Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 4, pp. 510–525
doi: https://doi.org/10.15407/mag15.04.510

Three Anholonomy Densities According to Bishop Frame in Euclidean 3-Space

Nevin Gürbüz

In this paper, we obtain three anholonomy densities using three transformations of Bishop frame in Euclidean 3-space.

Key words: geometric phase, Bishop phase.

Mathematical Subject Classification 2010: 53Z05, 81Q70.

1. Introduction

Space curves have many physical applications such as vortex filament motion, twisted optical fiber, etc. Over the last four decades pioneering results have spawned a huge research area exploring the connection between the motion of space curves, surfaces and some important nonlinear equations of mathematical physics.

Hasimoto presented the motion of a vortex filament and its relation to elastica [13, 14]. Lakshmanan showed the connection of the nonlinear Schrödinger equation with the continuum Heisenberg ferromagnetic spin chain system [15]. Lakshmanan, Myrzakulov, Vijayalakshmi and Danlybaeva investigated the motion of curves, surfaces and nonlinear evolution equations in (2+1) dimensions [16]. Langer and Perline studied the Hasimoto transformation and integrable flows on curves [17]. Murugesh and Balakrishnan presented new connections between moving curves and soliton equations in terms of Frenet frame in Euclidean space [19]. Munijara and Lakshmanan investigated the motion of space curves in a three-dimensional Minkowski space [20]. Gürbüz studied three classes of curve evolution in terms of Bishop frame in Minkowski 3-space [10]. Guha studied the connection of moving space curves with KdV-type equations in Euclidean 3-space [9].

A time evolution of a space curve is associated with the geometric phase. Berry was first to study the quantum geometric phase in the adiabatic approximation [6]. Later this topic was generalized by Aharonov and Anandan [1, 2]. Tomita and Chiao investigated the angle of rotation of linearly polarized light in this fiber and gave a direct measure of Berry's phase [21]. Mostafazadeh presented relativistic adiabatic approximation and geometric phase [18].

Balakrishnan, Bishop and Dandoloff used Lamb′s formalism to derive the anholonomy density and the geometric phase in terms of Frenet frame in Euclidean 3-space [4]. Gürbüz studied three formulations of curve evolution and three geometric phases according to the Frenet frame in Minkowski space [11]. Balakrishnan discussed the first class of the curve evolution associated with the geometric phase according to the Darboux frame in Euclidean space [3]. Gürbüz introduced three classes of curve evolution associated with three geometric phases according to the Darboux frame in Minkowski 3-space [12].

This paper considers the temporal motion of the so-called Bishop frame instead of the natural Frenet frame of a space curve. Basically, three transformations of the Bishop frame are given and these three transformations yield to the nonlinear Schrödinger, the coupled KdV and the Belavin–Polyakov equations [5] through the usual computations involving curve motions, Darboux vector formulations, Heisenberg spin chain equations etc. Later the corresponding anholonomy densities are also computed.

In this section some preliminaries will be given.

The Frenet frame $\{T, N, B\}$ formulas in Euclidean 3-space are given by

$$\frac{\partial T}{\partial s} = \kappa N, \quad \frac{\partial N}{\partial s} = -\kappa T + \tau B, \quad \frac{\partial B}{\partial s} = -\tau N, \tag{1.1}$$

where κ, τ are the curvature and the torsion of the curve in $\mathbb{E}^3$ [8]. The Bishop frame is defined as a moving frame that well-defined even when the second derivative of the curve has vanished.

The Bishop frame $\{T, E_1, E_2\}$ formulas are given in $\mathbb{E}^3$ as follows [7]:

$$\frac{\partial T}{\partial s} = \xi_1 E_1 + \xi_2 E_2, \quad \frac{\partial E_1}{\partial s} = -\xi_1 T, \quad \frac{\partial E_2}{\partial s} = -\xi_2 T, \tag{1.2}$$

where ξ_1, ξ_2 are the first and the second Bishop curvatures, s is the arc length of the curve. The connection between the Frenet frame and the Bishop frame is given by

$$T = T, \quad N = E_1 \cos\alpha + E_2 \cos\alpha, \quad B = E_1 \sin\alpha + E_2 \cos\alpha.$$

2. Anholonomy according to Bishop frame in $\mathbb{E}^3$

Case I. The first new frame $\{U_1, U_2, U_2^*\}$ for the curve evolution associated with the nonlinear Schrödinger equation, the coupled KdV equation and the Belavin–Polyakov equation in terms of Bishop frame is given by

$$U_1 = E_1, \tag{2.1}$$

$$U_2 = \frac{(T + iE_2)}{\sqrt{2}} e^{i\int \xi_2}, \tag{2.2}$$

$$U_2^* = \frac{(T - iE_2)}{\sqrt{2}} e^{-i\int \xi_2}. \tag{2.3}$$

The first transformation for the first case of curve evolution associated with the nonlinear Schrödinger equation, the coupled KdV equation and the Belavin–Polyakov equation in terms of Bishop frame in $\mathbb{E}^3$ is given by

$$\theta = \frac{\xi_1}{\sqrt{2}} e^{i\int \xi_2}. \tag{2.4}$$

We get

$$\frac{\partial U_1}{\partial t} = \zeta_1 U_2 + \zeta_2 U_2^* + \zeta_3 U_1, \tag{2.5}$$

$$\frac{\partial U_2}{\partial t} = \eta_1 U_2 + \eta_2 U_2^* + \eta_3 U_1, \tag{2.6}$$

$$\frac{\partial U_2^*}{\partial t} = \eta_1^* U_2 + \eta_2^* U_2^* + \eta_3^* U_1. \tag{2.7}$$

