Journal of Mathematical Physics, Analysis, Geometry
2020, Vol. 16, No. 1, pp. 66–85
doi: https://doi.org/10.15407/mag16.01.066

An Iterative Regularization Method for a Class of Inverse Boundary Value Problems of Elliptic Type

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This paper deals with the problem of determining an unknown source and an unknown boundary condition $u(0)$ in a boundary value problem of elliptic type from extra measurements at internal points. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. For solving the considered problem an iterative method is proposed. Using this method a regularized solution is constructed and an a priori error estimate between the exact solution and its regularization approximation is obtained. Moreover, the numerical results are presented to illustrate the accuracy and efficiency of this method.

Key words: inverse problems, ill-posed problems, elliptic problems, iterative regularization method.

Mathematical Subject Classification 2010: 35J25, 47A52, 65M32.

1. Introduction

Let H be a separable Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$. Consider the following problem:

$$\begin{cases} u_{yy}(y) - Au(y) = f, & 0 < y < +\infty, \\ u(0) = g, & \| u(+\infty) \| < +\infty, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a positive self-adjoint linear operator with a compact resolvent. We denote by $\sigma(A)$ the spectrum of the operator A. Our purpose is to identify an unknown boundary condition $u(0)$ and an unknown source f from the input data

$$u(T_1) = \psi_1 \in H, \quad u(T_2) = \psi_2 \in H, \quad 0 < T_1 < T_2 < +\infty. \tag{1.2}$$

This problem is an abstract version of an inverse boundary value problem, which generalizes inverse problems for second-order elliptic partial differential equations in a cylindrical domain. A simple example of (1.1) is the boundary value problem for the Poisson equation in the strip $(0, 1) \times (0, +\infty)$. The operator A is taken to be $-\frac{\partial^2}{\partial x^2}$ with the domain $D(A) = H_0^1(0,1) \cap H^2(0,1)$. Then we can formulate (1.1) in the form

$$\begin{cases} -u_{yy} - u_{xx} = f(x), & 0 < x < 1,\ 0 < y < +\infty, \\ u(0,y) = u(1,y) = 0, & 0 \le y < +\infty, \\ u(x,0) = g(x), & 0 \le x \le 1, \\ u(x,y) \text{ is bounded as } y \to \infty, & 0 \le x \le 1. \end{cases} \tag{1.3}$$

The last problem has extensive applications to engineering problems dealing with steady state heat conduction in heat generating media, groundwater flow with recharge or depletion [15]. For other physical motivation and models we refer the reader to [7, 9, 13].

The main difficulty in the study of the inverse problem (1.1)–(1.2) is that it is ill-posed, i.e., even if a solution exists, it does not depend continuously on the data. In other words, a small error in the data measurement can induce a large error in the calculated solutions. Thus, special regularization methods that restore the stability with respect to measurement errors are needed. In the mathematical literature various methods have been proposed for solving ill-posed problems. We can notably mention the iterative method introduced by Kozlov and Maz'ya [10, 11], which is based on solving a sequence of well-posed boundary value problems such that the sequence of solutions converges to the solution of the original problem. It has been successfully used for solving various classes of ill-posed elliptic, parabolic and hyperbolic problems [1–5, 16, 17].

In the present work, we apply an iterative method proposed by G. Bastay [2] for studying a class of inverse parabolic problems. We should notice that the author only established theoretical results and did not give a numerical implementation. We point out that although the elliptic equation is very popular and widely studied in the literature of inverse problems for PDEs, the results on the simultaneous identification of the source term f and the boundary condition $u(0)$ are very scarce.

The paper is organized as follows. In Section 2, we give some tools which are useful for this study. In Section 3, we introduce some basic results and show the ill-posedness of the inverse problem. In Section 4, we present the iterative method and give the convergence estimates. The numerical implementation is described in Section 5 to illustrate the accuracy and efficiency of this method.

