

Toeplitz Operators with Radial Symbols on Bergman Space and Schatten-von Neumann Classes

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In the present paper, we study spectral properties of Toeplitz operators with (quasi-) radial symbols on Bergman space. More precisely, the problem we are interested in is to understand when a given Toeplitz operator belongs to a Schatten–von Neumann class. The methods of the approximation theory (i.e., Legendre polynomials) are used to advance in this direction.

Key words: Toeplitz operators, (quasi-) radial symbols, Bergman spaces, Schatten–von Neumann classes, Legendre polynomials.

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1. Introduction

The study of Toeplitz operators is an important topic of the modern operator theory. In forties-eighties of the last century, the research was focused on Toeplitz operators on Hardy spaces, see Nikolski [19] for a complete account of results on the topic and the references. Since mid-eighties, the interest shifted to operators defined on different spaces of analytic functions, such as Bergman, Fock and other spaces. For instance, we quote the articles by Axler [4], Luecking [17, 18], Zhu [26] and later papers by Ahern–Cuckovic [2], Cuckovic–Rao [5], Grudsky–Vasilevski [11], Grudsky–Karapetyants–Vasilevski [12], Stroethoff [22], Korenblum–Zhu [13] and Zorboska [24]. These articles focused on the study of the analytical features of a Toeplitz operator (i.e., its boundedness, compactness, etc.) related to the properties of its symbol, see (1.1) for the definitions, and they serve a basis for our approach.

In the present article, we pursue in this direction and we give a series of sufficient conditions for Toeplitz operators with radial complex symbols to belong to the so called Schatten–von Neumann classes. In particular, we suppose the symbols neither real nor sign-definite. What seems to be more important is that we suggest a method of solving the *direct* and *inverse* spectral problems for operators of this class. It relies on the techniques of approximation theory and it makes an extensive use of Legendre orthogonal polynomials on an interval.

To make all this more precise, we introduce the notation and give some definitions. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, $\mathbb{T} = \{z : |z| = 1\}$ the unit circle and

$$dA(z) = \frac{1}{\pi} dx dy$$

be the probability area measure on \mathbb{D} . For $1 \leq p \leq \infty$, consider

$$L^p(\mathbb{D}) = L^p(\mathbb{D}, dA) = \left\{ f : \|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < +\infty \right\},$$

and the corresponding subspaces of analytic functions

$$L_a^p(\mathbb{D}) = L_a^p(\mathbb{D}, dA) = \left\{ f : f \text{ is analytic on } \mathbb{D}, \|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < +\infty \right\}.$$

These spaces $L_a^p(\mathbb{D})$ are called Bergman spaces, see Hedenmalm–Korenblum–Zhu [8] and Zhu [25] for more details on the topic.

We are mainly concerned with the Hilbert case $p = 2$. As usual, the scalar product in $L^2(\mathbb{D})$ is defined as

$$(f, g) = \int_{\mathbb{D}} f(z)\bar{g}(z) dA(z), \quad f, g \in L^2(\mathbb{D}).$$

It is well known that there is a bounded orthogonal projection (called Bergman projection) $P_+ : L^2(\mathbb{D}) \rightarrow L_a^2(\mathbb{D})$. This projection admits the integral representation

$$(P_+f)(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \bar{z}w)^2} dA(z), \quad f \in L^2(\mathbb{D}),$$

see [8, Sect. 1.1].

It is easy to see that polynomials with respect to z (denoted by \mathcal{P}), form a dense subset of $L_a^2(\mathbb{D})$. Now, let $\varphi \in L^2(\mathbb{D})$ and define the Toeplitz operator with symbol φ as

$$T_\varphi : L_a^2(\mathbb{D}) \rightarrow L_a^2(\mathbb{D}), \quad T_\varphi p = P_+(\varphi p), \quad p \in \mathcal{P}. \quad (1.1)$$

We say that the operator T_φ is bounded if one can extend it by continuity to the whole $L_a^2(\mathbb{D})$ and the resulting operator is still bounded. We denote the extended operator by T_φ as well.

The class of compact operators is denoted by \mathcal{S}_∞ . The Schatten–von Neumann classes are defined as

$$\mathcal{S}_p = \left\{ A \in \mathcal{S}_\infty : \|A\|_{\mathcal{S}_p}^p = \sum_k s_k(A)^p < \infty \right\}, \quad 0 < p < \infty, \quad (1.2)$$

where $\{s_k(A)\}$ are the singular numbers of the operator A . We refer to Simon [21], Gohberg–Goldberg–Kaashoek [10] for more information on the subject.

Write $z = |z|e^{i\theta} = re^{i\theta} \in \mathbb{D}$. One says that the symbol φ is radial (or homogeneous), if $\varphi(z) = \varphi(|z|) = \varphi(r)$, i.e., φ is rotationally symmetric with

respect to the origin. With a slight abuse of notation, we consider a radial symbol φ as a function on \mathbb{D} and, keeping the same letter, as a function on $[0, 1)$. When needed, we extend it to a function defined on $[0, 1]$ letting $\varphi(1) = 0$. One says that the symbol is quasi-radial (or quasi-homogeneous), if $\varphi(z) = \varphi(|z|)e^{im\theta}$ for some fixed $m \in \mathbb{Z}$.

As we already mentioned, we wish to understand when a Toeplitz operator T_φ with a radial symbol lies in the Schatten–von Neumann class \mathcal{S}_p , $0 < p < \infty$ (1.2). We pay a particular attention to the study of complex-valued or sign-indefinite symbols. The case of quasi-homogeneous symbols follows similarly. A sample of known results in this direction is given in Subsection 2.1.

The spectrum of a Toeplitz operator of the above type is rather easy to compute. Indeed, taking the standard orthogonal basis $e_n(z) = \sqrt{n+1}z^n$ for $L_a^2(\mathbb{D})$, one sees that the matrix elements $(T_\varphi)_{nm} = (T_\varphi e_n, e_m)$ are given by (2.1), and so the spectrum $\sigma(T_\varphi)$ is given by $\{\overline{c_n(\varphi)}\}$, where

$$c_n(\varphi) := (T_\varphi)_{nm} = 2(n+1) \int_0^1 x^{2n+1} \varphi(x) dx.$$

We sketch now the idea on the construction of the spectral maps for radial Toeplitz operators, see Subsections 4.1, 4.2 for more details. To this end, we consider the Legendre polynomials, i.e., the polynomials which are orthogonal in $L^2[-1, 1] = L^2([-1, 1], ds)$, see also Subsection 2.2.

Let $\varphi \in L^2([0, 1], xdx)$ be the symbol of a radial Toeplitz operator. We set $\psi(s) = \varphi(\sqrt{s})$, $s \in [0, 1]$ and

$$\psi_e(s) = \begin{cases} \psi(x), & x \in [0, 1] \\ \psi(-x), & x \in [-1, 0] \end{cases}, \quad \psi_o(s) = \begin{cases} \psi(x), & x \in [0, 1] \\ -\psi(-x), & x \in [-1, 0] \end{cases}.$$

It is plain that $\psi_e, \psi_o \in L^2([-1, 1], ds)$ and we can write

$$\psi_e(s) = \sum_{k=0}^{\infty} a_{2k} P_{2k}(s), \quad \psi_o(s) = \sum_{k=0}^{\infty} a_{2k+1} P_{2k+1}(s),$$

where

$$a_{2k} = \frac{1}{\|P_{2k}\|_{L^2[-1,1]}^2} (\psi_e, P_{2k})_{L^2[-1,1]} = \frac{(4k+1)}{2} (\psi_e, P_{2k})_{L^2[-1,1]},$$

$$a_{2k+1} = \frac{1}{\|P_{2k+1}\|_{L^2[-1,1]}^2} (\psi_o, P_{2k+1})_{L^2[-1,1]} = \frac{(4k+3)}{2} (\psi_o, P_{2k+1})_{L^2[-1,1]}.$$

Above, $\|\cdot\|_{L^2[-1,1]}$ and $(\cdot, \cdot)_{L^2[-1,1]}$ stay for the usual norm and the scalar product in $L^2[-1, 1]$, respectively. We have that

$$\{c_n(\varphi)\} = \mathcal{D}(\{a_k\}), \quad \{a_k\} = \mathcal{I}(\{c_n(\varphi)\}),$$

where the maps \mathcal{D} and \mathcal{I} are given by (4.5), (4.6), see (4.2), (4.4) for the definitions of the corresponding coefficients.