Also,

$$\left\langle \frac{\partial U_1}{\partial t}, U_2 \right\rangle = \zeta_2, \quad \left\langle \frac{\partial U_2^*}{\partial t}, U_1 \right\rangle = \eta_3^*, \quad \left\langle \frac{\partial U_1}{\partial t}, U_1 \right\rangle = \zeta_3 = 0. \tag{2.8}$$

From (2.8), it follows

$$\eta_3 = -\zeta_2 \quad \eta_3^* = -\zeta_1. \tag{2.9}$$

With the aid of (2.9), we have

$$\frac{\partial U_1}{\partial t} = -\eta_3^* U_2 - \eta_3 U_2^*, \tag{2.10}$$

$$\frac{\partial U_2}{\partial t} = i\chi_1 U_2 + \eta_3 U_1. \tag{2.11}$$

Taking the derivatives of (2.1), (2.2), (2.3) with respect to s, we obtain

$$\frac{\partial U_1}{\partial s} = -\theta^* U_2 - \theta U_2^*, \tag{2.12}$$

$$\frac{\partial U_2}{\partial s} = \theta U_1, \tag{2.13}$$

$$\frac{\partial U_2^*}{\partial s} = \theta^* U_1, \tag{2.14}$$

where

$$\theta^* = \frac{\xi_1}{\sqrt{2}} e^{-i\int \xi_2}. \tag{2.15}$$

The compatibility conditions

$$\frac{\partial^2 U_2}{\partial t \partial s} = \frac{\partial^2 U_2}{\partial s \partial t}$$

give

$$\frac{\partial \chi_1}{\partial s} = -i\eta_3 \theta^* + i\eta_3^* \theta, \tag{2.16}$$

$$\frac{\partial\theta}{\partial t} - i\chi_1\theta - \frac{\partial\eta_3}{\partial s} = 0. \tag{2.17}$$

The Darboux vector for the first case in terms of Bishop frame in $\mathbb{E}^3$ is presented as

$$\mathcal{Z}_1 = AT - \xi_2^* E_1 + CE_2. \tag{2.18}$$

By using (2.18), the derivatives of $\{T, E_1, E_2\}$ with respect to t can be written in the form

$$\frac{\partial T}{\partial t} = \mathcal{Z}_1 \times T = -uE_1 + \xi_2^* E_2, \tag{2.19}$$

$$\frac{\partial E_1}{\partial t} = \mathcal{Z}_1 \times E_1 = uT + wE_2, \tag{2.20}$$

$$\frac{\partial E_2}{\partial t} = \mathcal{Z}_1 \times E_2 = -\xi_2^* T - wE_1, \tag{2.21}$$

where $u = -C$, $w = A$. Moreover,

$$\eta_3 = -\frac{(u + iw)}{\sqrt{2}} e^{i\int \xi_2} \tag{2.22}$$

satisfies (2.10) and (2.20). Taking the derivatives of (2.2) with respect to t and using (2.11) and (2.22), the following equalities are derived for the first case in terms of Bishop frame in Euclidean 3-space

$$\frac{\partial T}{\partial t} = -uE_1 + \left(\int_{-\infty}^{s} \frac{\partial \xi_2}{\partial t}\, ds - \chi_1\right) E_2, \tag{2.23}$$

$$\frac{\partial E_2}{\partial t} = \left(\chi_1 - \int_{-\infty}^{s} \frac{\partial \xi_2}{\partial t}\, ds\right) T - wE_1. \tag{2.24}$$

(2.19) and (2.23) give

$$\xi_2^* = -\chi_1 + \int_{-\infty}^{s} \frac{\partial \xi_2}{\partial t}\, ds. \tag{2.25}$$

From (2.25),

$$-\frac{\partial \chi_1}{\partial s} = \frac{\partial \xi_2^*}{\partial s} - \frac{\partial \xi_2}{\partial t}$$

is derived. Using (2.4), (2.15), (2.16) and (2.22),

$$\frac{\partial \chi_1}{\partial s} = -\xi_1 w \tag{2.26}$$

is obtained.

The natural Bishop frame vectors T and E_2 rotate around E_1 with the angular velocity $\xi_2(s)$. When moving from s_0 to s_1 along a spatial curve, a geometric phase

$$\Omega_1 = \int_{s_0}^{s_1} \xi_2(s)\, ds$$

arises between the natural Bishop frame T, E_1 and the corresponding nonrotating Bishop frame for the first case in $\mathbb{E}^3$.

