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Fractional Boundary Value Problem on the Half-Line

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We consider the semilinear fractional boundary value problem

$$D^{\beta}\left(\frac{1}{b(x)}D^{\alpha}u\right) = a(x)u^{\sigma} \quad \text{in } (0,\infty)$$

with the conditions $\lim_{x\to 0} x^{2-\beta}\frac{1}{b(x)}D^{\alpha}u(x) = \lim_{x\to\infty} x^{1-\beta}\frac{1}{b(x)}D^{\alpha}u(x) = 0$ and $\lim_{x\to 0} x^{2-\alpha}u(x) = \lim_{x\to\infty} x^{1-\alpha}u(x) = 0$, where $\beta, \alpha \in (1,2)$, $\sigma \in (-1,1)$ and D^{β}, D^{α} stand for the standard Riemann–Liouville fractional derivatives. The functions $a, b : (0,\infty) \to \mathbb{R}$ are nonnegative continuous functions satisfying some appropriate conditions. The existence and the uniqueness of a positive solution are established. Also, a description of the global behavior of this solution is given.

Key words: fractional differential equation, positive solution, Schauder fixed point theorem.

Mathematical Subject Classification 2010: 34A08, 35B09, 47H10.

1. Introduction

In this paper, we consider the nonlinear boundary value problem of the fractional differential equation

$$\begin{cases} D^{\beta}(\dfrac{1}{b(x)}D^{\alpha}u) = a(x)u^{\sigma}, \quad x \in (0,\infty), \\ \lim\limits_{x\to 0} x^{2-\beta}\dfrac{1}{b(x)}D^{\alpha}u(x) = \lim\limits_{x\to\infty} x^{1-\beta}\dfrac{1}{b(x)}D^{\alpha}u(x) = 0, \\ \lim\limits_{x\to 0} x^{2-\alpha}u(x) = \lim\limits_{x\to\infty} x^{1-\alpha}u(x) = 0, \end{cases} \tag{1.1}$$

where $\beta, \alpha \in (1,2)$ and $\sigma \in (-1,1)$. The functions a and b are positive and continuous in $(0,\infty)$. They may be singular at $x = 0$ and satisfy some conditions related to Karamata's regular variation theory. Our goal is to study the existence, uniqueness and exact asymptotic behavior of positive solutions for problem (1.1).

Many papers on fractional differential equations have been recently received much attention. The motivation for those works stems from the fact that fractional equations serve as an excellent tool to describe many phenomena in various fields of science and engineering such as viscoelasticity, electro-chemistry, control theory, porous media, electromagnetism, etc. For examples and details, see [15, 17, 24].

Therefore, the existence, uniqueness and asymptotic behavior of positive continuous solutions related to fractional differential equations have been developed very quickly by many researchers. Most of the results are focused on developing the global existence and uniqueness of the solution on the finite intervals (see [3, 5, 6, 9, 10, 14, 16, 19, 20, 25, 29]). However, to the best of our knowledge, there are few papers devoted to the study of fractional differential equations on the half-line, see, for instance, [1, 2, 4, 7, 8, 13, 18, 21, 22, 27, 28].

This work is motivated by recent advances in the study of fractional differential equations involving nonlinearities with different boundary conditions. Namely, in [20], Liu considered the fractional differential equation

$$D^{\beta}(\rho(x)\Phi_p(D^{\alpha}u(x))) = f(x, u(x)), \quad x \in (0,1),$$

where $0 < \alpha, \beta \leq 1$, $\rho \in \mathcal{C}(0,1)$, and f is a nonnegative function on $(0,1] \times \mathbb{R}$ allowed to be singular at $x = 0$. The author proved the existence of positive solution with fractional nonlocal integral boundary conditions.

In [8], Bachar and Mâagli considered the problem on the half-line

$$\begin{cases} D^{\alpha}u(x) = -a(x)u^{\sigma}, \quad x \in (0,\infty),\ 1 < \alpha < 2, \\ \lim\limits_{x\to 0^+} x^{2-\alpha}u(x) = \lim\limits_{x\to\infty} x^{1-\alpha}u(x) = 0, \end{cases} \tag{1.2}$$

where $-1 < \sigma < 1$ and the function a is a nonnegative continuous function on $(0,\infty)$ that may be singular at 0. To describe the result of [8] in more details, we need some notations. We first introduce the following Karamata's classes.

Definition 1.1. The classes $\mathcal{K}$ and $\mathcal{K}^{\infty}$ are the sets of all Karamata's functions defined respectively on $(0,\eta]$, $(\eta > 1)$ and $[1,\infty)$ by

$$\mathcal{K} := \left\{ L(t) = c\exp\left(\int_t^{\eta} \frac{z(s)}{s}\,ds\right) :\ c > 0,\ z \in \mathcal{C}[0,\eta],\ z(0) = 0 \right\} \tag{1.3}$$

and

$$\mathcal{K}^{\infty} := \left\{ L(t) = c\exp\left(\int_1^{t} \frac{z(s)}{s}ds\right) :\ c > 0,\ z \in \mathcal{C}[1,\infty),\ \lim_{t\to\infty} z(t) = 0 \right\}. \tag{1.4}$$

For $\lambda \leq 2 + (\alpha-2)\sigma$, $\mu \geq 1 + (\alpha-1)\sigma$, $L \in \mathcal{K}$ defined on $(0,\eta]$, $(\eta > 1)$ and $\tilde{L} \in \mathcal{K}^{\infty}$, we define the functions $\Psi_{L,\lambda,\sigma}$ and $\Phi_{\tilde{L},\mu,\sigma}$ respectively on $(0,\eta)$ and $[1,\infty)$ by

$$\Psi_{L,\lambda,\sigma}(t) := \begin{cases} 1, & \text{if } \lambda < 1 + (\alpha-1)\sigma \\ \left(\int_t^{\eta} \frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 1 + (\alpha-1)\sigma \\ (L(t))^{\frac{1}{1-\sigma}} & \text{if } 1 + (\alpha-1)\sigma < \lambda < 2 + (\alpha-2)\sigma \\ \left(\int_0^{t} \frac{L(s)}{s}ds\right)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 2 + (\alpha-2)\sigma \end{cases}.$$

and

$$\Phi_{\tilde{L},\mu,\sigma}(t) := \begin{cases} 1 & \text{if } \mu > 2+(\alpha-2)\sigma \\ \left(\int_1^{t+1} \frac{\tilde{L}(s)}{s} ds\right)^{\frac{1}{1-\sigma}} & \text{if } \mu = 2+(\alpha-2)\sigma \\ \left(\tilde{L}(t)\right)^{\frac{1}{1-\sigma}} & \text{if } 1+(\alpha-1)\sigma < \mu < 2+(\alpha-2)\sigma \\ \left(\int_t^{\infty} \frac{\tilde{L}(s)}{s} ds\right)^{\frac{1}{1-\sigma}} & \text{if } \mu = 1+(\alpha-1)\sigma \end{cases}.$$

Also, we denote by $\mathcal{C}_{2-\alpha}[0,\infty)$ the set of all functions f such that $t \mapsto t^{2-\alpha} f(t)$ is continuous on $[0,\infty)$. For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $\frac{1}{c} f(x) \leq g(x) \leq c f(x)$ for all $x \in S$.