2. Preliminaries

Let $(\varphi_n)_{n\ge1} \subset H$ be an orthonormal eigenbasis corresponding to the eigenvalues $(\lambda_n)_{n\ge1}$ such that

$$A\varphi_n = \lambda_n \varphi_n, \quad n \in \mathbb{N}^*,$$
$$0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lim_{n\to\infty} \lambda_n = +\infty,$$
$$\forall b \in H \quad b = \sum_{n=1}^{\infty} b_n \varphi_n, \quad b_n = (b, \varphi_n).$$

For $\alpha \in \mathbb{R}$, we introduce a Hilbert scale H^α induced by $\sqrt{A}$ as follows:

$$H^\alpha = \{b \in H : \sum_{n=1}^{\infty} \lambda_n^\alpha |(b, \varphi_n)|^2 < +\infty\},$$

with the norm

$$\|b\|_{H^\alpha} = \left(\sum_{n=1}^{\infty} \lambda_n^\alpha |(b, \varphi_n)|^2 \right)^{\frac{1}{2}}, \qquad b \in H^\alpha.$$

For $s > 0$, $C^s([0, +\infty); H)$ denotes the space of s-times continuously differentiable functions on $[0, +\infty)$ with values in H. Finally, we denote by $\{S(y) = e^{-y\sqrt{A}}\}_{y \geq 0}$ the C_0-semigroup generated by $-\sqrt{A}$ on H,

$$S(y)b = \sum_{n=1}^{\infty} e^{-y\sqrt{\lambda_n}} (b, \varphi_n)\varphi_n, \quad \forall b \in H.$$

Theorem 2.1 ([14]). *For the family of operators $\{S(y)\}_{y\geq 0}$, we have the following properties:*

1. $\|S(y)\| \leq 1$ *for every* $y \geq 0$;
2. *the function* $y \to S(y)$, $y > 0$, *is analytic*;
3. $S(y) : H \to D(A^{r/2})$ *for every* $y > 0$ *and* $r \geq 0$;
4. *for every* $b \in D(A^{r/2})$ *and* $r \geq 0$, $S(y)A^{r/2}b = A^{r/2}S(y)b$;
5. *for every* $y > 0$, *the operator* $A^{r/2}S(y)$ *is bounded.*

The next result will be used to study the regularity of the solution to the direct problem corresponding to the inverse problem (1.1)–(1.2).

Theorem 2.2 ([8, Theorem 1.4]). *For each $\xi \in H$, the problem*

$$\begin{cases} v'(y) + \sqrt{A}v(y) = \xi, & 0 < y < +\infty, \\ v(0) = \psi \end{cases} \tag{2.1}$$

has a unique solution $v \in C([0, +\infty), H) \cap C^1((0, +\infty), H)$ for each $\psi \in H$. Moreover, if $\psi \in D(\sqrt{A})$, then $v \in C^1([0, +\infty), H)$.

We complete this section by giving a result concerning nonexpansive operators.

Definition 2.3. A linear bounded operator $L : H \to H$ is called nonexpansive if $\|L\| \leq 1$.

Let L be a nonexpansive operator. To solve the equation

$$(I - L)\varphi = \psi, \tag{2.2}$$

we state a convergence theorem for a successive approximation method.

Theorem 2.4 ([12]). *Let L be a nonexpansive, self-adjoint positive operator on H. Let $\psi \in H$ be such that equation (2.2) has a solution. If 1 is not an eigenvalue of L, then the successive approximations*

$$\varphi_{k+1} = L\varphi_k + \psi, \quad k = 0, 1, 2, \dots$$

converge to a solution to (2.2) for any initial data $\varphi_0 \in H$. Moreover, $L^k \varphi \to 0$ for every $\varphi \in H$ as $k \to +\infty$.

3. Basic results

3.1. The direct problem. For given functions $\psi, \xi \in H$, consider the direct problem

$$\begin{cases} w''(y) - Aw(y) = \xi, & 0 < y < +\infty, \\ w(0) = \psi, & \| w(+\infty) \| < +\infty. \end{cases} \tag{3.1}$$

For problem (3.1), we introduce the following theorem.