A geometric phase for the first case corresponding to the Bishop frame

$$\Omega_2 = \int_{t_1}^{t_2} \xi_2^*(t)\,dt$$

appears between the Bishop frame and the nonrotating Bishop frame along a temporal curve for the first case in $\mathbb{E}^3$. As a β curve moves from (s,t) to $(s+\Delta s, t+\Delta t)$ in terms of Bishop frame in $\mathbb{E}^3$, the rotation angle Ω is given as follows:

$$\Omega_1 = \xi_2(s,t)\Delta s + \xi_2^*(s+\Delta s,t)\Delta t, \quad \Omega_2 = \xi_2^*(s,t)\Delta t + \xi_2(s,t+\Delta t)\Delta s.$$

The geometric phase difference for the first case in terms of Bishop frame is given by

$$\delta\Omega = \mathcal{AD}_1(s,t)\Delta s\Delta t = \Omega_1 - \Omega_2 = (\frac{\partial \xi_2^*}{\partial s} - \frac{\partial \xi_2}{\partial t})\Delta s\Delta t.$$

Here, $\mathcal{AD}_1(s,t) = (\frac{\partial \xi_2^*}{\partial s} - \frac{\partial \xi_2}{\partial t})$ is a measure anholonomy density for the first case in terms of Bishop frame in Euclidean 3-space. $\mathcal{AD}_1(s,t)$ is given by

$$\mathcal{AD}_1(s,t) = -\frac{\partial \chi_1}{\partial s} = \xi_1 w. \tag{2.27}$$

The total anholonomy Ω for the first case in terms of Bishop frame is

$$\begin{aligned} \Omega &= \int_{t_1}^{t_2}\int_{-s_0}^{s} \mathcal{AD}_1(s,t) = \int_{t_1}^{t_2}\int_{-s_0}^{s} \xi_1 w\,ds\,dt \\ &= \int_{t_1}^{t_2}\int_{-s_0}^{s} \left\langle E_1, \frac{\partial E_1}{\partial s} \times \frac{\partial E_1}{\partial t} \right\rangle ds\,dt. \end{aligned} \tag{2.28}$$

The compatibility conditions

$$\frac{\partial^2 U_1}{\partial t \partial s} = \frac{\partial^2 U_1}{\partial s \partial t}$$

give

$$\begin{aligned} \frac{\partial \xi_1}{\partial t} &= w\xi_2 - \frac{\partial u}{\partial s}, \\ \frac{\partial w}{\partial s} + u\xi_2 &= -\xi_1 \left(\int_{-\infty}^{s} \frac{\partial \xi_2}{\partial t}\,ds - \chi_1 \right). \end{aligned}$$

The geometric phase for the first case in terms of Bishop frame is expressed as

$$\Omega = -i \int_{-s_0}^{s} \frac{\partial}{\partial s} \int_{t_1}^{t_2} \left\langle \frac{\partial U_2}{\partial t}, \frac{\partial U_2^*}{\partial t} \right\rangle ds\,dt.$$

Example 2.1. The equation

$$\frac{\partial E_1}{\partial t} = E_1 \times E_1 \times \frac{\partial^2 E_1}{\partial s^2} - \left\langle \frac{\partial E_1}{\partial s}, \frac{\partial E_1}{\partial s} \right\rangle \frac{\partial E_1}{\partial s} \tag{2.29}$$

satisfies the coupled KDV-type equation for the first case in Euclidean 3-space. The total anholonomy associated with the coupled-KdV equation for the first case is given by

$$\Omega = \int_{t_1}^{t_2}\int_{-s_0}^{s} \frac{\partial(\xi_1^2 \xi_2)}{\partial s}\,ds\,dt.$$

Proof. Using (2.29), we obtain

$$\frac{\partial E_1}{\partial t} = T\left(\frac{\partial^2 \xi_1}{\partial s^2} - \xi_1\xi_2^2\right) + E_2\left(\frac{\partial(\xi_1\xi_2)}{\partial s} + \frac{\partial \xi_1}{\partial s}\xi_2\right). \tag{2.30}$$

From (2.30), it follows

$$u = \left(\frac{\partial^2 \xi_1}{\partial s^2} - \xi_1\xi_2^2\right), \quad w = \left(\frac{\partial(\xi_1\xi_2)}{\partial s} + \frac{\partial \xi_1}{\partial s}\xi_2\right). \tag{2.31}$$

From (2.16), (2.22) and (2.31), we obtain

$$\chi_{1s} = i\left(\frac{\partial^2\theta}{\partial s^2}\theta^* - \frac{\partial^2\theta^*}{\partial s^2}\theta\right), \tag{2.32}$$

$$\eta_3 = -\frac{\partial^2\theta}{\partial s^2}. \tag{2.33}$$

Using (2.17), (2.32) and (2.33), the coupled KdV equation

$$\frac{\partial\theta}{\partial t} + \frac{\partial^3\theta}{\partial s^3} + 2\left|\theta\right|^2\frac{\partial\theta}{\partial s} - \frac{\partial\left(\left|\theta\right|^2\right)}{\partial s}\theta = 0$$

is obtained. The total phase Ω associated with the coupled KdV equation for the first case in terms of Bishop frame using (2.28) is

$$\Omega = \int_{t_1}^{t_2}\int_{-s_0}^{s}\xi_1 w = \int_{t_1}^{t_2}\int_{-s_0}^{s}\frac{\partial(\xi_1^2\xi_2)}{\partial s}\,ds\,dt. \qquad \square$$

Example 2.2. The Heisenberg spin chain equation

$$\frac{\partial E_1}{\partial t} = E_1 \times \frac{\partial^2 E_1}{\partial s^2} \tag{2.34}$$

satisfies the nonlinear Schrödinger equation (NLS) for the first case in Euclidean 3-space. The anholonomy density associated with the NLS is

$$\Omega = \frac{1}{2}\int_{t_1}^{t_2}\int_{-s_0}^{s}\frac{\partial(\xi_1^2)}{\partial s}\,ds\,dt.$$