In [8], Bachar and Mâagli studied problem (1.2) where a satisfies the following condition:

(H$_0$) $a \in C(0,\infty)$ such that for each $x \in (0,\infty)$,

$$a(x) \approx x^{-\lambda}(1+x)^{\lambda-\mu} L(\min(x,1)) \tilde{L}(\max(x,1)),$$

where $\lambda \leq 2+(\alpha-2)\sigma$, $\mu \geq 1+(\alpha-1)\sigma$, $L \in \mathcal{K}$ and $\tilde{L} \in \mathcal{K}^{\infty}$ satisfy

$$\int_0^{\eta} \frac{L(t)}{t^{\lambda-(\alpha-2)\sigma-1}} dt < \infty \quad \text{and} \quad \int_1^{\infty} \frac{\tilde{L}(t)}{t^{\mu-(\alpha-1)\sigma}} dt < \infty.$$

In [8], the authors, basing on the Schauder fixed-point theorem, showed the following result.

Theorem 1.2. *Assume that a satisfies* **(H$_0$)**. *Then problem* (1.2) *has a unique positive solution $u \in C_{2-\alpha}[0,\infty)$ satisfying for $x \in (0,\infty)$,*

$$u(x) \approx x^{\alpha-2+\nu}(1+x)^{\zeta-\nu} \Psi_{L,\lambda,\sigma}(\min(x,1)) \Phi_{\tilde{L},\mu,\sigma}(\max(x,1)), \tag{1.5}$$

where $\nu = \min\left(1, \frac{2-\lambda+(\alpha-2)\sigma}{1-\sigma}\right)$ and $\zeta = \max\left(0, \frac{2-\mu+(\alpha-2)\sigma}{1-\sigma}\right)$.

In this paper, we improve and extend the above results on the boundary behavior of solutions to problem (1.1). Let us consider the following hypotheses.

(H$_1$) $a, b \in C(0,\infty)$ satisfy for each $x \in (0,\infty)$,

$$a(x) \approx x^{-\lambda}(1+x)^{\lambda-\mu} L_1(\min(x,1)) L_2(\max(x,1))$$

and

$$b(x) \approx x^{\lambda-\beta-2-(\alpha-2)\sigma}(1+x)^{-(\lambda-\beta-2-(\alpha-2)\sigma)-r} \tilde{L}_1(\min(x,1)) \tilde{L}_2(\max(x,1)),$$

where $1 < \lambda-(\alpha-2)\sigma < 2$, $\mu, r \in \mathbb{R}$, $L_1, \tilde{L}_1 \in \mathcal{K}$ and $L_2, \tilde{L}_2 \in \mathcal{K}^{\infty}$ satisfy

$$\int_0^{\eta} \frac{L_1(t)\tilde{L}_1(t)}{t} dt < \infty \quad \text{and} \quad \int_1^{\infty} \frac{L_2(t)\tilde{L}_2(t)}{t} dt < \infty.$$

Our main result is the following.

Theorem 1.3. *Assume* **(H$_1$)**. *Then problem* (1.1) *has a unique positive solution* $u \in C_{2-\alpha}[0,\infty)$ *satisfying for* $x \in (0,\infty)$,

$$u(x) \approx x^{\alpha-2}\left(\int_0^{\min(x,1)} \frac{L_1(s)\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma}}\varphi(x),$$

where the function φ *is defined on* $(0,\infty)$ *by*

- *if* $1 < \mu - (\alpha-2)\sigma - \sigma < 2$ *and* $r = 1 - \mu + \beta + (\alpha-2)\sigma + \sigma$, *then*

$$\varphi(x) = \max(x,1)\left(\int_{\max(x,1)}^{\infty} \frac{L_2(s)\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}};$$

- *if* $1 < \mu - (\alpha-2)\sigma - (2 - \frac{\mu-\alpha\sigma+r-\beta}{1-\sigma})\sigma < 2$ *and* $\beta - \mu + \alpha\sigma - \sigma + 1 < r < \beta - \mu + \alpha\sigma - 2\sigma + 2$, *then*

$$\varphi(x) = (\max(x,1))^{2-\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma}}\left(L_2(\max(x,1))\tilde{L}_2(\max(x,1))\right)^{\frac{1}{1-\sigma}};$$

- *if* $1 < \mu - (\alpha-2)\sigma < 2$ *and* $r = 2 - \mu + (\alpha-2)\sigma + \beta$, *then*

$$\varphi(x) = \left(\int_1^{1+\max(x,1)} \frac{L_2(s)\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}};$$

- *if* $1 < \mu - (\alpha-2)\sigma < 2$ *and* $2 - \mu + (\alpha-2)\sigma + \beta < r$, *then*

$$\varphi(x) = 1.$$

The rest of the paper is as follows. In Section 2, we give some necessary definitions and lemmas from the fractional calculus theory and already known results on the functions in Karamata's classes as well as the estimates on Green's function. In Section 3, we present some necessary conditions to the existence result and prove our main results stated in Theorem 1.3. The last section is reserved to an example.

2. Preliminary Results

2.1. Fractional calculus. We begin this subsection with some definitions and fundamental facts of the fractional calculus theory, which can be found in [17, 24].

Definition 2.1. Let $\gamma > 0$, the Riemann–Liouville fractional integral of order γ of a measurable function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I^{\gamma}f(x) = \frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}f(t)dt,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here Γ is the Euler gamma function.

Definition 2.2. The Riemann–Liouville derivative of order $\gamma > 0$ of a measurable function $f: (0,\infty) \to \mathbb{R}$ is given by

$$\begin{aligned} D^\gamma f(x) &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\gamma-1} f(t)dt \\ &= \left(\frac{d}{dx}\right)^n I^{n-\gamma} f(x), \end{aligned}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here $n = [\gamma] + 1$, where $[\gamma]$ means the integer part of the number γ.

