of γ at an arbitrary point $p \in \gamma$ is spanned and oriented by the first two vectors of the Frenet fame of γ at p. Equivalently, the same plane is spanned and oriented by the vectors $\frac{dx}{dt}(t)$, $\frac{d^2x}{dt^2}(t)$, where $x = x(t)$ is the position-vector of $p \in \gamma$. Being translated to the origin $O \in \mathbb{R}^4$, this osculating plane represents a point in the Grassmann manifolds $G(2,4)$. (For definitions and geometric properties of $G(2,4)$, see [2, Chap. 8], [3].) By moving p along γ, one obtains the one-dimensional family of osculating planes of γ, which generates a curve $\tilde{\gamma}$ in $G(2,4)$. This curve is called the *osculating indicatrix* of γ.

The Grassmann manifold $G(2,4)$ can be isometrically embedded into $\mathbb{R}^6$ via the Plücker coordinates, see [2, Chap. 8.2], [3]. Consequently, $\tilde{\gamma} \subset G(2,4)$ can be viewed as a curve in $\mathbb{R}^6$. If $x = x(t)$ is the position-vector of γ, then $\tilde{\gamma}$ is represented in $\mathbb{R}^6$ by the position-vector

$$\tilde{x} = \frac{\left[\frac{dx}{dt}, \frac{d^2x}{dt^2}\right]}{\left|\left[\frac{dx}{dt}, \frac{d^2x}{dt^2}\right]\right|}, \tag{3.1}$$

where the brackets $[\cdot,\cdot]$ denote the exterior product of vectors. More precisely, one has

$$\left[\frac{dx}{dt}, \frac{d^2x}{dt^2}\right] = (\tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{14}, \tilde{x}_{23}, \tilde{x}_{24}, \tilde{x}_{34}), \tag{3.2}$$

where

$$\tilde{x}_{ij} = \frac{dx_i}{dt}\frac{d^2x_j}{dt^2} - \frac{dx_j}{dt}\frac{d^2x_i}{dt^2}, \quad 1 \le i < j \le 4. \tag{3.3}$$

Recall that the position-vector $x(t)$ of γ is given by (2.1). By substituting (2.1) into (3.1)–(3.3), we can easily derive the position-vector $\tilde{x}(t)$ of the osculating indicatrix $\tilde{\gamma}$ in $\mathbb{R}^6$:

$$\tilde{x} = \left(\frac{a_1^2\alpha_1^3}{\lambda}, \frac{-a_1a_2\alpha_1\alpha_2}{\lambda}\left(\alpha_1\cos\alpha_1 t \sin\alpha_2 t - \alpha_2 \sin\alpha_1 t\cos\alpha_2 t\right), \right.$$

$$\frac{-a_1a_2\alpha_1\alpha_2}{\lambda}\left(\alpha_1\cos\alpha_1 t\cos\alpha_2 t+\alpha_2\sin\alpha_1 t\sin\alpha_2 t\right),$$
$$\frac{a_1a_2\alpha_1\alpha_2}{\lambda}\left(\alpha_1\sin\alpha_1 t\sin\alpha_2 t+\alpha_2\cos\alpha_1 t\cos\alpha_2 t\right),$$
$$\left.\frac{a_1a_2\alpha_1\alpha_2}{\lambda}\left(\alpha_1\sin\alpha_1 t\cos\alpha_2 t-\alpha_2\cos\alpha_1 t\sin\alpha_2 t\right),\frac{-a_2^2\alpha_2^3}{\lambda}\right), \tag{3.4}$$

where $\lambda=\sqrt{a_1^2\alpha_1^2+a_2^2\alpha_2^2}\sqrt{a_1^2\alpha_1^4+a_2^2\alpha_2^4}$.

Clearly, the curve $\tilde{\gamma}$ lies in the four-dimensional affine subspace $\mathbb{R}^4\subset\mathbb{R}^6$ given by the equations $\tilde{x}^{12}=\frac{a_1^2\alpha_1^3}{\lambda}$, $\tilde{x}^{34}=\frac{-a_2^2\alpha_2^3}{\lambda}$.

By using standard formulae of the classical theory of curves, see [1, Chap. 32], the following expressions for the arc length $\tilde{s}$ and the curvatures $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ of $\tilde{\gamma}$ can be found from (3.4).