Proof. With the aid of (2.34),

$$\frac{\partial E_1}{\partial t} = -\xi_1\xi_2 T + \frac{\partial\xi_1}{\partial s}E_2 \tag{2.35}$$

is obtained. Here,

$$u = -\xi_1\xi_2, \quad w = \frac{\partial\xi_1}{\partial s}. \tag{2.36}$$

From (2.16), (2.22) and (2.36), we obtain

$$\frac{\partial\chi_1}{\partial s} = -\frac{1}{2}\frac{\partial}{\partial s}\left(\left|\theta\right|^2\right), \tag{2.37}$$

$$\eta_3 = -\frac{i\partial\theta}{\partial s}. \tag{2.38}$$

Using (2.17), (2.37) and (2.38), the nonlinear Schrödinger equation

$$\frac{\partial\theta}{\partial t} + i\frac{\partial^2\theta}{\partial s^2} + \frac{i}{2}|\theta|^2\theta = 0$$

is obtained. From (2.27), (2.36), we obtain the anholonomy density associated with the NLS for the first case:

$$\mathcal{AD}_1(s,t) = \xi_1\frac{\partial\xi_1}{\partial s}. \tag{2.39}$$

From (2.28), the total phase Ω associated with the NLS for the first case in terms of Bishop frame is

$$\Omega = \int_{t_1}^{t_2}\int_{-s_0}^{s}\mathcal{AD}_1(s,t)dsdt = \frac{1}{2}\int_{t_1}^{t_2}\int_{-s_0}^{s}\frac{\partial(\xi_1^2)}{\partial s}\,ds\,dt. \qquad \square \tag{2.40}$$

Case II. The second frame $\{V_1, V_2, V_2^*\}$ for the second case of the curve evolution associated with the coupled KdV equation, the nonlinear Schrödinger equation and the Belavin–Polyakov equation in Euclidean 3-space in terms of Bishop frame is given by

$$V_1 = E_2, \tag{2.41}$$

$$V_2 = \frac{(T+iE_1)}{\sqrt{2}}e^{i\int\xi_1}, \tag{2.42}$$

$$V_2^* = \frac{(T-iE_1)}{\sqrt{2}}e^{-i\int\xi_1}. \tag{2.43}$$

The second transformation of the Bishop frame associated with the coupled KdV equation, the nonlinear Schrödinger equation and the Belavin–Polyakov equation in $\mathbb{E}^3$ is expressed by

$$\psi = \frac{\xi_2}{\sqrt{2}}e^{i\int\xi_1}. \tag{2.44}$$

The derivatives of $\{V_1, V_2, V_2^*\}$ with respect to t are given by

$$\frac{\partial V_1}{\partial t} = kV_2 + lV_2^* + mV_1, \tag{2.45}$$

$$\frac{\partial V_2}{\partial t} = fV_2 + gV_2^* + hV_1, \tag{2.46}$$

$$\frac{\partial V_2^*}{\partial t} = f^*V_2 + g^*V_2^* + h^*V_1. \tag{2.47}$$

If the derivatives of (2.41), (2.42) and (2.43) are taken with respect to s, we obtain

$$\frac{\partial V_1}{\partial s} = -\psi^*U_2 - \psi U_2^*, \tag{2.48}$$

$$\frac{\partial V_2}{\partial s} = \psi U_1, \tag{2.49}$$

$$\frac{\partial V_2^*}{\partial s} = \psi^* U_1, \tag{2.50}$$

where $\psi^* = \xi_2^{-i\int \xi_1}$. From (2.45), (2.46), (2.47), we have

$$l = \langle V_{1t}, V_2 \rangle \Rightarrow l = -h, \quad \langle V_{2t}^*, V_1 \rangle = h^* \Rightarrow k = -h^*, \quad \langle V_{1t}, V_1 \rangle = m = 0. \tag{2.51}$$

Using (2.45), (2.46), (2.47) and (2.51), we derive

$$\frac{\partial V_1}{\partial t} = -h^* V_2 - h V_2^*, \tag{2.52}$$

$$\frac{\partial V_2}{\partial t} = i\chi_2 V_2 + h V_1, \tag{2.53}$$

where χ_2 is a real function. From the equality

$$\frac{\partial^2 V_2}{\partial t \partial s} = \frac{\partial^2 V_2}{\partial s \partial t}$$

we obtain

$$\frac{\partial \chi_2}{\partial s} = ih^* \psi - ih\psi^*, \tag{2.54}$$

$$\frac{\partial \psi}{\partial t} - \frac{\partial h}{\partial s} - i\psi\chi_2 = 0. \tag{2.55}$$