Lemma 2.3. *Let $\alpha, \gamma > 0$ and $f \in L^1(0,\infty)$. Then we have:*

(i) $I^\alpha I^\gamma f(x) = I^{\alpha+\gamma} f(x)$ *for* $x \in [0,\infty)$ *and* $\alpha + \gamma \geq 1$*;*

(ii) $D^\alpha I^\alpha f(x) = f(x)$ *for almost all* $x \in [0,\infty)$*;*

(iii) $D^\alpha f(x) = 0$ *if and only if* $f(x) = \sum_{j=1}^n c_j x^{\alpha-j}$*, where n is the smallest integer greater than or equal to α and $(c_1, \dots, c_n) \in \mathbb{R}^n$.*

2.2. Karamata's properties. In this subsection, we quote some fundamental properties of functions belonging to the classes $\mathcal{K}$ and $\mathcal{K}^\infty$ collected from [12, 23, 26].

Proposition 2.4. *The following assertions hold.*

(i) *A function L is in $\mathcal{K}$ if and only if L is a positive function in $\mathcal{C}^1(0,\eta]$, for some $\eta > 1$, such that*

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{2.1}$$

(ii) *A function L is in $\mathcal{K}^\infty$ if and only if L is a positive function in $\mathcal{C}^1[1,\infty)$ such that*

$$\lim_{t\to +\infty} \frac{tL'(t)}{L(t)} = 0. \tag{2.2}$$

Proposition 2.5. *The following assertions hold.*

(i) *Let $L_1,\ L_2 \in \mathcal{K}$ (respectively, $\mathcal{K}^\infty$) and $p \in \mathbb{R}$. Then we have*

$$L_1 + L_2, L_1 L_2 \text{ and } L_1^p \text{ belong to } \mathcal{K} \quad (\text{respectively, } \mathcal{K}^\infty).$$

(ii) *Let $L \in \mathcal{K}$ (respectively, $\mathcal{K}^\infty$) and $\varepsilon > 0$. Then we have*

$$\lim_{t\to 0^+} t^\varepsilon L(t) = 0, \quad (\text{respectively, } \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0).$$

Lemma 2.6 ([12])**.** *The following assertions hold.*

(i) *Let L be a function in $\mathcal{K}$. Then we have*

$$\lim_{t\to 0^+}\frac{L(t)}{\displaystyle\int_t^\eta \frac{L(s)}{s}\,ds}=0.$$

In particular,

$$t\longmapsto \int_t^\eta \frac{L(s)}{s}\,ds\in\mathcal{K}.$$

(ii) *If $\int_0^\eta \frac{L(s)}{s}\,ds$ converges, then*

$$\lim_{t\to 0^+}\frac{L(t)}{\displaystyle\int_0^t \frac{L(s)}{s}\,ds}=0.$$

In particular,

$$t\longmapsto \int_0^t \frac{L(s)}{s}\,ds\in\mathcal{K}.$$

Lemma 2.7 ([11])**.** *The following assertions hold.*

(i) *Let L be a function in $\mathcal{K}^\infty$. Then we have*

$$\lim_{t\to \infty}\frac{L(t)}{\displaystyle\int_1^t \frac{L(s)}{s}ds}=0.$$

In particular,

$$t\longmapsto \int_1^{t+1} \frac{L(s)}{s}ds\in\mathcal{K}^\infty.$$

(ii) *If $\int_1^\infty \frac{L(s)}{s}\,ds$ converges, then*

$$\lim_{t\to \infty}\frac{L(t)}{\displaystyle\int_t^\infty \frac{L(s)}{s}ds}=0.$$

In particular,

$$t\longmapsto \int_t^\infty \frac{L(s)}{s}ds\in\mathcal{K}^\infty.$$

Applying Karamata's theorem, we get the next results.

Lemma 2.8. *Let $\gamma\in\mathbb{R}$ and let L be a function in $\mathcal{K}$ defined on $(0,\eta]$ for some $\eta>1$. We have*

(i) *If $\gamma>-1$, then $\int_0^\eta s^\gamma L(s)\,ds$ converges and*

$$\int_0^t s^\gamma L(s)\,ds \underset{t\to 0^+}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

(ii) *If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s)\, ds$ diverges and*

$$\int_t^\eta s^\gamma L(s)\, ds \underset{t\to 0^+}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

Lemma 2.9. *Let $\gamma \in \mathbb{R}$ and let L be a function in $\mathcal{K}^\infty$. We have:*

(i) *if $\gamma > -1$, then $\int_1^\infty s^\gamma L(s)\, ds$ diverges and*

$$\int_1^t s^\gamma L(s) ds \underset{t\to\infty}{\sim} \frac{t^{1+\gamma}L(t)}{1+\gamma};$$

(ii) *if $\gamma < -1$, then $\int_1^\infty s^\gamma L(s) ds$ converges and*

$$\int_t^\infty s^\gamma L(s)\, ds \underset{t\to\infty}{\sim} -\frac{t^{1+\gamma}L(t)}{1+\gamma}.$$

2.3. Estimates on Green's function. Since our approach is based on the potential theory, we should recall some basic tools. For $\gamma \in (1,2)$ and $f \in L^1(0,\infty)$, by $G_\gamma(x,t)$, we denote Green's function for the following boundary value problem:

$$\begin{cases} D^\gamma u(x) = -f(x), \quad x \in (0,\infty), \\ \lim\limits_{x\to 0^+} x^{2-\gamma}u(x) = \lim\limits_{x\to\infty} x^{1-\gamma}u(x) = 0. \end{cases}$$

From [8], we have

$$G_\gamma(x,t) = \frac{1}{\Gamma(\gamma)}[x^{\gamma-1} - ((x-t)^+)^{\gamma-1}],$$

where $x^+ = \max(x,0)$.

Proposition 2.10 ([8])**.** *Let $1 < \gamma < 2$ and let f be a nonnegative measurable function on $(0,\infty)$. Then we have:*

(i) *for $x,t \in (0,\infty)$,*

$$G_\gamma(x,t) \approx x^{\gamma-2}\min(x,t);$$

(ii) *for $x \in (0,\infty)$,*

$$x \to G_\gamma f(x) := \int_0^\infty G_\gamma(x,t)f(t)\, dt$$

belongs to $C_{2-\gamma}[0,\infty)$ if and only if

$$\int_0^\infty \min(1,t)f(t)dt < \infty;$$

(iii) *if the map $t \to \min(1,t)f(t)$ is continuous and integrable on $(0,\infty)$, then $G_\gamma f$ is the unique solution in $C_{2-\gamma}[0,\infty)$ of the boundary value problem*

$$\begin{cases} D^\gamma u(x) = -f(x), \quad x \in (0,\infty), \\ \lim\limits_{x\to 0^+} x^{2-\gamma}u(x) = \lim\limits_{x\to\infty} x^{1-\gamma}u(x) = 0. \end{cases}$$

To this end, we need the following lemma.