As an application of this construction, we prove the following theorem.

Theorem 1.1. *Take an arbitrary $\delta_0 > 0$. There exists a $\varphi \in L^2([0, 1], xdx)$ such that*

1. $\varphi' \notin L^2[0, 1]$;
2. *the Toeplitz operator T_φ (1.1) with radial symbol φ lies in \mathcal{S}_{δ_0} .*

Further results of this type are in progress and they are to be published elsewhere. Notice that this theorem gives a different point of view on results of Propositions 3.1, 3.3, and the related examples.

The article is organized as follows. The first part of Section 2 contains a reminder on notions related to Bergman spaces and basic computations on radial Toeplitz operators there. To put the things in perspective, we give a brief account of known results on the topic as well. The second part of Section 2 gives a summary of facts on Legendre polynomials. Section 3 presents certain sufficient conditions for radial Toeplitz operators to lie in the classes \mathcal{S}_p , $0 < p < \infty$ and a related example. The first part of Section 4 explains the construction of the spectral maps (4.5), (4.6). Its second part deals with the proof of Theorem 1.1.

In this paper, C stays for a constant changing from one relation to another; we give usually the list of parameters the constant depends on.

2. Preliminaries

2.1. Notions related to Bergman spaces and some results on Toeplitz operators. An extensive account on Bergman spaces can be found in Hedenmalm–Korenblum–Zhu [8] and Zhu [25].

Recall the spaces $L^p(\mathbb{D})$ and $L_a^p(\mathbb{D})$, $1 \leq p \leq \infty$, defined in the previous section. An important property of the space $L_a^2(\mathbb{D})$ is that for any $w \in \mathbb{D}$, there exists the so called reproducing kernel, i.e., $k_w \in L_a^2(\mathbb{D})$ given by

$$k(z, w) = k_w(z) = \frac{(1 - |w|^2)}{(1 - \bar{w}z)^2}, \quad z, w \in \mathbb{D}.$$

One has that for any $f \in L_a^2(\mathbb{D})$

$$(1 - |w|^2)f(w) = (f, k_w),$$

where (\cdot, \cdot) is the scalar product in $L^2(\mathbb{D})$. For more information on the reproducing kernels, see [8, Sect. 1.1] and [25, Chap. 4].

For a function $g \in L^1(\mathbb{D})$, one considers its Berezin transform [8, Chap. 2] and [25, Chap. 6] defined as

$$\tilde{g}(w) = (gk_w, k_w), \quad w \in \mathbb{D}.$$

Similarly, for a Toeplitz operator T_φ (1.1), one sets

$$\tilde{T}_\varphi(w) = \tilde{\varphi}(w) = (T_\varphi k_w, k_w) = (P_+ \varphi k_w, k_w) = (\varphi k_w, k_w), \quad w \in \mathbb{D}.$$

As we will see, many properties of T_φ can be expressed conveniently in terms of $\tilde{\varphi}$.

The next two results hold for general Toeplitz operators T_φ (i.e., the symbols φ are not required to be radial). Let $\beta(z, w)$, $z, w \in \mathbb{D}$, be the Bergman (or, equivalently, hyperbolic) metric on \mathbb{D} , see [25, Chap. 4], and

$$D(w, r) = \{z \in \mathbb{D} : \beta(z, w) < r\},$$

where $r \geq 0$. For a fixed r , define the averaging of φ as

$$\hat{\varphi}_r(w) = \frac{1}{A(D(w, r))} \int_{D(w, r)} \varphi(z) dA(z),$$

where $A(D(w, r)) = \int_{D(w, r)} dA(z)$.

Theorem 2.1 ([25, Chap. 7]).

1. Let $\varphi \in C(\mathbb{D})$ (i.e., φ is continuous on $\bar{\mathbb{D}}$). Then T_φ is compact (i.e., $T_\varphi \in \mathcal{S}_\infty$) iff

$$\lim_{|z| \rightarrow 1^-} \varphi(z) = 0.$$

2. Let $\varphi(z) \geq 0$ on \mathbb{D} . Then the following is equivalent:

- $T_\varphi \in \mathcal{S}_\infty$,
- $\lim_{|z| \rightarrow 1^-} \tilde{\varphi}(z) = 0$,
- for any fixed $r > 0$, $\lim_{|z| \rightarrow 1^-} \hat{\varphi}_r(z) = 0$.

For assertion 1 of this theorem, see [25, Chap. 7, Corollary 7.9]; for assertion 2, see [25, Chap. 7, Corollary 7.9].

Put also $d\lambda(z) = dA(z)/(1 - |z|^2)^2$.

Theorem 2.2 ([25, Chap. 7]). Let $\varphi(z) \geq 0$ and $p \geq 1$. Then the following is equivalent:

- $T_\varphi \in \mathcal{S}_p$,
- $\tilde{\varphi} \in L^p(\mathbb{D}, d\lambda)$,
- for any fixed $r > 0$, $\hat{\varphi}_r \in L^p(\mathbb{D}, d\lambda)$.

From now on, we suppose the symbol φ to be radial. Then the matrix of the Toeplitz operator T_φ is simple to compute. Taking the standard basis $e_n(z) = \sqrt{n+1} z^n$ of $L_a^2(\mathbb{D})$, one has

$$(T_\varphi)_{nm} = \frac{(n+1)^{1/2}(m+1)^{1/2}}{\pi} \int_0^{2\pi} e^{i\theta(n-m)} d\theta \int_0^1 r^{n+m+1} \varphi(r) dr.$$

So

$$(T_\varphi)_{nm} = \begin{cases} 2(n+1) \int_0^1 r^{2n+1} \varphi(r) dr, & m = n \\ 0, & n \neq m \end{cases}, \quad (2.1)$$

and the matrix of the operator is diagonal. It is convenient to set

$$c_n(\varphi) := (T_\varphi)_{nn} = 2(n+1) \int_0^1 r^{2n+1} \varphi(r) dr. \quad (2.2)$$

For a radial symbol φ , its Berezin transform is also radial and it is computed as

$$\tilde{\varphi}(z) = (1 - |z|^2)^2 \sum_{n=0}^{\infty} (n+1) c_n(\varphi) |z|^{2n},$$

see Zorboska [24].

Notice that one does not suppose φ to be sign-definite in the following theorem.

Theorem 2.3 (Korenblum–Zhu [13]). *Let T_φ be a Toeplitz operator with a radial bounded symbol φ . Then the following is equivalent:*

1. $T_\varphi \in \mathcal{S}_\infty$;
2. $\lim_{n \rightarrow +\infty} 2(n+1) \int_0^1 \varphi(x) x^{2n+1} dx = 0$;
3. $\lim_{t \rightarrow 1^-} \frac{1}{1-t} \int_t^1 \varphi(x) dx = 0$.

We finish this subsection mentioning interesting related papers by Louhichi–Zakariasy [14], Louhichi–Strouse–Zakariasy [15], and Louhichi–Randrimahaleo–Zakariasy [16].