The second Darboux vector for the Bishop frame for the second case in Euclidean 3-space has the form

$$\mathcal{Z}_2 = AT + BE_1 + \xi_1^* E_2. \tag{2.56}$$

By using (2.56), the temporal evolution equations can be written as

$$\frac{\partial T}{\partial t} = \mathcal{Z}_2 \times T = -uE_2 + \xi_1^* E_1, \tag{2.57}$$

$$\frac{\partial E_1}{\partial t} = \mathcal{Z}_2 \times E_1 = -\xi_1^* T - vE_2, \tag{2.58}$$

$$\frac{\partial E_2}{\partial t} = \mathcal{Z}_2 \times E_2 = uT + vE_1, \tag{2.59}$$

where $u = B$, $v = -A$. The quantity

$$h = -\frac{(u + iv)}{\sqrt{2}} e^{i\int \xi_1} \tag{2.60}$$

satisfies (2.52) and (2.59). Taking the derivative of (2.42) with respect to t, the following expressions are obtained:

$$\frac{\partial T}{\partial t} = -uE_2 + \left(\int_{-\infty}^{s} \frac{\partial \xi_1}{\partial t}\, ds - \chi_2 \right) E_1, \tag{2.61}$$

$$\frac{\partial E_1}{\partial t} = \left(-\int_{-\infty}^{s} \frac{\partial \xi_1}{\partial t} ds + \chi_2 \right) T - vE_2. \tag{2.62}$$

From (2.57) and (2.61), we obtain

$$-\xi_1^* = \chi_2 - \int_{-\infty}^{s} \frac{\partial \xi_1}{\partial t}\, ds,$$
$$\frac{\partial \chi_2}{\partial s} = -\frac{\partial \xi_1^*}{\partial s} + \frac{\partial \xi_1}{\partial t}.$$

Using (2.44), (2.54) and (2.60), we have the equation

$$\frac{\partial \chi_2}{\partial s} = -\xi_2 v. \tag{2.63}$$

The compatibility conditions

$$\frac{\partial^2 V_1}{\partial t \partial s} = \frac{\partial^2 V_1}{\partial s \partial t}$$

give

$$-\frac{\partial \xi_2}{\partial t} = -v\xi_1 + \frac{\partial u}{\partial s},$$
$$\frac{\partial v}{\partial s} + u\xi_1 = -\xi_2 \left(\int_{-\infty}^{s} \frac{\partial \xi_1}{\partial t}\, ds - \chi_2 \right).$$

The natural Bishop frame vectors T and E_1 rotate around E_2 with the angular velocity $\xi_1(s)$. When moving along a spatial curve from s_0 to s_1, a geometric phase

$$\Lambda_1 = \int_{s_0}^{s_1} \xi_1(s)\, ds$$

arises between the natural Bishop frame vectors T, E_1 and the corresponding second nonrotating Bishop frame in Euclidean 3-space. The geometric phase for the second case in terms of Bishop frame

$$\Lambda_2 = \int_{t_1}^{t_2} \xi_1^*(t)\, dt$$

appears between the Bishop frame and the nonrotating Bishop frame along a temporal curve for the first case in $\mathbb{E}^3$. As the β curve moves from (s,t) to $(s+\Delta s, t+\Delta t)$ according to Bishop frame in $\mathbb{E}^3$, the rotation angle Λ is given as follows:

$$\Lambda_1 = \xi_1(s,t)\Delta s + \xi_1^*(s+\Delta s, t)\Delta t, \qquad \Lambda_2 = \xi_1^*(s,t)\,\Delta t + \xi_1(s, t+\Delta t)\,\Delta s.$$

The geometric phase difference for the second case in terms of Bishop frame is given by

$$\delta\Lambda = \mathcal{A}\mathcal{D}_2(s,t)\Delta s \Delta t = \Lambda_1 - \Lambda_2 = (\xi_{1s}^* - \xi_{1t})\Delta s \Delta t.$$

The second anholonomy density for the second case in terms of Bishop frame is

$$\mathcal{AD}_2(s,t) = \frac{\partial \xi_1^*}{\partial s} - \frac{\partial \xi_1}{\partial t}.$$

The total phase Λ for the second case in terms of Bishop frame is

$$\begin{aligned}\Lambda &= \int_{t_1}^{t_2}\int_{-s_0}^{s} \mathcal{AD}_2(s,t) = -\int_{t_1}^{t_2}\int_{-s_0}^{s} \frac{\partial \chi_2}{\partial s}\, ds\, dt \\ &= \int_{t_1}^{t_2}\int_{-s_0}^{s} \xi_2 v\, ds\, dt = -\int_{t_1}^{t_2}\int_{-s_0}^{s} \left\langle E_2, \frac{\partial E_2}{\partial s} \times \frac{\partial E_2}{\partial t}\right\rangle ds\, dt. \qquad (2.64)\end{aligned}$$

Example 2.3. The Heisenberg spin chain equation

$$\frac{\partial E_2}{\partial t} = E_2 \times \frac{\partial^2 E_2}{\partial s^2} \qquad (2.65)$$

satisfies the nonlinear Schrödinger equation for the second case in $\mathbb{E}^3$. Using (2.64), the total phase Λ is given by

$$\Lambda = \int_{t_1}^{t_2}\int_{-s_0}^{s} \mathcal{AD}_2(s,t) ds dt = -\frac{1}{2}\int_{t_1}^{t_2}\int_{-s_0}^{s} \frac{\partial(\xi_2^2)}{\partial s}\, ds\, dt. \qquad (2.66)$$

Proof. From (2.65), we obtain

$$\frac{\partial E_2}{\partial t} = -\frac{\partial \xi_2}{\partial s} E_1 + \xi_1 \xi_2 T. \qquad (2.67)$$

Here,

$$u = \xi_1 \xi_2, \qquad v = -\frac{\partial \xi_2}{\partial s}. \qquad (2.68)$$