Theorem 3.1. *Let $(\xi, \psi) \in H \times H$. Then problem (3.1) admits a unique solution $w \in C([0, +\infty), H) \cap C^1((0, +\infty), H)$.*

Proof. First, we determine the fundamental solutions. By using the method of diagonalization, we write

$$w(y) = \sum_{n=1}^{\infty} w_n(y)\varphi_n, \tag{3.2}$$

where $w_n(y) = (w(y), \varphi_n)$. We also have

$$\xi = \sum_{n=1}^{\infty} \xi_n \varphi_n \quad \text{and} \quad w(0) = \sum_{n=1}^{\infty} w_n(0)\varphi_n = \psi = \sum_{n=1}^{\infty} \psi_n \varphi_n. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we get a second-order family of differential equations

$$\begin{cases} w_n''(y) - \lambda_n w_n(y) = \xi_n, \quad 0 < y < \infty, \\ w_n(0) = \psi_n, \\ \displaystyle\sum_{n=1}^{+\infty} | w_n(\infty) |^2 < +\infty. \end{cases} \tag{3.4}$$

For each fixed n, the general solution to the homogenous equation

$$w_n'' - \lambda_n w_n = 0 \tag{3.5}$$

is given by

$$c_{1n} e^{y\sqrt{\lambda_n}} + c_{2n} e^{-y\sqrt{\lambda_n}}, \quad c_{1n}, c_{2n} \in \mathbb{R}, \tag{3.6}$$

where $w_{1n}(y) = c_{1n}e^{\sqrt{\lambda_n}y}$ and $w_{2n}(y) = c_{2n}e^{-\sqrt{\lambda_n}y}$ are a pair of fundamental solutions. It is easily verified that $-\frac{1}{\lambda_n}\xi_n$ is a particular solution to the nonhomogeneous equation

$$w_n'' - \lambda_n w_n = \xi_n. \tag{3.7}$$

So, the general solution to equation (3.7) is given by

$$w_n(y) = c_{1n}e^{y\sqrt{\lambda_n}} + c_{2n}e^{-y\sqrt{\lambda_n}} - \frac{1}{\lambda_n}\xi_n.$$

From the convergence of the series $\sum_{n=1}^{+\infty}|w_n(\infty)|^2$ and the condition $w_n(0) = \psi_n$, it follows that $c_{1n} = 0$ and $c_{2n} = \psi_n + \frac{1}{\lambda_n}\xi_n$. Hence, the solution to problem (3.4) is given by

$$w_n(y) = e^{-y\sqrt{\lambda_n}}\psi_n - \frac{1}{\lambda_n}(1 - e^{-y\sqrt{\lambda_n}})\xi_n.$$

Thus, the solution of problem (3.1) takes the form

$$w(y) = \sum_{n=1}^{\infty}(e^{-y\sqrt{\lambda_n}}\psi_n - \frac{1}{\lambda_n}(1 - e^{-y\sqrt{\lambda_n}})\xi_n)\varphi_n = S(y)\psi - K(y)\xi, \tag{3.8}$$

where $S(y) = e^{-y\sqrt{A}}$ and $K(y) = A^{-1}(I - e^{-y\sqrt{A}})$. It is clear that the expression (3.8) solves the problem

$$\begin{cases} v'(y) + \sqrt{A}v(y) = -A^{-\frac{1}{2}}\xi, & 0 < y < \infty, \\ v(0) = \psi. \end{cases} \tag{3.9}$$

By virtue of Theorem 2.2, we can easily check that

$$w \in C([0, +\infty), H) \cap C^1((0, +\infty), H).$$ □

3.2. Instability of the inverse problem. Now we wish to solve the inverse problem, i.e., to find the pair of functions (f, g) in the system (1.1). Making use of the supplementary conditions (1.2), we have

$$\begin{cases} u(T_1) = S(T_1)g - K(T_1)f = \psi_1, \\ u(T_2) = S(T_2)g - K(T_2)f = \psi_2. \end{cases} \tag{3.10}$$

From (3.10), we derive the system

$$\begin{cases} (K(T_2) - K(T_1))f = S(T_2)\psi_1 - S(T_1)\psi_2. \\ (K(T_2) - K(T_1))g = K(T_2)\psi_1 - K(T_1)\psi_2. \end{cases} \tag{3.11}$$

Hence, we look for a solution (f, g) to the system

$$\begin{cases} Bf = S(T_2)\psi_1 - S(T_1)\psi_2, \\ Bg = K(T_2)\psi_1 - K(T_1)\psi_2, \end{cases} \tag{3.12}$$

where

$$B = K(T_2) - K(T_1) = A^{-1}(S(T_1) - S(T_2)).$$

It is easily seen that B is a linear, injective, compact and self-adjoint operator with the singular values

$$\sigma_n = \frac{e^{-T_1\sqrt{\lambda_n}} - e^{-T_2\sqrt{\lambda_n}}}{\lambda_n}, \quad n = 1, 2, 3, \dots.$$