2.2. Few facts on Legendre polynomials. To advance on the problem presented in the Introduction, the idea is to use a “natural” basis of $L^2[-1, 1] = L^2([-1, 1], dx)$. Of course, the system $(x^n)_{n \in \mathbb{N}}$ does not suit this purpose, since it is complete, but, as we will see, it is very far from an orthogonal (or Riesz) basis.

A much better choice are the so called orthogonal Legendre polynomials $\{P_n(x)\}$. Below, we give a short list of their properties needed for our construction presented in Section 4. For extensive information on the subject, see the books by Erdélyi–Magnus–Oberhettinger et al. [6, Chap. 3] and Arfken–Weber [3, Chap. 11], as well as Abramowitz–Stegun [7, Chap. 8] for a list of useful formulas. The most results of this section are well-known and we give their brief proofs for the sake of completeness.

Let $P_0(x) = 1$, $P_1(x) = x$. The Legendre polynomials $\{P_n(x)\}$ are defined by recurrence relations

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n \geq 1. \quad (2.3)$$

It is important for us that

$$(P_n, P_m)_{L^2[-1,1]} = \int_{-1}^1 P_n(x) \overline{P_m(x)} dx = \frac{2}{2n+1} \delta_{nm}, \quad (2.4)$$

where δ_{nm} is the Kronecker delta. As usual, $(\cdot, \cdot)_{L^2[-1,1]}$ stays for the scalar product in $L^2[-1, 1]$. By the Gram–Schmidt orthonormalization procedure applied to $\{x^n\}_{n \geq 0}$, the polynomials $\{p_n\}_{n \geq 0}$

$$p_n(x) = \sqrt{\frac{2}{2n+1}} P_n(x)$$

form an orthonormal basis in $L^2[-1, 1]$. To stick with more standard objects, we work with $\{P_n\}_{n \geq 0}$ keeping in mind the above renormalization condition.

We have for the derivatives $P'_n(x)$, see [6, Sect. 10.10]

$$P'_{n+1}(x) = \sum_{\substack{k \geq 0 \\ n-2k \geq 0}} \frac{2}{\|P_{n-2k}\|^2} P_{n-2k}(x) = \sum_{\substack{k \geq 0 \\ n-2k \geq 0}} (2(n-2k)+1) P_{n-2k}(x). \quad (2.5)$$

It is important that the systems $\{x^n\}_{n \geq 0}$ and $\{P_n(x)\}_{n \geq 0}$ can be recalculated in terms of each other in a (relatively) simple manner. Let $[x]$ be the entire (integer) part of $x \in \mathbb{R}$, and $m!!$ be the odd (or even) factorial, i.e.,

$$(2m+1)!! = (2m+1)(2m-1) \cdots 1, \quad (2m)!! = (2m)(2m-2) \cdots 2.$$

The proof of the following proposition is standard and is omitted.

Proposition 2.4 ([6, Sect. 10.10]). *We have*

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, \quad (2.6)$$

and

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k}^2 (x-1)^{n-k} (x+1)^k. \quad (2.7)$$

It is rather remarkable that these formulas can be *explicitly* inverted. Despite its neat form, we were unable to find this result in the literature, and for this reason we prefer to give its proof; see notes Hui [9] and Schmied [20].

Proposition 2.5 ([9, 20]). *For $n \geq 1$, we have*

$$x^n = \sum_{k=0}^{[n/2]} \frac{n! (2n-4k+1)}{2^k k! (2n-2k+1)!!} P_{n-2k}(x). \quad (2.8)$$

Proof. We set $P_{-1}(x) = 0$. Relation (2.8) yields obviously $P_0(x) = 1$ and $P_1(x) = x$. To argue by recurrence, we suppose that formula (2.8) holds true. We want to show that

$$x^{n+1} = \sum_{k=0}^{[(n+1)/2]} \frac{(n+1)!(2(n+1)-4k+1)}{2^k k! (2(n+1)-2k+1)!!} P_{(n+1)-2k}(x). \quad (2.9)$$

Due to (2.3), we see

$$xP_{n-2k}(x) = \frac{n-2k}{2n-4k+1}P_{n-2k-1}(x) + \frac{n-2k+1}{2n-4k+1}P_{n-2k+1}(x).$$

Writing $x^{n+1} = x(x^n)$ and plugging it in the formula, we obtain

$$\begin{aligned} x^{n+1} &= \sum_{k=0}^{[n/2]} \frac{n!(2n-4k+1)}{2^k k! (2n-2k+1)!!} xP_{n-2k}(x) \\ &= \sum_{k=0}^{[n/2]} \frac{n!(2n-4k+1)}{2^k k! (2n-2k+1)!!} \left(\frac{n-2k}{2n-4k+1} P_{n-2k-1}(x) \right. \\ &\quad \left. + \frac{n-2k+1}{2n-4k+1} P_{n-2k+1}(x) \right) \\ &= \sum_{k=1}^{[(n+1)/2]} \frac{(n+1)!}{n+1} \frac{2k}{2^k k!} \frac{((n+1)-2k+1)}{(2(n+1)-2k+1)} P_{(n+1)-2k}(x) \\ &\quad + \sum_{k=0}^{[n/2]} \frac{(n+1)!}{n} \frac{1}{2^k k!} \frac{((n+1)-2k)}{(2(n+1)-2k-1)} P_{(n+1)-2k}(x). \end{aligned}$$

After a series of elementary transformations, we come to formula (2.9). The proposition is proved. \square

For the sequel, it will be convenient to specify the formula of the above proposition to the case on even and odd n . That is, for n even, $n = 2m$, we have

$$\begin{aligned} x^{2m} &= \sum_{k=0}^m \frac{(4k+1)(2m)!}{2^{m-k}(m-k)!(2k+2m+1)!!} P_{2k}(x) \\ &= \sum_{k=0}^m \frac{(4k+1)(2m)!}{(2m-2k)!!(2k+2m+1)!!} P_{2k}(x) = \sum_{k=0}^m b_{2m,2k} P_{2k}(x), \end{aligned}$$

where

$$b_{2m,2k} = \frac{(4k+1)(2m)!}{(2m-2k)!!(2k+2m+1)!!}, \quad k = 0, \dots, m.$$

Similarly, for n odd, $n = 2m + 1$,

$$\begin{aligned} x^{2m+1} &= \sum_{k=0}^m \frac{(4k+3)(2m+1)!}{2^{m-k}(m-k)!(2k+2m+3)!!} P_{2k+1}(x) \\ &= \sum_{k=0}^m \frac{(4k+3)(2m+1)!}{(2m-2k)!!(2k+2m+3)!!} P_{2k+1}(x) = \sum_{k=0}^m b_{2m+1,2k+1} P_{2k+1}(x), \end{aligned}$$

where

$$b_{2m+1,2k+1} = \frac{(4k+3)(2m+1)!}{(2m-2k)!!(2k+2m+3)!!}, \quad k = 0, \dots, m.$$

3. Some sufficient conditions for radial Toeplitz to lie in Schatten-von Neumann classes \mathcal{S}_p , $0 < p < \infty$

Let the function φ be defined on $I = [0, 1]$. Theorem 2.3 suggests is that if the symbol φ is continuous in a (left) neighborhood of point $x = 1$ and $\varphi(1) = 0$, then obviously $\lim_{n \rightarrow +\infty} c_n(\varphi) = 0$.

Suppose now that φ is l times continuously differentiable on I , $l \in \mathbb{N}$. One can easily see that the conditions $\varphi^{(k)}(1) = 0$, $k = 0, \dots, l - 1$, imply a faster decay of $c_n(\varphi)$ when $n \rightarrow +\infty$.

To formulate the coming proposition, denote by $W^{l,p}(I)$ the Sobolev space on the interval I , i.e., $f \in W^{l,p}(I)$ iff

$$\|f\|_{W^{l,p}} = \sum_{j=0}^l \|f^{(j)}\|_p < \infty,$$

where the derivatives $f^{(j)}$ are understood in the distributional sense, and the norms $\|\cdot\|_p$ refer to $L^p(I)$, $p \geq 1$. See Adams [1] for more information on the subject.