From (2.60), (2.63), and (2.68), we get

$$h = i\frac{\partial \psi}{\partial s}, \qquad (2.69)$$

$$\frac{\partial \chi_2}{\partial s} = \frac{1}{2}\frac{\partial(|\psi|^2)}{\partial s}. \qquad (2.70)$$

From (2.55), the nonlinear Schrödinger equation

$$\frac{\partial \psi}{\partial t} - i\frac{\partial^2 \psi}{\partial s^2} - \frac{i\,|\psi|^2\,\psi}{2} = 0$$

is obtained. The anholonomy density is found as

$$\mathcal{AD}_2(s,t) = -\xi_2 \frac{\partial \xi_2}{\partial s}. \qquad (2.71)$$

The geometric phase associated with the NLS for the second case has the form

$$\Lambda = -\frac{1}{2}\int_{t_1}^{t_2}\int_{-s_0}^{s} \frac{\partial(\xi_2^2)}{\partial s}\, ds\, dt. \qquad \square$$

Example 2.4. The equation

$$\frac{\partial E_2}{\partial t} = E_2 \times E_2 \times \frac{\partial^2 E_2}{\partial s^2} - \left\langle \frac{\partial E_2}{\partial t}, \frac{\partial E_2}{\partial t} \right\rangle \frac{\partial E_2}{\partial t} \tag{2.72}$$

satisfies the coupled KDV-type equation for the second case in terms of Bishop frame in Euclidean 3-space. The geometric phase is

$$\Lambda = \int_{t_1}^{t_2} \int_{-s_0}^{s} \frac{\partial(\xi_2^2 \xi_1)}{\partial s}\, ds\, dt.$$

Proof. With the aid of (2.72),

$$\frac{\partial \Lambda_1}{\partial t} = \left(\frac{\partial^2 \xi_2}{\partial s^2} - \xi_1^2 \xi_2 \right) T + E_1 \left(2 \frac{\partial \xi_2}{\partial s} \xi_1 + \frac{\partial \xi_1}{\partial s} \xi_2 \right) \tag{2.73}$$

is obtained. From (2.73), we get

$$u = (\frac{\partial^2 \xi_2}{\partial s^2} - \xi_1^2 \xi_2), \qquad v = (2 \frac{\partial \xi_2}{\partial s} \xi_1 + \frac{\partial \xi_1}{\partial s} \xi_2). \tag{2.74}$$

Using (2.54) , (2.60) and (2.74), we have

$$h = -\frac{\partial^2 \psi}{\partial s^2}, \tag{2.75}$$

$$\frac{\partial \chi_2}{\partial s^2} = i \left(-\frac{\partial^2 \psi^*}{\partial s^2} \psi + \frac{\partial^2 \psi}{\partial s^2} \psi^* \right). \tag{2.76}$$

Using (2.55), (2.75) and (2.76), the coupled KdV equation

$$\frac{\partial \psi}{\partial t} + \frac{\partial^3 \psi}{\partial s^3} + 2 \left| \psi \right|^2 \frac{\partial \psi}{\partial s} - \frac{\partial (\left| \psi \right|^2)}{\partial s} \psi = 0$$

is obtained. The anholonomy density associated with the coupled KdV equation of the curve evolution according to the Bishop frame for the second case is given by

$$\mathcal{AD}_2(s,t) = \xi_2 v = \frac{\partial(\xi_2^2 \xi_1)}{\partial s}. \tag{2.77}$$

The total phase Λ associated with the coupled KdV equation for the Bishop frame using (2.28) is

$$\Lambda = \int_{t_1}^{t_2} \int_{-s_0}^{s} \frac{\partial(\xi_2^2 \xi_1)}{\partial s}\, ds\, dt. \qquad \square$$

Example 2.5. The antiferromagnetic chain equation

$$\frac{\partial E_2}{\partial t} = -E_2 \times \frac{\partial E_2}{\partial s} \tag{2.78}$$

satisfies the Belavin–Polyakov equation for the second case in terms of Bishop frame in Euclidean 3-space. The geometric phase is obtained as $\Lambda = \int_{t_1}^{t_2} \int_{-s_0}^{s} \xi_2^2 ds dt$.

Proof. From (2.78), we obtain

$$\frac{\partial E_2}{\partial t} = \xi_2 E_1.$$

Using (2.54) and (2.60), we derive

$$\frac{\partial \chi_2}{\partial s} = -\xi_2^2, \qquad h = -i\psi. \tag{2.79}$$

From (2.55) and (2.79), the Belavin–Polyakov equation

$$\frac{\partial \psi}{\partial t} + i\frac{\partial \psi}{\partial s} + i\int |\psi|^2 = 0$$

is obtained for the second case in terms of Bishop frame in Euclidean 3-space. The second anholonomy density is found as

$$\mathcal{AD}_2(s,t) = \xi_2^2. \tag{2.80}$$

From (2.80), the geometric phase for the second case is obtained:

$$\Lambda = \int_{t_1}^{t_2}\int_{-s_0}^{s} \xi_2^2 \, ds\, dt. \qquad \square$$

Case III. The third frame $\{W_1, W_2, W_2^*\}$ associated with the nonlinear Schrödinger equation, the coupled KdV equation and the Belavin–Polyakov equation in terms of Bishop frame is given by