Remark 3.2. As for many ill-posed boundary inverse value problems for partial differential equations, the study of problem (1.1) is reduced to the study of operator equations of the first kind of the form $Bb = \eta$. From the injectivity of B, we obtain

$$b = B^{-1}\eta = \sum_{n=1}^{\infty} \frac{1}{\sigma_n}(\eta, \varphi_n)\varphi_n.$$

Since $\frac{1}{\sigma_n} \to \infty$ as $n \to \infty$, the inverse problem is ill-posed, i.e., the solution does not depend continuously on the given data. Our purpose is to construct a stable approximation to the solution by using the iterative method. The main idea is to write the equation $Bb = \eta$ in the following way:

$$b = (I - \gamma B)b + \gamma\eta = Lb + \gamma\eta,$$

where γ is a positive number satisfying $\gamma < 1/\|B\|$. Then we will show that the operator L is nonexpansive and 1 is not an eigenvalue of L. Thus, it follows from Theorem 2.4 that $(b_k)_{k\in\mathbb{N}^*}$ converges, and for every $b \in H$, $(I - \gamma B)^k b \to 0$ as $k \to +\infty$.

4. Iterative procedure and convergence results

The alternating iterative method is based on reducing the ill-posed problem (1.1), (1.2) to a sequence of well-posed boundary value problems and consists of the following steps. First, we start by letting f_0, $g_0 \in H$ be arbitrary. Let u_0 be a solution to the direct problem

$$\begin{cases} u_0'' - Au_0 = f_0, & 0 < y < \infty, \\ u_0(0) = g_0, & \\ \|u_0(+\infty)\| < +\infty. & \end{cases}$$

Then the initial approximate solution is

$$u_0(y) = S(y)g_0 - K(y)f_0.$$

Let $\eta_1 = \nu_2(T_1) - \nu_1(T_2)$ and $\eta_2 = \omega_1(T_2) - \omega_2(T_1)$, where ν_i, for $i = 1, 2$, are the solutions to the problem

$$\begin{cases} \nu_i'' - A\nu_i = \psi_i, & 0 < y < \infty, \\ \nu_i(0) = 0, & \\ \|\nu_i(+\infty)\| < +\infty, & \end{cases} \tag{4.1}$$

and ω_i are the solutions to the problem

$$\begin{cases} \omega_i'' - A\omega_i = 0, & 0 < y < \infty, \\ \omega_i(0) = \psi_i, & \\ \|\omega_i(+\infty)\| < +\infty. & \end{cases} \tag{4.2}$$

That is,

$$\eta_1 = K(T_2)\psi_1 - K(T_1)\psi_2 \tag{4.3}$$

and

$$\eta_2 = S(T_2)\psi_1 - S(T_1)\psi_2. \tag{4.4}$$

If the k-th approximate solution has been constructed, we let v_k be the solution to the problem

$$\begin{cases} v_k'' - Av_k = g_k, & 0 < y < \infty, \\ v_k(0) = 0, & \\ \|v_k(+\infty)\| < +\infty. & \end{cases} \tag{4.5}$$

Then

$$v_k(y) = -K(y)g_k.$$

Furthermore, let w_k be the solution to the problem

$$\begin{cases} w_k'' - Aw_k = f_k, & 0 < y < \infty, \\ w_k(0) = 0, & \\ \|w_k(+\infty)\| < +\infty, & \end{cases} \tag{4.6}$$

that is,

$$w_k(y) = -K(y)f_k.$$

Then, let

$$g_{k+1} = g_k - \gamma(v_k(T_1) - v_k(T_2) - \eta_1), \tag{4.7}$$

$$f_{k+1} = f_k - \gamma(w_k(T_1) - w_k(T_2) - \eta_2), \tag{4.8}$$

where γ is such that

$$0 < \gamma < \frac{1}{\|B\|} \tag{4.9}$$

and $\|B\| = \sup_{n\in\mathbb{N}^*} \frac{e^{-T_1\sqrt{\lambda_n}} - e^{-T_2\sqrt{\lambda_n}}}{\lambda_n}$. Finally, we get u_{k+1} by solving the problem

$$\begin{cases} u_{k+1}'' - Au_{k+1} = f_{k+1}, & 0 < y < \infty, \\ u_{k+1}(0) = g_{k+1}, & \\ \|u_{k+1}(+\infty)\| < +\infty. & \end{cases} \tag{4.10}$$

Hence,

$$u_{k+1}(y) = S(y)g_{k+1} - K(y)f_{k+1}.$$

Now we introduce some properties and tools which are useful for our main theorems.