The proposition is completely elementary, and we give it for the sake of completeness only.

Proposition 3.1. *Let $l \in \mathbb{N}$, $p \geq 1$. Let $\varphi \in W^{l,p}(I)$ and $\varphi^{(k)}(1) = 0$, $k = 0, \dots, l - 1$. Then*

$$|c_n(\varphi)| \leq \frac{C}{n^{l-1/p}} \|\varphi\|_{W^{l,p}},$$

where $C = C(l, p)$.

Proof. As usual, for $p \geq 1$, we set $q = p/(p - 1)$, $q \geq 1$. Starting with $l = 1$, we see that

$$\begin{aligned} c_n(\varphi) &= 2(n+1) \int_0^1 \varphi(x) x^{2n+1} dx = \int_0^1 \varphi(x) d(x^{2(n+1)}) \\ &= - \int_0^1 \varphi'(x) x^{2(n+1)} dx. \end{aligned}$$

By Hölder inequality

$$|c_n(\varphi)| \leq \|\varphi'\|_p \|x^{2(n+1)}\|_q \leq \frac{C}{n^{1/q}} \|\varphi\|_{W^{1,p}},$$

where $C = C(l, p)$. For $l = 1$, one has $1/q = 1 - 1/p = l - 1/p$.

For general l , the integration by parts applied $(l - 1)$ times to the above integral gives

$$c_n(\varphi) = \frac{(-1)^l}{(2n+3)(2n+4) \cdots (2n+1+l)} \int_0^1 \varphi^{(l)}(x) x^{2n+1+l} dx.$$

Consequently, we get by Hölder inequality

$$|c_n(\varphi)| \leq \frac{C}{n^{l-1+1/q}} \|\varphi^{(l)}\|_p \leq \frac{C}{n^{l-1/p}} \|\varphi\|_{W^{l,p}},$$

where $C = C(l, p)$. □

Suppose that $l - 1/p > 0$. It follows immediately that $T_\varphi \in \mathcal{S}_\alpha, \alpha > 1/(l - 1/p)$, for symbols φ from the above proposition, and, moreover

$$\|T_\varphi\|_{\mathcal{S}_\alpha} \leq C\|\varphi\|_{W^{l,p}}.$$

Let us observe also that Proposition 3.1 is a sufficient condition for the decay of $\{c_n(\varphi)\}$, but it is not at all a necessary one. For instance, it explains how the coefficients $\{c_n(\varphi)\}$ can decay when $\varphi \geq 0$ while no cancellations in integral (2.2) defining $c_n(\varphi)$ are possible. When φ is complex (or real and sign-indefinite), the integral giving $c_n(\varphi)$ can become small because of very fast oscillations of φ in a (left) neighborhood of $x = 1$, see Proposition 3.3.

The lemma is a simple calculation based on well-known properties Γ -function.

Lemma 3.2. *Let $\delta > -1$. Then*

$$I_\delta(n) = \int_0^1 s^n(1-s)^\delta ds \leq \frac{C\Gamma(\delta+1)}{n^{\delta+1}},$$

where C is an absolute constant.

Proof. We give a brief sketch of the proof for the completeness. Take $a_n = 1/\sqrt{n+1}$. Making the change of variable $s = e^{-t}$ in $I_\delta(n)$ and writing the integral on $[0, a_n)$ and $[a_n, +\infty)$, we get

$$\begin{aligned} \int_0^1 s^n(1-s)^\delta ds &= \int_0^\infty e^{-nt} (1-e^{-t})^\delta dt = I_1 + I_2 \\ &:= \int_0^{a_n} e^{-nt} (1-e^{-t})^\delta dt + \int_{a_n}^\infty e^{-nt} (1-e^{-t})^\delta dt. \end{aligned}$$

One sees easily that

$$I_2 \leq \frac{C}{\sqrt{n+1}} e^{-(n+1)a_n}.$$

For any $\epsilon > 0$ and t small enough, we have $(1 - e^{-t}) \leq (1 + \epsilon)t$, and so

$$I_1 \leq (1 + \epsilon)^\delta \int_0^{a_n} e^{-nt} t^\delta dt = \frac{(1 + \epsilon)^\delta}{(n + 1)^\delta} \int_0^{(n+1)a_n} e^{-u} u^\delta du \rightarrow \frac{(1 + \epsilon)^\delta \Gamma(1 + \delta)}{(n + 1)^\delta},$$

where $u = (n + 1)t$ and $n \rightarrow +\infty$. The lemma is proved. \square

Of course, one can prove with a little more effort that $I_\delta(n) \asymp \Gamma(\delta + 1)/n^{\delta+1}$ as $n \rightarrow +\infty$.

The first two points of the following proposition are borrowed from Grudsky–Vasilevski [11, Theorem 3.3]. The third point addresses the fact whether the Toeplitz operator T_φ lies in $\mathcal{S}_p, 0 < p < \infty$, and it is obtained with the help of techniques similar to those of [11]. Notice that neither the continuity, nor even the boundedness of the symbol φ in the neighborhood of $x = 1$ are required. We also warn the reader that our notation differs from that one of the latter paper.

We set $\Phi(x) = \int_x^1 \varphi(s) ds$.

Proposition 3.3. *Assume that $\varphi \in L^1[0, 1]$. We have the following implications (one understands that $s \rightarrow 1 - 0$):*

1. *If $|\Phi(s)| \leq C(1-s)$, then $\sup_n |c_n(\varphi)| < \infty$ (i.e., the operator T_φ is bounded).*

2. *If*

$$\lim_{s \rightarrow 1-0} \frac{|\Phi(s)|}{(1-s)} = 0,$$

then $\lim_{n \rightarrow +\infty} |c_n(\varphi)| = 0$ (i.e., the operator T_φ is compact).

3. *Let*

$$|\Phi(s)| \leq C(1-s)^\delta \tag{3.1}$$

for a $\delta > 1$. Then

$$|c_n(\varphi)| \leq \frac{C}{n^{(\delta-1)}},$$

i.e., the operator T_φ is in the Schatten–von Neumann class $\mathcal{S}_{p+\epsilon}$, $p = 1/(\delta - 1)$ for any $\epsilon > 0$.

Proof. For **1** and **2**, see Grudsky–Vasilevski [11, Theorem 3.3]. Turning to **3**, we make integration by parts in (2.2)

$$c_n(\varphi) = (2n+2)(2n+1) \int_0^1 \Phi(x)x^{2n} dx.$$

By hypothesis of **3**, there is a $\delta_0 > 0$ such that $|\Phi(x)| \leq C(1-x)^\delta$ for $1 - \delta_0 \leq x < 1$. We re-write the latter integral as

$$\begin{aligned} c_n(\varphi) &= (2n+2)(2n+1) \int_0^1 \Phi(x)x^{2n} dx \\ &= (2n+2)(2n+1) \left(\int_0^{1-\delta_0} \Phi(x)x^{2n} dx + \int_{1-\delta_0}^1 \Phi(x)x^{2n} dx \right) =: I_1 + I_2, \end{aligned}$$

the factor $(2n+2)(2n+1)$ being absorbed in the integrals. Since $\varphi \in L^1[0, 1]$, Φ is bounded and so I_1 decays with exponential rate, i.e.,

$$I_1 \leq C(2n+2)(2n+1)(1-\delta_0)^{2n}.$$

Turning to I_2 , we see

$$\begin{aligned} I_2 &\leq C(2n+2)(2n+1) \int_{1-\delta_0}^1 (1-x)^\delta x^{2n} dx \\ &\leq C(2n+2)(2n+1) \int_0^1 (1-x)^\delta x^{2n} dx \leq \frac{C \Gamma(\delta+1)n^2}{n^{\delta+1}} \leq \frac{C}{n^{\delta-1}}, \end{aligned}$$

where at the last step we used the above lemma. \square

Now, we consider

$$\varphi(x) = \varphi_{\alpha,\beta}(x) = (1-x)^{-\beta} \sin(1-x)^{-\alpha}, \quad (3.2)$$

where $\alpha > 0$ and $\beta < 1$. Of course, the condition on β guarantees that $\varphi \in L^1[0, 1]$. The computations from [11, p. 23] (see equation (3.3) and relations below) give that

$$\Phi(x) = \frac{1}{\alpha} (1-x)^{\alpha-\beta+1} \cos(1-x)^{-\alpha} + O\left((1-x)^{2\alpha-\beta+1}\right),$$

so condition (3.1) is satisfied with $\delta = \alpha - \beta + 1 > 0$, and, consequently, point (3) of Proposition 3.3 holds. Notice that we may pick δ as large as we want taking $\alpha > 0$ big enough.