Lemma 2.11. *For $x, t \in (0, \infty)$,*

$$\min(1,x)\min(1,t) \leq \min(x,t) \leq \max(1,x)\min(1,t).$$

Remark 2.12. By Proposition 2.10 (i) and Lemma 2.11, there exists a positive constant c such that for $x, t \in (0, \infty)$,

$$\frac{1}{c}x^{\gamma-2}\min(1,x)\min(1,t) \leq G_\gamma(x,t) \leq cx^{\gamma-2}\max(1,x)\min(1,t).$$

Lemma 2.13. *Let $\alpha, \beta \in (1,2)$. Let f be a nonnegative measurable function in $(0,\infty)$ such that $x \longmapsto \min(1,x)f(x)$ and $x \longmapsto \min(1,x)b(x)G_\beta f(x)$ are continuous and integrable on $(0,\infty)$. Then the boundary value problem*

$$\begin{cases} D^\beta(\frac{1}{b(x)}D^\alpha u) = f, \quad x \in (0,\infty), \\ \lim\limits_{x\to 0} x^{2-\beta}\frac{1}{b(x)}D^\alpha u(x) = \lim\limits_{x\to\infty} x^{1-\beta}\frac{1}{b(x)}D^\alpha u(x) = 0, \\ \lim\limits_{x\to 0} x^{2-\alpha}u(x) = \lim\limits_{x\to\infty} x^{1-\alpha}u(x) = 0 \end{cases} \tag{2.3}$$

has a unique positive solution in $\mathcal{C}_{2-\alpha}[0,\infty)$ given by

$$u(x) = G_\alpha\Big(bG_\beta f\Big)(x) := \int_0^\infty G_\alpha(x,t)b(t)\int_0^\infty G_\beta(t,s)f(s)\,ds\,dt.$$

Proof. Since $x \longmapsto \min(1,x)f(x)$ is continuous and integrable on $(0,\infty)$, we deduce by Proposition 2.10 (iii) that for $x \in (0,\infty)$ we have

$$\frac{1}{b(x)}D^\alpha u(x) = -G_\beta f(x).$$

Thus,

$$D^\alpha u(x) = -b(x)G_\beta f(x).$$

In addition, using the fact that $x \longmapsto \min(1,x)b(x)G_\beta f(x)$ is continuous and integrable on $(0,\infty)$, we conclude again by Proposition 2.10 (iii) that problem (2.3) has a unique solution u in $\mathcal{C}_{2-\alpha}([0,\infty)$ given by

$$u(x) = G_\alpha\Big(bG_\beta f\Big)(x), \quad x > 0. \qquad \square$$

Below we provide a crucial property on the continuity.

Lemma 2.14. *Let h and b be two nonnegative measurable functions on $(0,\infty)$ such that*

$$\int_0^\infty \min(1,t)b(t)G_\beta h(t)\,dt < \infty.$$

Then the family

$$\mathcal{F} = \{Sf : x \longmapsto \frac{x^{2-\alpha}}{x+1}G_\alpha(bG_\beta f))(x);\ |f| \leq h\}$$

is relatively compact in $C_0[0,\infty)$.

Proof. Let f be a measurable function in $(0,\infty)$ such that $|f(x)| \leq h(x)$ for all $x \in (0,\infty)$.

Using Remark 2.12 1, there exists $c > 0$ such that for all $x, t > 0$ we have

$$\frac{x^{2-\alpha} G_\alpha(x,t)}{1+x} \leq c \min(1,t).$$

This implies that for all $x \in (0,\infty)$,

$$\begin{aligned} |Sf(x)| = \left| \frac{x^{2-\alpha}}{x+1} G_\alpha(bG_\beta f)(x) \right| &= \frac{x^{2-\alpha}}{x+1} \left| \int_0^\infty G_\alpha(x,t) b(t) G_\beta f(t)\, dt \right| \\ &\leq c \int_0^\infty \min(1,t) b(t) G_\beta h(t)\, dt < \infty. \end{aligned}$$

Thus $\mathcal{F}$ is uniformly bounded. Now, let us prove that $\mathcal{F}$ is equicontinuous in $[0,\infty]$. Let $x, y \in (0,\infty)$, then we have

$$\begin{aligned} |Sf(x) - Sf(y)| &= \left| \frac{x^{2-\alpha}}{x+1} G_\alpha(bG_\beta f)(x) - \frac{y^{2-\alpha}}{y+1} G_\alpha(bG_\beta f)(y) \right| \\ &= \left| \int_0^\infty \left(\frac{x^{2-\alpha} G_\alpha(x,t)}{1+x} - \frac{y^{2-\alpha} G_\alpha(y,t)}{1+y} \right) b(t) G_\beta f(t)\, dt \right| \\ &\leq \int_0^\infty \left| \frac{x^{2-\alpha} G_\alpha(x,t)}{1+x} - \frac{y^{2-\alpha} G_\alpha(y,t)}{1+y} \right| b(t) G_\beta h(t)\, dt. \end{aligned}$$

For every $t \in (0,1)$, we have

$$\left| \frac{x^{2-\alpha} G_\alpha(x,t)}{1+x} - \frac{y^{2-\alpha} G_\alpha(y,t)}{1+y} \right| \to 0 \quad \text{as } |x-y| \to 0$$

and

$$\left| \frac{x^{2-\alpha} G_\alpha(x,t)}{1+x} - \frac{y^{2-\alpha} G_\alpha(y,t)}{1+y} \right| \leq 2c \min(1,t).$$

Then, by Lebesgue's theorem, we obtain that

$$|Sf(x) - Sf(y)| \to 0 \quad \text{as } |x-y| \to 0.$$

Now, let $x \in (0,\infty)$. Using the fact that $x \to \frac{x^{2-\alpha} G_\alpha(x,t)}{1+x}$ is in $\mathcal{C}_0([0,\infty))$, again by Lebesgue's theorem, we get that

$$\big|Sf(x)\big| \to 0 \quad \text{as } x \to 0 \text{ or } x \to \infty.$$

Finally, we conclude that the family $\mathcal{F}$ is equicontinuous in $[0,\infty]$. Hence, by Ascoli's theorem, we deduce that $\mathcal{F}$ is relatively compact in $\mathcal{C}_0[0,\infty)$. □

The following Lemma is due to [8].