Lemma 4.1. *Let T be a positive constant, then we have the estimate*

$$\frac{(1-e^{-\sqrt{\lambda}T})}{\lambda} \geq e^{-\sqrt{\lambda}T}T^2, \quad \lambda > 0. \tag{4.11}$$

Proof. To prove (4.11), it suffices to establish that

$$F_1(\mu) = 1 - (1+\mu^2)e^{-\mu} \geq 0, \quad \mu > 0. \tag{4.12}$$

We have

$$F_1'(\mu) = (\mu - 1)^2 e^{-\mu} \geq 0, \quad \mu > 0.$$

Since F_1 is nondecreasing, it follows that $F_1(\mu) \subset (0,1)$. So $F_1(\mu) \geq 0$, $\mu > 0$. Choosing $\mu = T\sqrt{\lambda}$ in (4.12), we obtain (4.11). □

Lemma 4.2. *The norm of the operator $K(y)$ is given by*

$$\|K(y)\| = \sup_{n\geq 1} \frac{1-e^{-y\sqrt{\lambda_n}}}{\lambda_n} = \frac{1-e^{-y\sqrt{\lambda_1}}}{\lambda_1}.$$

Proof. We aim to find the supremum of the function $\frac{1-e^{-y\sqrt{\lambda_n}}}{\lambda_n}$, $n \in \mathbb{N}^*$. For this purpose, fixing y, letting $\mu = y\sqrt{\lambda}$ and defining the function

$$F_2(\mu) = \frac{1-e^{-\mu}}{\mu^2} \quad \text{for} \quad \mu \geq \mu_1 = y\sqrt{\lambda_1},$$

we compute

$$F_2'(\mu) = \frac{(\mu+2)e^{-\mu} - 2}{\mu^3}.$$

Put

$$F_3(\mu) = (\mu + 2)e^{-\mu} - 2.$$

Hence,

$$F_2'(\mu) = \frac{F_3(\mu)}{\mu^3}.$$

To study the monotony of F_2, it is sufficient to determine the sign of F_3. We have

$$F_3'(\mu) = -(\mu+1)e^{-\mu} < 0, \qquad \mu \geq \mu_1 > 0.$$

Then F_3 is decreasing, moreover, $F_3(\mu) \subset (-2, 0)$. Hence $F_3(\mu) < 0$, $\mu \geq \mu_1$, which implies that F_2 is decreasing and

$$\sup_{\mu\geq\mu_1} F_2(\mu) = F_2(\mu_1).$$

Therefore,

$$\sup_{n\geq 1} \frac{1-e^{-y\sqrt{\lambda_n}}}{\lambda_n} = \frac{1-e^{-y\sqrt{\lambda_1}}}{\lambda_1}.$$

Moreover,

$$\sup_{y\in[0,+\infty)} \|K(y)\| = \sup_{y\geq 0} \frac{1-e^{-y\sqrt{\lambda_1}}}{\lambda_1} \leq \frac{1}{\lambda_1}. \tag{4.13}$$

The lemma is proved. □

Proposition 4.3. *For the linear operator $L = I - \gamma B$, we have the following properties:*

1. *L is positive and self-adjoint,*

2. *L is nonexpansive,*

3. *1 is not an eingenvalue of L.*

Proof. From the properties of the operator A and the definition of L, it follows that L is a self-adjoint nonexpansive positive operator. From the inequality

$$0 < 1 - \gamma \frac{\left(e^{-\sqrt{\lambda}T_1} - e^{-\sqrt{\lambda}T_2}\right)}{\lambda} < 1 \quad \text{for } \lambda \in \sigma(A),$$

it follows that the point spectrum of L, $\sigma_p(L) \subset (0, 1)$. Then 1 is not an eigenvalue of the operator L. □