Corollary 3.4. *There is a symbol $\varphi = \varphi_{\alpha,\beta}$ (3.2) with obvious choice of parameters α, β having the properties:*

1. $\varphi \in L^1[0, 1]$,
2. the operator T_φ is compact,
3. for any $\delta_0 > 0$, $\sup_{[1-\delta_0, 1]} |\varphi(x)| = +\infty$,
4. moreover, for any $p > 0$, there are $\alpha > 0$ and $\beta < 1$, such that $T_{\varphi_{\alpha,\beta}} \in S_p$.

4. The solution to the spectral problem for radial Toeplitz operators and Legendre polynomials

4.1. Direct and inverse spectral problems for radial Toeplitz operators. Recall that for a given radial function $\varphi \in L^1([0, 1], xdx)$, the Toeplitz operator (1.1) has a diagonal matrix with coefficients $\{c_n(\varphi)\}$ (2.2) in the standard basis of $L_a^2(\mathbb{D})$. So the spectrum $\sigma(T_\varphi)$ coincides with $\{c_n(\varphi)\}$.

The spectral problem for these operators consists of two parts. The first part (i.e., the direct spectral problem) is to understand the properties of $\sigma(T_\varphi)$ (or to compute the spectrum in a rather explicit manner) for a given φ . The second part (i.e., the inverse direct problem) is to reconstruct the symbol φ (if possible), from the spectrum $\sigma(T_\varphi)$ of the Toeplitz operator. As we show, this problem admits a neat solution in terms of Legendre polynomials $\{P_n(x)\}$ in $L^2[-1, 1]$.

Before giving the construction, we make some reductions. First, recalling formula (2.2), we make the change of variable $x = \sqrt{s}$ and we set $\psi(s) = \phi(\sqrt{s})$, so

$$C_n(\psi) := c_n(\varphi) = 2(n+1) \int_0^1 \varphi(x) x^{2n+1} dx = (n+1) \int_0^1 \psi(s) s^n ds. \quad (4.1)$$

We accept the notation from the above relation for the rest of the paper.

Second, we wish to represent the function ψ as a series of Legendre polynomials on $[-1, 1]$. The rescaling the polynomials $\{P_n\}$ to the interval $[-1, 1]$ leads

to some cumbersome relations; to keep the formulas we work with reasonably simple, we prefer to extend the function ψ to $[-1, 1]$ in appropriate ways.

Namely, for a $\psi \in L^p[0, 1]$, $p \geq 1$, we set

$$\psi_e(s) = \begin{cases} \psi(x), & x \geq 0 \\ \psi(-x), & x < 0 \end{cases}, \quad \psi_o(s) = \begin{cases} \psi(x), & x \geq 0 \\ -\psi(-x), & x < 0 \end{cases}.$$

Of course, we have $\psi = \psi_e + \psi_o$ on $[-1, 1]$, where ψ is extended by zero to $[-1, 0)$, and $\|\psi_e\|_{L^p[-1,1]}^p = \|\psi_o\|_{L^p[-1,1]}^p = 2\|\psi\|_{L^p[0,1]}^p$. It follows

$$C_n(\psi) = (n+1) \int_0^1 \psi(s) s^n ds = \frac{(n+1)}{2} \int_{-1}^1 \psi_e(s) s^n ds =: C_n(\psi_e)$$

for n even and $C_n(\psi_e) = 0$ for n odd. Similarly,

$$C_n(\psi) = (n+1) \int_0^1 \psi(s) s^n ds = \frac{(n+1)}{2} \int_{-1}^1 \psi_o(s) s^n ds =: C_n(\psi_o)$$

for n odd and $C_n(\psi_o) = 0$ for n even. In particular, $C_n(\psi) = C_n(\psi_e)$ for n even, and $C_n(\psi) = C_n(\psi_o)$ for n odd.

Assume now $\psi(s) = \sum_{k=0}^N a_k P_k(s)$ on $[-1, 1]$. Then, we get for even $m = 2n$

$$\begin{aligned} C_{2n}(\psi) &= \frac{(2n+1)}{2} \int_{-1}^1 \psi_e(x) x^{2n} dx \\ &= \frac{(2n+1)}{2} \int_{-1}^1 \left(\sum_{j=0}^N a_j P_j(x) \right) \left(\sum_{k=0}^n b_{2n,2k} P_{2k}(x) \right) dx \\ &= \frac{(2n+1)}{2} \int_{-1}^1 \sum_{j=0}^{[N/2]} \sum_{k=0}^n b_{2n,2k} a_{2j} = P_{2j}(x) P_{2k}(x) dx \\ &= \sum_{k=0}^{\min\{n, [N/2]\}} \left(\frac{2n+1}{4k+1} b_{2n,2k} \right) a_{2k} =: \sum_{k=0}^{\min\{n, [N/2]\}} d_{2n,2k} a_{2k}, \end{aligned} \quad (4.2)$$

where

$$d_{2n,2k} = \frac{(2n+1)!}{(2n-2k)!!(2n+2k+1)!!}. \quad (4.3)$$

For odd $m = 2n+1$, we obtain

$$\begin{aligned} C_{2n+1}(\psi) &= \frac{(2n+2)}{2} \int_{-1}^1 \psi_o(x) x^{2n+1} dx \\ &= \frac{(2n+2)}{2} \int_{-1}^1 \left(\sum_{j=0}^N a_j P_j(x) \right) \left(\sum_{k=0}^n b_{2n+1,2k+1} P_{2k+1}(x) \right) dx \\ &= \frac{(2n+2)}{2} \int_{-1}^1 \sum_{j=0}^{[N/2]} \sum_{k=0}^n b_{2n+1,2k+1} a_{2j+1} P_{2j+1}(x) P_{2k+1}(x) dx \end{aligned}$$

$$= \sum_{k=0}^{\min\{n, [N/2]\}} \left(\frac{2n+2}{4k+3} b_{2n+1, 2k+1} \right) a_{2k+1} =: \sum_{k=0}^{\min\{n, [N/2]\}} d_{2n+1, 2k+1} a_{2k+1}, \quad (4.4)$$

where

$$d_{2n+1, 2k+1} = \frac{(2n+2)!}{(2n-2k)!!(2n+2k+3)!!}.$$

In the sequel, we will be concerned with bounds on the coefficients $C_n(\psi)$ for different functions ψ . Observe that the coefficients $C_{2n}(\psi)$ (4.2) depend on $d_{2n, 2k}$ and a_{2k} only, and the same applies to $C_{2n+1}(\psi)$ (4.4) and $d_{2n+1, 2k+1}$, a_{2k+1} , respectively. Hence, we have a nice separation of variables. Since the computations of $\{C_n(\psi)\}$ for n even or odd are similar, we concentrate on the case of even n . The case of odd n goes through likewise and is omitted.