Lemma 2.15. *Let $\tilde{L}_1 \in \mathcal{K}$, $\tilde{L}_2 \in \mathcal{K}^\infty$ and let for $x \in (0,\infty)$,*

$$b(x) = x^{-\lambda_1}(1+x)^{\lambda_1-\mu_1}\tilde{L}_1(\min(x,1))\tilde{L}_2(\max(x,1))$$

with $\lambda_1 \leq 2$ and $\mu_1 \geq 1$. Assume that

$$\int_0^\eta t^{1-\lambda_1}\tilde{L}_1(t)\,dt < \infty \quad \text{and} \quad \int_1^\infty t^{-\mu_1}\tilde{L}_2(t)\,dt < \infty.$$

Then we have for $x \in (0,\infty)$,

$$\begin{aligned} x^{2-\beta}G_\beta b(x) \approx\, & (\min(x,1))^{\min(2-\lambda_1,1)}(\max(x,1))^{\max(2-\mu_1,0)} \\ & \times \Psi_{\tilde{L}_1,\lambda_1,0}(\min(x,1))\Phi_{\tilde{L}_2,\mu_1,0}(\max(x,1)). \end{aligned}$$

Remark 2.16. We need to verify the conditions $\int_0^\eta t^{1-\lambda_1}\tilde{L}_1(t)\,dt < \infty$ and $\int_1^\infty t^{-\mu_1}\tilde{L}_2(t)\,dt < \infty$ in Lemma 2.15 only if $\lambda_1 = 2$ and $\mu_1 = 1$. This is due to Lemmas 2.8 and 2.9.

3. Proof of main results

We begin this section by stating the proposition that will play a crucial role in proving our main result.

Proposition 3.1. *Assume* **($\mathbf{H}_1$)** *and suppose that there exists a nonnegative function θ in $C[0,\infty)$ such that $t \to w(t) := t^{(\alpha-2)\sigma}a(t)\theta^\sigma(t)$ satisfies*

$$\int_0^\infty \min(1,t)w(t)dt < \infty, \tag{3.1}$$

and

$$x^{2-\alpha}G_\alpha(bG_\beta w)(x) \approx \theta(x). \tag{3.2}$$

Then problem (1.1) has a unique solution $u \in C_{2-\alpha}[0,\infty)$ satisfying for each $x \in (0,\infty)$,

$$u(x) \approx x^{\alpha-2}\theta(x). \tag{3.3}$$

Proof. Let $m \geq 1$ and let θ be a nonnegative function satisfying for each $x \in (0,\infty)$,

$$\frac{1}{m}\theta(x) \leq x^{2-\alpha}G_\alpha(bG_\beta w)(x) \leq m\theta(x). \tag{3.4}$$

Then, for each $x \in (0,\infty)$,

$$\frac{1}{m}\frac{\theta(x)}{x+1} \leq \frac{x^{2-\alpha}}{x+1}G_\alpha(bG_\beta w)(x) \leq m\frac{\theta(x)}{x+1}. \tag{3.5}$$

Existence: Put $c_0 := m^{\frac{1}{1-|\sigma|}}$. We consider the closed convex set given by

$$Y := \left\{ v \in \mathcal{C}_0[0,\infty);\ \frac{1}{c_0(1+x)}\theta(x) \leq v(x) \leq \frac{c_0}{1+x}\theta(x) \right\}.$$

Using Remark 1 and (3.2), there exists $c > 0$ such that

$$\frac{x^{2-\alpha}G_\alpha(x,t)}{1+x} \le c\min(1,t) \quad \text{for all } x,t \in (0,\infty)$$

and

$$\int_0^\infty \min(1,t)b(t)G_\beta w(t)\,dt < \infty. \tag{3.6}$$

This implies with Lemma 2.14 that the function $x \mapsto \frac{x^{2-\alpha}}{x+1}G_\alpha(bG_\beta w)(x)$ belongs to $\mathcal{C}_0([0,\infty))$ and satisfies (3.5). Thus Y is not empty. In order to use Schauder's fixed point theorem, we denote $\tilde{a}(x) = x^{(\alpha-2)\sigma}(1+x)^\sigma a(x)$ and define the operator T on Y by

$$Tv(x) = \frac{x^{2-\alpha}}{1+x}G_\alpha(bG_\beta(\tilde{a}v^\sigma))(x).$$

We need to check that the operator T has a fixed point v in Y. For this choice of c_0, we will prove that T maps Y into itself. Indeed, let $v \in Y$. By using (3.5), we have

$$Tv(x) \le c_0^{|\sigma|}\frac{x^{2-\alpha}}{1+x}G_\alpha(bG_\beta w))(x) \le c_0^{|\sigma|}m\frac{\theta(x)}{1+x} = c_0\frac{\theta(x)}{1+x}$$

and

$$Tv(x) \ge c_0^{-|\sigma|}\frac{x^{2-\alpha}}{1+x}G_\alpha(bG_\beta w)(x) \ge \frac{c_0^{-|\sigma|}}{m}\frac{\theta(x)}{1+x} = \frac{1}{c_0}\frac{\theta(x)}{1+x}.$$

Then, using (3.6) and the fact that

$$\tilde{a}(x)v^\sigma(x) \le c_0^{|\sigma|}x^{(\alpha-2)\sigma}a(x)\theta^\sigma(x) = c_0^{|\sigma|}w(x),$$

by Lemma 2.14, we deduce that TY is relatively compact in $\mathcal{C}_0[0,\infty)$. Thus Y is invariant under T. Next, we shall prove the continuity of T. Let $(v_k)_k$ be a sequence in Y which converges uniformly to v in Y. For $x \in (0,\infty)$, we have

$$\begin{aligned}|Tv_k(x) - Tv(x)| &= \frac{x^{2-\alpha}}{x+1}\left|G_\alpha(bG_\beta(\tilde{a}v_k^\sigma))(x) - G_\alpha(bG_\beta(\tilde{a}v^\sigma))(x)\right| \\ &\le \frac{x^{2-\alpha}}{x+1}\int_0^\infty G_\alpha(x,t)b(t)\left|G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t)\right|dt.\end{aligned}$$

For $t \in (0,\infty)$, we have

$$\left|G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t)\right| \le \int_0^\infty G_\beta(t,s)\left|(\tilde{a}v_k^\sigma)(s) - (\tilde{a}v^\sigma)(s)\right|ds$$

and, by Remark 2.16, we have that for every $s \in (0,\infty)$,

$$G_\beta(t,s)\left|(\tilde{a}v_k^\sigma)(s) - (\tilde{a}v^\sigma)(s)\right| \le 2cc_0^{|\sigma|}t^{\beta-2}\max(1,t)\min(1,s)w(s).$$

Hence, using (3.1), by Lebesgue's theorem, we obtain that

$$\left|G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t)\right| \to 0 \quad \text{as } k \to \infty.$$

We have

$$b(t)\left|G_\beta(\tilde{a}v_k^\sigma)(t) - G_\beta(\tilde{a}v^\sigma)(t)\right| \le 2c_0^{|\sigma|}b(t)G_\beta w(t)).$$

Using (3.6), by Lebesgue's theorem, we obtain that for $x \in (0,\infty)$,

$$Tv_k(x) \to Tv(x) \quad \text{as } k \to \infty.$$

Since TY is relatively compact in $\mathcal{C}_0([0,\infty))$, we have the uniform convergence, namely,

$$\|Tv_k - Tv\|_\infty \to 0 \quad \text{as } k \to \infty.$$

Thus, we have proved that T is a compact mapping from Y into itself. By Schauder's fixed-point theorem, it follows that there exists $v \in Y$ such that $Tv = v$. Put $u(x) = x^{\alpha-2}(1+x)v(x)$. Then $u \in \mathcal{C}_{2-\alpha}([0,\infty))$, and u satisfies the equation

$$u(x) = G_\alpha(bG_\beta(au^\sigma)))(x).$$

Then, due to Lemma 2.13, u is a positive solution in $\mathcal{C}_{2-\alpha}([0,\infty))$ of problem (1.1).