Proposition 3.1. 1. *The arc length $\tilde{s}$ of $\tilde{\gamma}$ is expressed as follows:*

$$\frac{d\tilde{s}}{dt}=\frac{1}{\lambda}\frac{a_1a_2\alpha_1\alpha_2}{|\alpha_1^2-\alpha_2^2|}. \tag{3.5}$$

2. *The curvatures $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ of $\tilde{\gamma}$ are expressed as follows:*

$$\tilde{k}_1=\lambda\frac{\sqrt{\alpha_1^2+\alpha_2^2}}{a_1a_2\alpha_1\alpha_2|\alpha_1^2-\alpha_2^2|}, \tag{3.6}$$
$$\tilde{k}_2=\lambda\frac{2}{a_1a_2|\alpha_1^2-\alpha_2^2|\sqrt{\alpha_1^2+\alpha_2^2}}, \tag{3.7}$$
$$\tilde{k}_3=\lambda\frac{1}{a_1a_2\alpha_1\alpha_2\sqrt{\alpha_1^2+\alpha_2^2}}. \tag{3.8}$$

Consequently, the curvatures $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ of $\tilde{\gamma}$ are constant. Moreover, by applying (3.5)–(3.8), one can easily verify that 4), 5) of Theorem 1.2 hold true, and this completes the proof of this theorem.

Notice that if $\alpha_1=\alpha_2$, then γ represented by (2.1) is a circle, and thus its first curvature is $k_1=\frac{1}{\sqrt{a_1^2+a_2^2}}$, the second curvature is zero, $k_2=0$, and the third curvature k_3 is undefined. In this case, the osculating plane of γ at every point is the two-dimensional plane containing γ. Hence the osculating indicatrix $\tilde{\gamma}$ degenerates to a point in $G(2,4)$.

4. Concluding remarks and questions

Remark 4.1. Theorem 1.1 can be extended to the case of closed curves of constant curvatures in $\mathbb{R}^{2n}$, $n>2$. Moreover, for closed curves with non-zero constant curvatures in $\mathbb{R}^{2n}$ one can consider the integrals

$$\int_\gamma\sqrt{k_{2m-1}^2+k_{2m}^2+k_{2m+1}^2}\,ds,\quad 1\le m\le n-1.$$

It turns out that these integrals satisfy the same sharp inequality (1.5), although the proof is technically more complicated.

Remark 4.2. Let γ be an arbitrary smooth curve with nowhere vanishing curvatures k_1, k_2 in $\mathbb{R}^4$. Then its osculating indicatrix $\tilde{\gamma}$ is a smooth curve in the Grassmann manifold $G(2,4)$. By considering the Plücker coordinates, $G(2,4)$ is embedded into the unit sphere $S^5 \subset \mathbb{R}^6$, see [2, Chap. 8.2], [3]. It turns out that the osculating indicatrix $\tilde{\gamma}$ is an *asymptotic* curve of $G(2,4) \subset S^5$ with nowhere vanishing geodesic curvature. The integral

$$\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds$$

is equal to the total curvature

$$\int_{\tilde{\gamma}} \tilde{k}_1\, d\tilde{s}$$

of $\tilde{\gamma}$, when $\tilde{\gamma}$ is viewed as a curve in $\mathbb{R}^6$. Hence, the sharpness problem for

$$\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds$$

gives rise to the problem on finding the sharp lower bound for

$$\int_{\tilde{\gamma}} \tilde{k}_1 d\tilde{s},$$

where $\tilde{\gamma}$ is a closed asymptotic curve with nowhere vanishing geodesic curvature in $G(2,4) \subset S^5$.

Remark 4.3. From a local point of view, a smooth curve $\tilde{\gamma} \subset G(2,4)$ is the osculating indicatrix of a smooth curve $\gamma \subset \mathbb{R}^4$ if and only if $\tilde{\gamma}$ is an asymptotic curve with nowhere vanishing geodesic curvature in $G(2,4) \subset S^5$, c.f. [7]. We are interested in a global version of this statement. Exactly, what the necessary and sufficient conditions should be imposed for a *closed* smooth curve in $G(2,4)$ to be the osculating indicatrix of a smooth *closed* curve in $\mathbb{R}^4$.

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Про точність однієї інтегральної нерівності для замкнутих кривих в $\mathbb{R}^4$

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Для замкнутих кривих з ненульовими кривинами в $\mathbb{R}^4$ досліджується оптимальність інтегральної нерівності $\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds > 2\pi$. Доведено, що довільна замкнута крива зі сталими додатними кривинами в $\mathbb{R}^4$ задовольняє нерівність $\int_\gamma \sqrt{k_1^2 + k_2^2 + k_3^2}\, ds \geq 2\sqrt{5}\pi$.

Ключові слова: замкнута крива, кривина, криві зі сталими кривинами.