Let $\psi(s) = \sum_{k=0}^N a_k P_k(s)$, $s \in [-1, 1]$. Consider the map given by

$$\{c_n(\varphi)\} = \{C_{2n}(\psi); C_{2n+1}(\psi)\} = \mathcal{D}(\{a_k\}), \quad (4.5)$$

where $\mathcal{D}(\cdot)$ is the multiplication by $\mathcal{D} = [d_{n,k}]_{n,k}$, see (4.2) and (4.4) for the definition of $d_{n,k}$. We call the map *the direct spectral map* for the operator T_φ ; to keep the notation simple, we denote the map and its matrix (in the fixed basis $\{P_n\}$) by the same letter \mathcal{D} . *The inverse spectral map* for the operator T_φ is

$$\{a_k\} = \mathcal{I}(\{c_n(\varphi)\}) = \mathcal{I}(\{C_{2n}(\psi); C_{2n+1}(\psi)\}), \quad (4.6)$$

and it can be readily written down by means of formulas (2.6), (2.7). One of the main points of the present section is to study the direct spectral map \mathcal{D} ; the properties of the inverse spectral map \mathcal{I} can be treated from a similar point of view, but we do not pursue in this direction here.

We assume that $\psi \in L^2[-1, 1]$, or, equivalently,

$$\|\psi\|_{L^2[-1,1]}^2 = \sum_{k=0}^{\infty} \frac{2}{2k+1} |a_k|^2 < +\infty.$$

The corresponding weighted l^2 -space is denoted by $l^2(\{2/(2k+1)\})$. It follows immediately from the definition of \mathcal{S}_p , $p > 0$, that $T_\varphi \in \mathcal{S}_p$ iff $\{c_n(\varphi)\} \in l^p$, $p > 0$.

We wish to understand what sequences $\{c_n(\varphi)\}_n$ can be generated by functions $\psi \in L^2[-1, 1]$ with the help of map \mathcal{D} , see (4.5).

4.2. Some auxiliary results and the proof of Theorem 1.1. As explained above, we do the computations for the sequences with even indices, i.e., we suppose $\psi = \sum_{k=0}^{\infty} a_{2k} P_{2k}$. We denote by \mathcal{D}_e the restriction of the map \mathcal{D} to the even-indexed subsequences, i.e.,

$$\{C_{2n}(\psi)\}_n = \mathcal{D}_e(\{a_{2k}\}_k).$$

Notice also that, for any n ,

$$d_{2n,0} = 1, \quad d_{2n,2n} = \frac{(2n+1)!}{(4n+1)!!} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$d_{2n,2k+2} = d_{2n,2k} \frac{2n - 2k}{2n + 2k + 3}. \quad (4.7)$$

Lemma 4.1.

1. We have the matrix of \mathcal{D}_e is lower-triangular, i.e., $d_{2n,2k} = 0$, $k > n$, and $d_{2n,2k} > 0$ for $0 \leq k \leq n$. Moreover:

- (a) for any fixed n , $d_{2n,2k}$ is decaying with respect to k ,
- (b) for any fixed k , $\lim_{n \rightarrow +\infty} d_{2n,2k} = 1$.

2. We have

$$d_{2n,2k} \leq \exp(-Ck^2/n) \quad (4.8)$$

with an absolute constant $C > 0$.

Proof. Claim 1 of the lemma is clear from (4.3). Claim 2 is also easy to prove and it follows from (4.7). Indeed, we have

$$\begin{aligned} d_{2n,2k} &= \prod_{j=0}^{k-1} \left(\frac{2n - 2j}{2n + 2j + 3} \right) = \exp \left(\sum_{j=0}^{k-1} \log \left(\frac{2n - 2j}{2n + 2j + 3} \right) \right) \\ &\leq \exp \left(- \sum_{j=0}^{k-1} \frac{4j + 3}{2n + 2j + 3} \right) \leq \exp \left(- \frac{1}{2n + 2k + 1} \sum_{j=0}^{k-1} (4j + 3) \right) \\ &\leq \exp \left(-C \frac{k^2}{n} \right), \end{aligned}$$

where we used that $0 \leq j \leq k \leq n$. \square

The form of the map \mathcal{D}_e suggests that it might have very nice cancellation properties, that is, for an appropriate choice a sequence $\{a_{2k}\} \in l^2(\{2/(2k+1)\})$, the corresponding sequence $\{C_{2n}(\psi)\} = \mathcal{D}_e(\{a_{2k}\})$ could be in l^p , $0 < p < \infty$.

Lemma 4.2. Fix a $p \in \mathbb{N}$. There is a set coefficients $(A_j(\cdot))_{j=0,\dots,p}$, $A_j(\cdot) = A_{p,j}(\cdot)$ considered as functions of k , with the following properties:

1. For all n, k , we have

$$\sum_{j=0}^p d_{2n,2(k+j)} A_j(k) = d_{2n,2k} \prod_{j=1}^p \frac{1}{(2n + 2(k+j) + 1)}. \quad (4.9)$$

2. $A_j(k) = S_j(k)/T_p(k)$, where $S_j = S_{j,p}, T_p$ are polynomials with respect to k , $j = 0, \dots, p$, $\deg T_p = p(p+1)/2$ and

$$\deg S_0 = \deg S_1 = p(p+1)/2 - 1, \quad \deg S_j = p(p+1)/2 - j,$$

where $j = 2, \dots, p$. In particular, the rational functions $\{A_j(\cdot)\}_{j=0,\dots,p}$ have the degrees $\deg A_0 = \deg A_1 = -1$, $\deg A_j = -j$, $j = 2, \dots, p$.

3. The zeroes of polynomials $\{S_j\}_{j=0,\dots,p}, T_p$ depend on p only; in particular, they do not depend neither on k , nor on n .
4. It follows that, for a fixed n , and k large enough

$$C_1 \leq k|A_0(k)| \leq C_2, \quad C_1 \leq k|A_1(k)| \leq C_2, \quad C_1 \leq k^j|A_j(k)| \leq C_2,$$

where $j = 2, \dots, p$ and $C_1, C_2 > 0$.

Proof. To illustrate the conclusion of the lemma, let us take $p = 1$ first. So, we want to pick $A_0(k)$ and $A_1(k)$ with the property

$$d_{2n,2k}A_0(k) + d_{2n,2(k+1)}A_1(k) = d_{2n,2k} \frac{1}{2n + 2k + 3}.$$

Rewriting the left-hand side of this relation as

$$d_{2n,2k}A_0(k) + d_{2n,2(k+1)}A_1(k) = d_{2n,2k} \left(A_0(k) + A_1(k) \frac{(2n - 2k)}{(2n + 2k + 3)} \right),$$

we see that

$$A_0(k) = \frac{1}{4k + 3}, \quad A_1(k) = -\frac{1}{4k + 3},$$

and these A_0, A_1 satisfy the claim of the lemma.

The computation for a general $p \in \mathbb{N}$ is rather lengthy and tedious, though completely elementary. We give a sketch of the proof in this situation leaving the technical details to the reader.

Pick an arbitrary $p \in \mathbb{N}$, $p > 1$. At the *first stage* of the construction, we seek for coefficients $\{B_j\}_{j=1,\dots,p}$, depending on p only, such that

$$F_p(n) := F_{p,k}(n) = \frac{1}{\prod_{j=1}^p (2n + 2(k + j) + 1)} = \sum_{j=1}^p \frac{B_j}{(2n + 2(k + j) + 1)}. \quad (4.10)$$

We consider the function F_p as a (rational) function of n ; k and p are fixed parameters. Equating the residues at the poles on the right-hand side and on the left-hand side of (4.10), we express the coefficients B_j as

$$B_j = \text{Res}_{(-2(k+j)-1)/2} F_p(n) = \frac{(-1)^{j-1}}{2^{j-1}(j-1)!} \frac{1}{2^{p-j}(p-j)!},$$

where $j = 1, \dots, p$.