Uniqueness: Finally, let us prove that u is the unique positive continuous solution satisfying (3.3). To this aim, we assume that (1.1) has two positive solutions u and v satisfying (3.3). Then there exists a constant $m > 1$ such that

$$\frac{1}{m}v \le u \le mv.$$

This implies that the set

$$J := \left\{t \in (1,\infty) : \frac{1}{t}v \le u \le tv\right\}$$

is not empty. Now, putting $c := \inf J$, we are to show that $c = 1$. Suppose that $c > 1$. Then, by simple calculus, we obtain that

$$\begin{cases} D^\beta\big(\frac{1}{b}(D^\alpha(c^{|\sigma|}v) - D^\alpha u)\big) = a(c^{|\sigma|}v^\sigma - u^\sigma) \ge 0, \\ \lim\limits_{x\to 0} x^{2-\beta}\dfrac{1}{b(x)}\big(D^\alpha(c^{|\sigma|}v) - D^\alpha u\big)(x) = 0, \\ \lim\limits_{x\to \infty} x^{1-\beta}\dfrac{1}{b(x)}\big(D^\alpha(c^{|\sigma|}v) - D^\alpha u\big)(x) = 0. \end{cases}$$

By Proposition 2.10 (iii), we conclude that

$$\frac{1}{b}(D^\alpha(c^{|\sigma|}v)) - D^\alpha u\big) = -G_\beta(a(c^{|\sigma|}v^\sigma - u^\sigma)) \le 0.$$

Then we have

$$D^\alpha(c^{|\sigma|}v) \le D^\alpha u,$$

which implies that

$$\begin{cases} D^{\alpha}(c^{|\sigma|}v-u)\leq 0, \\ \lim\limits_{x\to 0} x^{2-\alpha}(c^{|\sigma|}v-u)(x)=0. \end{cases}$$

Using again Proposition 2.10 (iii), we conclude that

$$c^{|\sigma|}v-u\geq 0.$$

By the symmetry, we obtain that $v\leq c^{|\sigma|}u$. Thus, $c^{|\sigma|}\in J$. Since $|\sigma|<1$ and $c>1$, we have $c^{|\sigma|}<c$. This leads to a contradiction with the fact that $c=\inf J$. Hence $c=1$ and, consequently, $u=v$. □

Proof of Theorem 1.3. Consider $1<\lambda-(\alpha-2)\sigma<2$ and L_1, L_2, $\tilde{L}_1$,$\tilde{L}_2$ satisfying

$$\int_0^{\eta}\frac{L_1(t)\tilde{L}_1}{t}\,dt<\infty \quad\text{and}\quad \int_1^{\infty}\frac{L_2(t)\tilde{L}_2(t)}{t}\,dt<\infty.$$

Let

$$\xi:=\begin{cases} 1 & if\ 1<\mu-(\alpha-2)\sigma-\sigma<2, \\ 2-\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma} & if\ \ 1<\mu-(\alpha-2)\sigma+(\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma}-2)\sigma<2, \\ 0, & if\ 1<\mu-(\alpha-2)\sigma<2, \end{cases}$$

and let θ be the function defined on $[0,\infty)$ by

$$\theta(x)=(\max(x,1))^{\xi}\left(\int_0^{\min(x,1)}\frac{L_1(s)\tilde{L}_1(s)}{s}ds\right)^{\frac{1}{1-\sigma}}\chi(\max(x,1)),$$

where

$$\chi(t):=\begin{cases} \left(\int_t^{\infty}\frac{L_2(s)\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}} & \text{if } 1<\mu-(\alpha-2)\sigma-\sigma<2, \\ & r=1-\mu+\beta+(\alpha-2)\sigma+\sigma, \\ \left(L_2(t)\tilde{L}_2(t)\right)^{\frac{1}{1-\sigma}} & \text{if } 1<\mu-(\alpha-2)\sigma+(\frac{\mu-\alpha\sigma-r+\beta}{1-\sigma}-2)\sigma<2, \\ & \beta-\mu+\alpha\sigma-\sigma+1<r, \\ & r<\beta-\mu+\alpha\sigma-2\sigma+2, \\ \left(\int_1^{t+1}\frac{L_2(s)\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}} & \text{if } 1<\mu-(\alpha-2)\sigma<2, \\ & r=2-\mu+(\alpha-2)\sigma+\beta, \\ 1 & \text{if } 1<\mu-(\alpha-2)\sigma<2, \\ & 2-\mu+(\alpha-2)\sigma+\beta<r. \end{cases}$$

Put

$$w(x)=x^{(\alpha-2)\sigma}a(x)\theta^{\sigma}(x).$$

Using $(\mathbf{H}_1)$, we deduce that

$$w(x)\approx x^{-(\lambda-(\alpha-2)\sigma)}(1+x)^{\lambda-\mu}(\max(x,1))^{\xi\sigma}L_1(\min(x,1))$$

$$\times \left(\int_0^{\min(x,1)} \frac{L_1(s)\tilde{L}_1(s)}{s}\, ds \right)^{\frac{\sigma}{1-\sigma}} L_2(\max(x,1))\chi^\sigma(\max(x,1)).$$

Put

$$m_1(x) = L_1(x) \left(\int_0^x \frac{L_1(s)\tilde{L}_1(s)}{s}\, ds \right)^{\frac{\sigma}{1-\sigma}} \quad \text{and} \quad m_2(x) = L_2(x)\chi^\sigma(x).$$

By Proposition 2.5, Lemmas 2.6 and 2.7, the functions m_1 and m_2 are in $\mathcal{K}$ and $\mathcal{K}^\infty$, respectively. For $t \in (0,1]$, $w(t) \approx t^{-(\lambda-(\alpha-2)\sigma)} m_1(t)$. Since $1 < \lambda - (\alpha - 2)\sigma < 2$, from Lemma 2.8 we deduce that

$$\int_0^1 t^{1-(\lambda-(\alpha-2)\sigma)} m_1(t)\, dt < \infty. \tag{3.7}$$

For $t \in [1,\infty)$, $w(t) \approx t^{-(\mu-(\alpha-2)\sigma-\xi\sigma)} m_2(t)$. By the expression to ξ, we remark that $1 < \mu - (\alpha - 2)\sigma - \xi\sigma < 2$, and from Lemma 2.9 we deduce that

$$\int_1^\infty t^{-(\mu-(\alpha-2)\sigma-\xi\sigma)} m_2(t)\, dt < \infty. \tag{3.8}$$

Combining (3.7) and (3.8), we conclude that the function θ satisfies (3.1):

$$\int_0^\infty \min(1,t) w(t)\, dt < \infty.$$

To reach the estimate (3.2), we distinguish the following cases.