Lemma 4.4 ([6])**.** *Let $k \in \mathbb{N}$, $k > 2$, $p > 0$. Then the function*

$$G(t) = (1-t)^k (1 + \ln(1/t))^{-p}, \tag{4.14}$$

defined on $[0, 1]$, satisfies

$$G(t) \le C(\ln k)^{-p}.$$

Proposition 4.5. *Let $k > 2$, $p > 0$, and γ satisfy (4.9). Then*

$$(1 - \gamma \frac{e^{-T_1\sqrt{\lambda_n}} - e^{-T_2\sqrt{\lambda_n}}}{\lambda_n})^k \lambda_n^{-\frac{p}{2}} \le T_2^p (\ln k)^{-p}, \qquad n \in \mathbb{N}^*. \tag{4.15}$$

Proof. Let $\lambda \in [\lambda_1, +\infty)$. Define the function

$$\phi(\lambda) = \left(1 - \gamma \frac{e^{-T_1\sqrt{\lambda}} - e^{-T_2\sqrt{\lambda}}}{\lambda}\right)^k (\sqrt{\lambda})^{-p}.$$

Write it as

$$\phi(\lambda) = \left(1 - \gamma_1 e^{-T_1\sqrt{\lambda}} \frac{(1 - e^{-(T_2 - T_1)\sqrt{\lambda}})}{\lambda (T_2 - T_1)^2}\right)^k (\sqrt{\lambda} T_2)^{-p} T_2^p, \tag{4.16}$$

where $\gamma_1 = \gamma (T_2 - T_1)^2$. Using inequality (4.11), we obtain

$$\phi(\lambda) \le (1 - \gamma_1 e^{-T_2\sqrt{\lambda}})^k (\sqrt{\lambda} T_2)^{-p} T_2^p. \tag{4.17}$$

Putting $t = e^{-T_2\sqrt{\lambda}}$, we obtain

$$\phi(\lambda) \le (1 - \gamma_1 e^{-T_2\sqrt{\lambda}})^k (\sqrt{\lambda} T_2)^{-p} T_2^p = (1 - \gamma_1 t)^k (\ln(1/t))^{-p} T_2^p. \tag{4.18}$$

Now, basing on some techniques similar to those used for establishing inequality (4.14), we can prove the estimate

$$G_1(t) = (1-\gamma_1 t)^k (\ln(1/t))^{-p} \leq (\ln k)^{-p}, \quad t \in [0,1], \tag{4.19}$$

with $0 < \gamma_1 t < 1$. The idea is to show that there exists a positive constant t_0 such that G_1 is monotonically increasing in $[0, t_0)$ and monotonically decreasing in $(t_0, 1]$. Since G_1 is continuously differentiable in $[0,1]$, $G_1(t) \geq 0$, and $G_1(0) = G_1(1) = 0$, it follows that the maximum of G_1 is attained at an interior point, which is a critical point of G_1. From

$$G_1'(t) = -\frac{1}{t}(1-\gamma_1 t)^{k-1}(\ln(1/t))^{-p-1}\left(\gamma_1 t\left(k\ln\left(\frac{1}{t}\right)+p\right)-p\right),$$

it follows that the critical point of G_1 in $(0,1)$ satisfies

$$\left(\gamma_1 t\left(k\ln\left(\frac{1}{t}\right)+p\right)-p\right) = 0.$$

We introduce the auxiliary function

$$\Gamma(t) = \left(\gamma_1 t\left(k\ln\left(\frac{1}{t}\right)+p\right)-p\right).$$

For k sufficiently large,

$$\Gamma(1/k) = \left(\gamma_1 \frac{k\ln(k)+p}{k} - p\right) > 0.$$

For $a > 1$ and k sufficiently large, we have

$$\Gamma(1/k^a) = \left(\gamma_1 \frac{k\ln(k^a)+p}{k^a} - p\right) < 0.$$

Therefore, there exists $k_0(a)$ such that

$$\begin{aligned} &\Gamma(k^{-a}) < 0 &&\text{for all} \quad k \geq k_0(a),\\ &\Gamma(k^{-1}) > 0 &&\text{for all} \quad k \geq k_0(a). \end{aligned}$$