Going to the *second stage* of the computation, we seek the coefficients $\{A_j(\cdot)\}_{j=1,\dots,p}$ with the property

$$\begin{aligned} \sum_{j=0}^p d_{2n,2(k+j)}A_j(k) &= d_{2n,2k} \frac{1}{\prod_{j=1}^p (2n + 2(k + j) + 1)} \\ &= d_{2n,2k} \sum_{s=1}^p \frac{B_s}{(2n + 2(k + s) + 1)}. \end{aligned} \quad (4.11)$$

We define the function $G_p(n) := G_{p,k}(n)$ as

$$G_p(n) = \frac{1}{d_{2n,2k}} \sum_{j=0}^p d_{2n,2(k+j)} A_j(k). \quad (4.12)$$

Relation (4.11) reads as

$$G_p(n) = \sum_{j=0}^p A_j(k) \prod_{s=1}^j \left(\frac{2n - 2(k + s - 1)}{2n + 2(k + s) + 1} \right). \quad (4.13)$$

Of course,

$$\frac{2n - 2(k + s - 1)}{2n + 2(k + s) + 1} = 1 - \frac{4(k + s) - 1}{2n + 2(k + s) + 1},$$

and hence (see (4.11))

$$\sum_{j=0}^p d_{2n,2(k+j)} A_j(k) = \sum_{j=0}^p A_j \left\{ \prod_{s=1}^j \left(1 - \frac{4(k + s) - 1}{2n + 2(k + s) + 1} \right) \right\}.$$

As at the first stage of the proof of the lemma, we equate the residues on the right-hand side and on the left-hand side of (4.13). That is, we have

$$\begin{aligned} & \text{Res}_{-(2(k+j)+1)/2} G_p(n) \\ &= A_j \left\{ (-1)^j \left(\frac{4(k+j)-1}{2} \right) \prod_{\substack{s=1 \\ s \neq j}}^j \left(\frac{4k+2(s+j)-1}{2(s-j)} \right) \right\} + \dots \\ &+ A_r \left\{ (-1)^r \left(\frac{4(k+j)-1}{2} \right) \prod_{\substack{s=1 \\ s \neq j}}^r \left(\frac{4k+2(s+j)-1}{2(s-j)} \right) \right\} + \dots \\ &+ A_p \left\{ (-1)^p \left(\frac{4(k+j)-1}{2} \right) \prod_{\substack{s=1 \\ s \neq j}}^p \left(\frac{4k+2(s+j)-1}{2(s-j)} \right) \right\} = \frac{B_j}{2}, \end{aligned}$$

where $j = 1, \dots, p$, and $j \leq r \leq p$.

Now, it is convenient to set

$$\mathcal{A} = [A_1, A_2, \dots, A_p]^t, \quad \mathcal{B} = [B_1, \dots, B_p]^t,$$

and $p \times p$ matrix \mathcal{C} with entries

$$\mathcal{C}_{j,r} = \begin{cases} (-1)^r (4(k+j)-1) \prod_{\substack{s=0 \\ s \neq j}}^r \left(\frac{4k+2(s+j)-1}{2(s-j)} \right), & t \geq j \\ 0, & r < j \end{cases}. \quad (4.14)$$

Notice that the matrix \mathcal{C} is upper triangular; in particular, we have

$$T_p(k) := \det \mathcal{C} = \prod_{j=1}^p \mathcal{C}_{jj} \neq 0$$

for k large enough. As relation (4.14) clearly indicates, every entry $\mathcal{C}_{jj} = \mathcal{C}_{jj}(k)$ is a polynomial with respect to k of degree j , $j = 1, \dots, p$. So, $\deg T_p = p(p+1)/2$, as claimed in the lemma.

The system of equations on $\{A_j\}_{j=1, \dots, p}$, see (4.11), reads as $\mathcal{C}\mathcal{A} = \mathcal{B}$ and one has to add equation $A_0 = -\sum_{j=1}^p A_j$ defining A_0 to it. Trivially, we solve this system with the help of Cramer's rule, i.e.,

$$A_j(k) = \frac{\det \mathcal{C}_{j,\mathcal{B}}}{\det \mathcal{C}} =: \frac{S_{j,p}(k)}{T_p(k)},$$

where $S_j(k) = S_{j,p}(k) = \det \mathcal{C}_{j,\mathcal{B}}$ is a polynomial of k , and $\mathcal{C}_{j,\mathcal{B}}$ is the $p \times p$ -matrix obtained from \mathcal{C} with its j -th column replaced by \mathcal{B} . It is easy to see that $\deg S_j = p(p+1)/2 - j$.

Claim 3 of the lemma follows at once from 1 and 2. \square

Proof of Theorem 1.1. For a given $\delta_0 > 0$, pick a $p \in \mathbb{N}$ so that $1/(p-1) < \delta_0$. Take N_0 large enough to guarantee that the coefficients $(A_j(k))_{j=0, \dots, p}$ are well-defined for $k \geq (p+1)N_0$, see Lemma 4.2, and especially its claims 2 and 3.

We define the sequence $\{a_{2k}\}_k$ using the coefficients $\{A_j(k)\}_{j=0, \dots, p}$ from the above lemma. We have $k = (p+1)s + r$, where $0 \leq r < p+1$, and we set

$$a_{2k} := A_r((p+1)s), \quad k \geq (p+1)N_0. \quad (4.15)$$

or, equivalently

$$\begin{aligned} & (a_{2(p+1)s}, a_{2((p+1)s+1)}, \dots, a_{2((p+1)s+p)}) \\ & := \begin{cases} (0, \dots, 0), & s < N_0 \\ (A_0((p+1)s), A_1((p+1)s), \dots, A_p((p+1)s)), & s \geq N_0 \end{cases}. \end{aligned} \quad (4.16)$$

We set now

$$\psi_e(x) = \sum_{n=0}^{\infty} a_{2n} P_{2n}(x),$$

recall that this ψ defines the function φ on $[0, 1]$ by means of (4.1). We claim that ψ (and hence φ) possesses the properties stated in the theorem. Indeed,

$$\begin{aligned} \|\psi\|_{L^2[0,1]}^2 &= \frac{1}{2} \|\psi_e\|_{L^2[-1,1]}^2 = \frac{1}{2} \sum_n |a_{2n}|^2 \|P_{2n}\|^2 \\ &\leq C \sum_n \frac{2}{n^2(2n+1)} \leq C \sum_n \frac{1}{n^3} < +\infty, \end{aligned}$$

where we used that, for $(s+1)p \leq k \leq (s+1)p+p$

$$|a_{2k}| \leq \max_{j=0, \dots, p} |A_j((p+1)s)| \leq \frac{C_2}{k},$$

see Lemma 4.2, 3.