Case 1. Let $x \in (0,1]$. We have

$$w(x) \approx x^{-(\lambda-(\alpha-2)\sigma)} m_1(x).$$

Since $1 < \lambda - (\alpha - 2)\sigma < 2$ and m_1 is in $\mathcal{K}$, by Lemma 2.15 with $\lambda_1 = \lambda - (\alpha - 2)\sigma$ and $L_3(x) = m_1(x)$, we obtain that

$$G_\beta w(x) \approx x^{\beta-\lambda+(\alpha-2)\sigma} m_1(x).$$

Then we have

$$b(x)G_\beta w(x) \approx x^{-2} L_1(x)\tilde{L}_1(x) \left(\int_0^x \frac{L_1(s)\tilde{L}_1(s)}{s}\, ds \right)^{\frac{\sigma}{1-\sigma}}.$$

Put

$$M(x) = L_1(x)\tilde{L}_1(x) \left(\int_0^x \frac{L_1(s)\tilde{L}_1(s)}{s}\, ds \right)^{\frac{\sigma}{1-\sigma}}.$$

In view of Proposition 2.5 and Lemma 2.6, the function M is in $\mathcal{K}$ and we have

$$\int_0^\eta \frac{M(t)}{t} dt \approx \left(\int_0^\eta \frac{L_1(t)\tilde{L}_1(t)}{t}\, ds \right)^{\frac{1}{1-\sigma}} < \infty.$$

Then, using again Lemma 2.15, we deduce that for $x \in (0,1]$,

$$x^{2-\alpha} G_\alpha\Big(bG_\beta w\Big)(x) \approx \int_0^x \frac{M(t)}{t}\, dt \approx \left(\int_0^x \frac{L_1(s)\tilde{L}_1(s)}{s}\, ds\right)^{\frac{1}{1-\sigma}}.$$

Hence, for $x \in (0,1]$,

$$x^{2-\alpha} G_\alpha\Big(bG_\beta w\Big)(x) \approx \theta(x). \tag{3.9}$$

Case 2. Let $x \in [1,\infty)$. We have

$$w(x) \approx x^{-(\mu-(\alpha-2)\sigma-\xi\sigma)} m_2(x).$$

Since $1 < \mu - (\alpha-2)\sigma - \xi\sigma < 2$ and m_2 is in $\mathcal{K}^\infty$, by Lemma 2.15 with $\mu_1 = \mu - (\alpha-2)\sigma - \xi\sigma$ and $L_4(x) = m_2(x)$, we obtain that

$$G_\beta w(x) \approx x^{\beta-(\mu-(\alpha-2)\sigma-\xi\sigma)} m_2(x).$$

Then we have

$$b(x) G_\beta w(x) \approx x^{\beta-r-(\mu-(\alpha-2)\sigma-\xi\sigma)} L_2(x)\tilde{L}_2(x)\chi^\sigma(x).$$

Put $N(x) = L_2(x)\tilde{L}_2(x)\chi^\sigma(x)$. In view of Proposition 2.5 and Lemma 2.7, the function N is in $\mathcal{K}^\infty$.

• Let $1 < \mu - (\alpha-2)\sigma - \sigma < 2$ and $r = 1 - \mu + \beta + (\alpha-2)\sigma + \sigma$. We have

$$\begin{aligned}\int_1^\infty \frac{N(t)}{t} dt &\approx \int_1^\infty \frac{L_2(t)\tilde{L}_2(t)}{t}\left(\int_t^\infty \frac{L_2(s)\tilde{L}_2(s)}{s} ds\right)^{\frac{\sigma}{1-\sigma}} dt\\ &\approx \left(\int_1^\infty \frac{L_2(t)\tilde{L}_2(t)}{t}\, dt\right)^{\frac{1}{1-\sigma}} < \infty.\end{aligned}$$

Using Lemma 2.15 with $\mu_1 := \mu - \beta - (\alpha-2)\sigma - \sigma + r = 1$ and

$$L_4(x) = L_2(x)\tilde{L}_2(x)\left(\int_x^\infty \frac{L_2(s)\tilde{L}_2(s)}{s} ds\right)^{\frac{\sigma}{1-\sigma}},$$

we obtain

$$\begin{aligned}x^{2-\alpha} G_\alpha\Big(bG_\beta w\Big)(x) &\approx x\int_x^\infty \frac{L_2(t)\tilde{L}_2(t)}{t}\left(\int_t^\infty \frac{L_2(s)\tilde{L}_2(s)}{s} ds\right)^{\frac{\sigma}{1-\sigma}} dt\\ &\approx x\left(\int_x^\infty \frac{L_2(s)\tilde{L}_2(s)}{s}\, ds\right)^{\frac{1}{1-\sigma}} = \theta(x).\end{aligned}$$

• Let $1 < \mu - (\alpha-2)\sigma - (2 - \frac{\mu-\alpha\sigma+r-\beta}{1-\sigma})\sigma < 2$ and $\beta - \mu + \alpha\sigma - \sigma + 1 < r < \beta - \mu + \alpha\sigma - 2\sigma + 2$. Then

$$1 < \mu + r - \beta - (\alpha-2)\sigma - (2 - \frac{\mu-\alpha\sigma+r-\beta}{1-\sigma})\sigma = \frac{\mu-\alpha\sigma+r-\beta}{1-\sigma} < 2.$$