$$W_1 = T, \tag{2.81}$$

$$W_2 = \frac{E_1 + iE_2}{\sqrt{2}}, \tag{2.82}$$

$$W_2^* = \frac{E_1 - iE_2}{\sqrt{2}}. \tag{2.83}$$

The third transformation λ is introduced as

$$\lambda = \frac{\xi_1 + i\xi_2}{\sqrt{2}}.$$

Taking the derivatives of (2.81), (2.82) and (2.83) with respect to s, we have

$$\frac{\partial W_1}{\partial s} = \lambda^* W_2 - \lambda W_{2,}^*$$

$$\frac{\partial W_2}{\partial s} = \lambda W_1,$$

$$\frac{\partial W_2^*}{\partial s} = -\lambda^* W_1.$$

Here $\lambda^* = \frac{\xi_1 - i\xi_2}{\sqrt{2}}$. Take the derivatives of W_1, W_2 and W_2^* with respect to t to get:

$$\frac{\partial W_1}{\partial t} = \mu_1 W_1 + \mu_2 W_2 + \mu_3 W_2^*, \tag{2.84}$$

$$\frac{\partial W_2}{\partial t} = \gamma_1 W_2 + \gamma_2 W_2^* + \gamma_3 W_1, \tag{2.85}$$

$$\frac{\partial W_2^*}{\partial t} = \gamma_1^* W_2 + \gamma_2^* W_2^* + \gamma_3^* W_1. \tag{2.86}$$

From (2.84) and (2.85), we obtain

$$\mu_1 = 0, \quad \gamma = 0, \quad \mu_3 = -\gamma_3, \quad \mu_2 = -\gamma_3^*. \tag{2.87}$$

Using (2.87), the time evolution of $\{W_1, W_2, W_2^*\}$ can be written in the form

$$\frac{\partial W_1}{\partial t} = T_t = -\gamma_3^* W_2 - \gamma_3 W_2^*, \tag{2.88}$$

$$\frac{\partial W_2}{\partial t} = \gamma_3 W_1 + i\chi_3 W_2. \tag{2.89}$$

The compatibility conditions

$$\frac{\partial^2 W_1}{\partial t \partial s} = \frac{\partial^2 W_1}{\partial s \partial t},$$

$$\frac{\partial^2 W_2}{\partial t \partial s} = \frac{\partial^2 W_2}{\partial s \partial t}$$

give

$$\frac{\partial \chi_3}{\partial s} = i\gamma_3 \lambda^* - i\lambda \gamma_3^*, \tag{2.90}$$

$$\frac{\partial \lambda}{\partial t} - i\lambda\chi_3 + \frac{\partial \gamma_3}{\partial s} = 0. \tag{2.91}$$

The Darboux vector for the third case of the curve evolution in terms of Bishop frame is defined as follows:

$$\mathcal{Z}_3 = \xi_3^* T + B E_1 + C E_2. \tag{2.92}$$

With the aid of (2.92), we have

$$\frac{\partial W_1}{\partial t} = (\xi_3^* T + B E_1 + C E_2) \times T = v E_1 + w E_2, \tag{2.93}$$

$$\frac{\partial W_2}{\partial t} = -vT - \chi_3 E_2, \tag{2.94}$$

$$\frac{\partial W_2^*}{\partial t} = -wT + \chi_3 E_1, \tag{2.95}$$

where $-B = w$, $v = C$ and the quantity

$$\gamma_3 = -\frac{(v + iw)}{\sqrt{2}} \tag{2.96}$$

satisfies (2.88) and (2.93). From (2.90) and (2.96), we have

$$\frac{\partial \chi_3}{\partial t} = \xi_1 w - v\xi_2 = -\frac{\partial \chi_1}{\partial s} + \frac{\partial \chi_2}{\partial s}. \tag{2.97}$$

The anholonomy density $\mathcal{AD}_3$ for the third case is given in terms of Bishop frame as

$$\begin{aligned}\mathcal{AD}_3(s,t) &= -v\xi_2 + \xi_1 w = ((\xi_{1s}^* - \xi_{2s}^*) - (\xi_{1t} - \xi_{2t})) \\ &= -\mathcal{AD}_2(s,t) + \mathcal{AD}_1(s,t).\end{aligned}$$

The total phase $\mathcal{P}$ for the third case in terms of Bishop frame is expressed as

$$P = \int_{t_1}^{t_2} \int_{-s_0}^{s_0} (\xi_1 w - v\xi_2)\, ds\, dt = \int_{t_1}^{t_2} \int_{-s_0}^{s} \left\langle T, \frac{\partial T}{\partial s} \times \frac{\partial T}{\partial t} \right\rangle ds\, dt. \tag{2.98}$$

Example 2.6. The ferromagnetic chain equation

$$\frac{\partial T}{\partial t} = -T \times \frac{\partial^2 T}{\partial s^2} \tag{2.99}$$

satisfies the nonlinear Schrödinger equation for the third case of the curve evolution according to Bishop frame in Euclidean 3-space. The geometric phase is

$$\mathcal{P} = \frac{1}{2} \int_{t_1}^{t_2} \int_{-s_0}^{s} \frac{\partial(\xi_1^2 + \xi_2^2)}{\partial s}\, ds\, dt.$$

Proof. From (2.99), we have

$$v = -\frac{\partial \xi_2}{\partial s}, \qquad w = \frac{\partial \xi_1}{\partial s}. \tag{2.100}$$