On the other hand, we have for ψ'_e by (2.5)

$$\begin{aligned} \psi' &= \sum_{k=0}^{\infty} a_{2k} P'_{2k} = \sum_{k=0}^{\infty} a_{2k} \left(\sum_{j=1}^k (4j-1) P_{2j-1} \right) \\ &= \sum_{j=1}^{\infty} (4j-1) P_{2j-1} \left(\sum_{k=j}^{\infty} a_{2k} \right). \end{aligned}$$

Set, for brevity, $R_j = \sum_{k=j}^{\infty} a_{2k}$. Recalling the definition of the sequence $\{a_{2k}\}_k$ (4.15), (4.16), we have

$$\sum_{k=(p+1)s}^{(p+1)s+p} a_{2k} = 0.$$

Consequently, for k large enough $k = (p+1)s + r$, $0 \leq r \leq p$, we get

$$\sum_{j=k}^{\infty} a_{2k} = \sum_{j=k}^{(p+1)s+p} a_{2k}.$$

In particular, the properties stated in Lemma 4.2 say that

$$\sum_{j=(p+1)s+1}^{(p+1)s+p} a_{2k} = -A_0((p+1)s),$$

and so

$$\left| \sum_{j=(p+1)s+1}^{(p+1)s+p} a_{2k} \right| = |A_0((p+1)s)| \geq \frac{C}{(p+1)s}.$$

Therefore,

$$\begin{aligned} \|\psi'_e\|_{L^2[-1,1]}^2 &= \sum_{j=1}^{\infty} (4j-1)^2 \|P_{2j-1}\|^2 |R_j|^2 \\ &\geq \sum_{s=N_0+1}^{\infty} (4(p+1)s+3)^2 \|P_{2(p+1)s+1}\|^2 |D_{(p+1)s+1}|^2 \\ &\geq C \sum_{s=N_0+1}^{\infty} \frac{(4(p+1)s+3)}{(p+1)s^2} = +\infty. \end{aligned}$$

It remains to show that $T_\psi \in \mathcal{S}_{\delta_0}$. For this purpose, we have to bound $C_{2n}(\psi)$ for n large enough. Once again, write $n = (p+1)l+r$, $0 \leq r \leq p$, and $l = [n/(p+1)]$. Recalling (4.2), we continue as

$$\begin{aligned} C_{2n}(\psi) &= \sum_{k=0}^n d_{2n,2k} a_{2k} = \sum_{s=0}^{l-1} \left(\sum_{j=0}^p d_{2n,2((p+1)s+j)} a_{2((p+1)s+j)} \right) \\ &\quad + \sum_{j=0}^r d_{2n,2((p+1)l+j)} a_{2((p+1)l+j)} =: (I) + (II). \end{aligned}$$

Since $d_{2n,2k} \leq \exp(-Ck^2/n)$, see (4.8), we easily obtain that $|(II)| \leq e^{-Cn}$. Recalling the definition of $\{a_{2k}\}_k$ (4.15), (4.16), and relation (4.9) from Lemma 4.2, we come to

$$\begin{aligned} (I) &= \sum_{s=0}^{l-1} \left(\sum_{j=0}^p d_{2n,2((p+1)s+j)} a_{2((p+1)s+j)} \right) \\ &= \sum_{s=0}^{l-1} \left(\sum_{j=0}^p d_{2n,2((p+1)s+j)} A_j((p+1)s) \right) \\ &= \sum_{s=0}^{[n/(p+1)]-1} d_{2n,2(p+1)s} \prod_{j=1}^p \frac{1}{(2n + 2((p+1)s + j) + 1)}. \end{aligned}$$

Take a small $0 < \delta' < 1/2$ and cut the above sum in two pieces: the first sum is taken for $s = 0, \dots, [n^{1/2+\delta'}/(p+1)]$, and second one is taken for $s = [n^{1/2+\delta'}/(p+1)] + 1, \dots, [n/(p+1)] - 1$. We have

$$\begin{aligned} &\left| \sum_{s=0}^{[n^{1/2+\delta'}/(p+1)]} d_{2n,2(p+1)s} \prod_{j=1}^p \frac{1}{(2n + 2((p+1)s + j) + 1)} \right| \\ &\leq C \sum_{s=0}^{[n^{1/2+\delta'}/(p+1)]} \frac{1}{n^p} \leq \frac{n^{1/2+\delta'}}{n^p} = \frac{1}{n^{p-1/2-\delta'}}, \end{aligned}$$

and, similarly,

$$\begin{aligned} &\left| \sum_{s=[n^{1/2+\delta'}/(p+1)]+1}^{[n/(p+1)]-1} d_{2n,2(p+1)s} \prod_{j=1}^p \frac{1}{(2n + 2((p+1)s + j) + 1)} \right| \\ &\leq C \sum_{s=[n^{1/2+\delta'}/(p+1)]+1}^{[n/(p+1)]-1} \frac{e^{-Cs^2/n}}{n^p} \\ &\leq C \sum_{s=[n^{1/2+\delta'}/(p+1)]+1}^{[n/(p+1)]-1} \frac{e^{-Cn^{2\delta'}}}{n^p} \leq C_\epsilon e^{-Cn^{2\delta'+\epsilon}} \end{aligned}$$

for any small $\epsilon > 0$. Hence, we obtain the bound

$$|C_{2n}(\psi)| \leq \frac{C}{n^{p-1/2-\delta'}}.$$

Since $0 < \delta' < 1/2$, it implies that $T_\varphi \in \mathcal{S}_{1/(p-1)}$; by the choice of p we have $1/(p-1) < \delta_0$ and therefore $T_\varphi \in \mathcal{S}_{\delta_0}$. The theorem is proved. \square

Few remarks are in order. First, using the asymptotics of Legendre polynomials $\{P_n\}$ [3, 6], one can show that the function φ constructed in the above theorem, is in $L^\infty[0, 1]$. Second, one can obviously get “a larger stock” of functions φ with the properties claimed in Theorem 1.1. That is, instead of definitions (4.15), (4.16), one can define the coefficients $\{a_{2k}\}_k$ as

$$\begin{aligned} & (a_{2(p+1)s}, a_{2((p+1)s+1)}, \dots, a_{2((p+1)s+p)}) \\ & := \begin{cases} (0, \dots, 0), & s < N_0 \\ ((p+1)s)^\delta (A_0((p+1)s), A_1((p+1)s), \dots, A_p((p+1)s)), & s \geq N_0 \end{cases} \end{aligned} \quad (4.17)$$

with $0 \leq \delta < 2$. Indeed, the computations show that, in this case

$$\begin{aligned} \|\psi_e\|_{L^2[-1,1]}^2 & \leq C \sum_n \frac{1}{n^{3-\delta}}, \\ \|\psi'_e\|_{L^2[-1,1]}^2 & \geq C \sum_n \frac{1}{n^{1-2\delta}}, \\ |C_{2n}(\psi)| & \leq \frac{C}{n^{p-(1/2+\delta')(1+\delta)}}. \end{aligned}$$

Finally, we would like to mention several open questions related to the subject of the present article. Suárez [23] proved that the C^* -algebra generated by the eigenvalues $\{c_n(\varphi)\}$ (2.2) of radial Toeplitz operators with bounded symbols φ , is dense in C^* -algebra of bounded and logarithmically oscillating sequences. Being given a bounded logarithmically oscillating sequence $\{\alpha_n\}$, is it possible to use the method of Section 4 to construct a bounded symbol φ such that $\{\alpha_n\} = \{c_n(\varphi)\}$? It is likely that the answer to the latter question is no; in that case, can one describe bounded logarithmically oscillating sequences coming as sequences of eigenvalues $\{c_n(\varphi)\}$ for a T_φ with a bounded symbol φ ?

It seems interesting to perform similar analysis for radial Toeplitz operators on weighted Bergman spaces $L_a^2(\mathbb{D}, dA_\alpha)$, $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA$, $\alpha > -1$. It is plausible that one will have to use Gegenbauer (or, more generally, Jacobi) orthogonal polynomials instead of Legendre polynomials in the construction of Section 4.

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Оператори Тепліца з радіальними символами на просторі Бергмана та класах Шаттена–фон Ноймана

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У цій роботі ми вивчаємо спектральні властивості операторів Тепліца з (квазі)радіальними символами на просторі Бергмана. Точніше, проблема, яка нас цікавить, полягає в тому, щоб зрозуміти коли даний оператор Тепліца належить класу Шаттена–фон Ноймана. Для розв'язання цього напрямку використано методи теорії апроксимації (тобто поліноми Лежандра).

Ключові слова: оператори Тепліца, (квазі)радіальні символи, простори Бергмана, класи Шаттена–фон Ноймана, поліноми Лежандра.