Using Lemma 2.15 with $\mu_1 = \mu + r - \beta - (\alpha-2)\sigma - (2 - \frac{\mu-\alpha\sigma+r-\beta}{1-\sigma})\sigma$ and $L_4(x) = L_2(x)\tilde{L}_2(x)\left(L_2(x)\tilde{L}_2(x)\right)^{\frac{\sigma}{1-\sigma}}$, we obtain

$$\begin{aligned} x^{2-\alpha}G_\alpha\left(bG_\beta w\right)(x) &\approx x^{2-(\mu+r-\beta-(\alpha-2)\sigma+(\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma}-2)\sigma)}\left(L_2(x)\tilde{L}_2(x)\right)^{\frac{1}{1-\sigma}} \\ &= x^{(2-\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma})}\left(L_2(x)\tilde{L}_2(x)\right)^{\frac{1}{1-\sigma}} = \theta(x). \end{aligned}$$

- Let $1 < \mu - (\alpha-2)\sigma < 2$ and $r = 2 - \mu + (\alpha-2)\sigma + \beta$. We have

$$b(x)G_\beta w(x) \approx x^{-2}L_2(x)\tilde{L}_2(x)\left(\int_1^{x+1}\frac{L_2(s)\tilde{L}_2(s)}{s}\,ds\right)^{\frac{\sigma}{1-\sigma}}.$$

Using Lemma 2.15 with $\mu_1 := r + \mu - (\alpha-2)\sigma - \beta = 2$ and

$$L_4(x) = L_2(x)\tilde{L}_2(x)\left(\int_1^{x+1}\frac{L_2(s)\tilde{L}_2(s)}{s}\,ds\right)^{\frac{\sigma}{1-\sigma}},$$

we obtain

$$\begin{aligned} x^{2-\alpha}G_\alpha\left(bG_\beta w\right)(x) &\approx \int_1^{x+1}\frac{L_2(t)\tilde{L}_2(t)}{t}\left(\int_1^{t+1}\frac{L_2(s)\tilde{L}_2(s)}{s}ds\right)^{\frac{\sigma}{1-\sigma}}\,dt \\ &\approx \left(\int_1^{x+1}\frac{L_2(s)\tilde{L}_2(s)}{s}ds\right)^{\frac{1}{1-\sigma}} = \theta(x). \end{aligned}$$

- Let $1 < \mu - (\alpha-2)\sigma < 2$ and $2 - \mu + (\alpha-2)\sigma + \beta < r$. We have

$$b(x)G_\beta w(x) \approx x^{-(r+\mu-\beta-(\alpha-2)\sigma)}L_2(x)\tilde{L}_2(x).$$

Using Lemma 2.15 with $\mu_1 := r + \mu - \beta - (\alpha-2)\sigma > 2$ and $L_4(x) = L_2(x)\tilde{L}_2(x)$, we obtain

$$x^{2-\alpha}G_\alpha\left(bG_\beta w\right)(x) \approx 1 = \theta(x).$$

Hence, for $x \in [1,\infty)$,

$$x^{2-\alpha}G_\alpha\left(bG_\beta w\right)(x) \approx \theta(x). \tag{3.10}$$

Combining (3.9) and (3.10), we conclude that for $x \in (0,\infty)$,

$$x^{2-\alpha}G_\alpha\left(bG_\beta w\right)(x) \approx \theta(x).$$

Then the function θ satisfies (3.1) and (3.2). It follows by Proposition 3.1 that problem (1.1) has a unique positive solution $u \in C_{2-\alpha}[0,\infty)$ satisfying for $x \in (0,\infty)$,

$$u(x) \approx x^{\alpha-2}\theta(x).$$

□

As an application of our main results, we give the following example.

Example 3.2. Let $\beta, \alpha \in (1,2)$ and $\sigma \in (-1,1)$. Let a and b be two positive continuous functions on $(0,\infty)$ such that

$$a(x) \approx x^{-\lambda}(1+x)^{\lambda-\mu}\left(\log(\frac{3}{\min(x,1)})\right)$$

and

$$b(x) \approx x^{\lambda-\beta-2-(\alpha-2)\sigma}(1++x)^{-(\lambda-\beta-2-(\alpha-2)\sigma)-r}\left(\log(\frac{3}{\min(x,1)})\right)^{-\gamma},$$

where $1<\lambda-(\alpha-2)\sigma<2$, $\gamma<2$ and $r\in\mathbb{R}$. Then, by Theorem 1.3, problem (1.1) has a unique positive solution $u \in C_{2-\alpha}[0,1]$ satisfying the following estimates:

- If $1<\mu-(\alpha-2)\sigma-(2-\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma})\sigma<2$, $\beta-\mu+\alpha\sigma-\sigma+1<r<\beta-\mu+\alpha\sigma-2\sigma+2$ and $\gamma<2$, then for $x\in(0,\infty)$,

$$u(x) \approx x^{\alpha-2}\Big(\max(x,1)\Big)^{2-\frac{\mu-\alpha\sigma+r-\beta}{1-\sigma}}\left(\log(\frac{3}{\min(x,1)})\right)^{\frac{2-\gamma}{1-\sigma}}.$$

- If $1<\mu-(\alpha-2)\sigma<2$ and $r=2-\mu+(\alpha-2)\sigma+\beta$, then for $x\in(0,\infty)$,

$$u(x) = x^{\alpha-2}\left(\log(\frac{3}{\min(x,1)})\right)^{\frac{2-\gamma}{1-\sigma}}\Big(\log(1+\max(x,1))\Big)^{\frac{1}{1-\sigma}}.$$

- If $1<\mu-(\alpha-2)\sigma<2$ and $2-\mu+(\alpha-2)\sigma+\beta<r$, then

$$u(x) = x^{\alpha-2}\left(\log(\frac{3}{\min(x,1)})\right)^{\frac{2-\gamma}{1-\sigma}}.$$

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Крайова задача з дробовими похідними на півосі

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У статті розглядається напівлінійна крайова задача з дробовими похідними

$$D^\beta\left(\frac{1}{b(x)}D^\alpha u\right) = a(x)u^\sigma \quad \text{на } (0,\infty)$$

з умовами $\lim_{x\to 0} x^{2-\beta}\frac{1}{b(x)}D^\alpha u(x) = \lim_{x\to\infty} x^{1-\beta}\frac{1}{b(x)}D^\alpha u(x) = 0$ та $\lim_{x\to 0} x^{2-\alpha}u(x) = \lim_{x\to\infty} x^{1-\alpha}u(x) = 0$, де $\beta,\alpha \in (1,2)$, $\sigma \in (-1,1)$ і D^β, D^α означають стандартні дробові похідні Рімана–Ліувілля. Функції $a,b : (0,\infty) \longrightarrow \mathbb{R}$ є невід'ємними неперервними функціями, які задовольняють деякі відповідні умови. Встановлено існування та єдиність позитивного розв'язку. Також надано опис глобальної поведінки цього розв'язку.

Ключові слова: рівняння з дробовими похідними, позитивний розв'язок, теорема Шаудера про нерухому точку.