With the aid of (2.90), (2.96) and (2.100), the following expressions are obtained:

$$\frac{\partial \chi_3}{\partial s} = -\left(\frac{\partial \xi_2}{\partial s} \xi_2 + \frac{\partial \xi_1}{\partial s} \xi_1 \right), \tag{2.101}$$

$$\gamma_3 = -i \frac{\partial \lambda}{\partial s}. \tag{2.102}$$

From (2.91), (2.100), (2.101), (2.102), the nonlinear Schrödinger equation

$$\frac{\partial \lambda}{\partial t} - \frac{i\partial^2 \lambda}{\partial s^2} - \frac{i\,|\lambda|^2\,\lambda}{2} = 0$$

is obtained. The third anholonomy density is derived as

$$\mathcal{AD}_3(s,t) = \left(\frac{\partial \xi_2}{\partial s} \xi_2 + \frac{\partial \xi_1}{\partial s} \xi_1 \right).$$

The geometric phase is obtained as follows:

$$\mathcal{P} = \frac{1}{2} \int_{t_1}^{t_2} \int_{-s_0}^{s} \frac{\partial(\xi_1^2 + \xi_2^2)}{\partial s}\, ds\, dt. \qquad \square$$

Supports. The author is supported by the Scientific Research Agency of Eskişehir Osmangazi University (ESOGU BAP Project Number: 2017-19029).

References

[1] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in the quantum theory,* Phys. Rev. **115** (1959), 485–490.

[2] Y. Aharonov and J. Anandan, *Phase change during a cylic quantum evolution*, Phys. Rev. Lett. **58** (1987), 1593–1596.

[3] R. Balakrishnan, *Space curves, anholonomy and nonlinearity*, Pramana J. Phys. **64** (2005), 607–615.

[4] R. Balakrishnan, R. Bishop, and R. Dandoloff, *Geometric phase in the classical continuous antiferromagnetic Heisenberg spin chain*, Phys Rev. Lett. **64** (1990), 2107–2110.

[5] A. Belavin and A.M. Polyakov, *Metastable states of two dimensional isotropic ferromagnets*, Pisma Zh. Eksp. Teor. Fiz. **22** (1975), 503–506.

[6] M.V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proceedings of the Royal Society of London Series A, Math. Phys. Sci. **392** (1984), 45–57.

[7] L.R. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly **82** (1975), 246–251.

[8] M.P. Do Carmo, *Differential Geometry of Curves and Surfaces*. Prentice Hall, Englewood Cliffs, NJ, 1976.

[9] P. Guha, *Moving space curve equations and a family of coupled KdV type systems*, Chaos, Solitons and Fractals **15** (2003), 41–46.

[10] N. Gürbüz, *Moving non-null curves according to Bishop frame in Minkowski 3-space*, Int. J. Geom. Methods Mod. Phys. **12** (2015), 1550052.

[11] N. Gürbüz, *Anholonomy according to three formulations of non-null curve evolution*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1750175.

[12] N. Gürbüz,*Three class of non-lightlike curve evolution according to Darboux frame and geometric phase*, Int. J. Geom. Methods Mod. Phys. **15** (2018), 1850023.

[13] H. Hasimoto, *Motion of a vortex filament and its relation to elastica*, J. Phys. Soc. Jpn. **31** (1971), 293–294.

[14] H. Hasimoto, *Soliton on a vortex filament*, J. Fluid Mech. **51** (1972), 477–485.

[15] M. Lakshmanan, *Continuum spin system as an exactly solvable dynamical system*, Phys. Lett. A **61** (1977), 53–54.

[16] M. Lakshmanan, *Continuum spin system as an exactly solvable dynamical system*, Phys. Lett. A **61** (1977), 53–54.

[17] J. Langer and R. Perline, *The Hasimoto transformation and integrable flows on curves*, Appl. Math. Lett. **3** (1990), 61–64.

[18] A. Mostafazadeh, *Relativistic adiabatic approximation and geometric phase*, J. Phys. A: Math. Gen. **31** (1998), 7829–7845.

[19] G. Munijara and M. Lakshmanan, *Motion of space curves in three dimensional Minkowski space* R_1^3, $SO(2,1)$ *spin equation and defocusing nonlinear Schrödinger equation*, Int. J. Geom. Methods Mod. Phys. **7** (2010) 1043–1049.

[20] S. Murugesh and R. Balakrishnan, *New connections between moving curves and soliton equations*, Phys. Lett. A **290** (2001), 81–87.

[21] A. Tomita and R.Y. Chiao, *Observation of Berry's topological phase by use of an optical fiber*, Phys. Rev. Lett. **57** (1986), 937–940.

Received April 23, 2018, revised October 30, 2018.

Nevin Gürbüz,

Eskisehir Osmangazi University, Sciences and Arts Faculty, Department of Mathematics-Computer, 26480, Eskisehir, Turkey,
E-mail: ngurbuz@ogu.edu.tr

Три неголгомні щільності відносно репера Бішопа у тривимірному евклідовому просторі

Nevin Gürbüz

У статті ми одержуємо три неголгомні щільності за допомогою трьох перетворень репера Бішопа у тривимірному евклідовому просторі.

Ключові слова: геометрична фаза, фаза Бішопа.