Consequently, a critical point t^* of G_1 must lie between k^{-a} and k^{-1}. Then, for $k \geq \max(k_0(a), 2)$, we have

$$G_1(t) = (1-\gamma_1 t)^k(\ln(1/t))^{-p} \leq G_1(t^*).$$

On the other hand,

$$G_1(t^*) = (1-\gamma_1 t^*)^k(\ln(1/t^*))^{-p} \leq (\ln(1/t^*))^{-p} \leq (\ln k)^{-p}. \tag{4.20}$$

So, for any $t \in [0,1]$, we have

$$G_1(t) \leq (\ln k)^{-p}. \tag{4.21}$$

From (4.18) and (4.21), we obtain

$$\phi(\lambda) \leq T_2^p (\ln k)^{-p}, \quad \lambda \in [\lambda_1, +\infty), \tag{4.22}$$

and from the estimate (4.22), there follows (4.15). □

Theorem 4.6. *Let u be a solution to the inverse problem (1.1), (1.2). Let f_0, $g_0 \in H$ be arbitrary data elements for the iterative procedure proposed above and u_k be the kth approximate solution. Then we have*

(a) *The method converges, i.e.,*

$$\sup_{y\in[0,+\infty)} \|u_k(y) - u(y)\| \to 0 \quad \text{as } k \to +\infty. \tag{4.23}$$

(b) *Moreover, for any $\alpha_1 > 0$, $\alpha_2 > 0$, and g, f such that*

$$\| g_0 - g \|_{H^{\alpha_1}} \leq E_1, \quad \| f_0 - f \|_{H^{\alpha_2}} \leq E_2 \tag{4.24}$$

with some $E_1 > 0$ and $E_2 > 0$, the rate of convergence of the method is given by

$$\sup_{y\in[0,+\infty)} \|u_k(y) - u(y)\| \leq T_2^{\beta_2}(1 + \lambda_1^{-1})(\ln k)^{-\beta_1}(E_1 + E_2), \tag{4.25}$$

where $\beta_1 = \min(\alpha_1, \alpha_2)$ and $\beta_2 = \max(\alpha_1, \alpha_2)$.

Proof. (a) Iterating in (4.7) backwards, we obtain

$$\begin{aligned} g_{k+1} &= g_k - \gamma(K(T_2) - K(T_1))g_k + \gamma\eta_1 \\ &= (I - \gamma B)g_k + \gamma\eta_1 = (I - \gamma B)^{k+1} g_0 + \gamma \sum_{j=0}^{k} (I - \gamma B)^j \eta_1. \end{aligned} \tag{4.26}$$

Furthermore,

$$g_{k+1} = (I - \gamma B)^{k+1}(g_0 - B^{-1}\eta_1) + B^{-1}\eta_1.$$

In the same manner, we get

$$f_{k+1} = (I - \gamma B)^{k+1}(f_0 - B^{-1}\eta_2) + B^{-1}\eta_2.$$

Thus, the approximate solution u_k is given by

$$\begin{aligned} u_k(y) &= S(y)(I - \gamma B)^k(g_0 - B^{-1}\eta_1) + S(y)B^{-1}\eta_1 \\ &\quad - K(y)(I - \gamma B)^k(f_0 - B^{-1}\eta_2) - K(y)B^{-1}\eta_2. \end{aligned} \tag{4.27}$$

From (3.12), (4.3) and (4.4), it follows that $g = B^{-1}\eta_1$ and $f = B^{-1}\eta_2$. Thus,

$$u_k(y) - u(y) = S(y)(I - \gamma B)^k(g_0 - g) - K(y)(I - \gamma B)^k(f_0 - f). \tag{4.28}$$

From the triangle inequality, we have

$$\begin{aligned} \|u_k(y) - u(y)\| &\leq \|S(y)(I - \gamma B)^k(g_0 - g)\| + \|K(y)(I - \gamma B)^k(f_0 - f)\| \\ &\leq \|S(y)\|\|(I - \gamma B)^k(g_0 - g)\| \\ &\qquad + \|K(y)\|\|(I - \gamma B)^k(f_0 - f)\|. \end{aligned} \tag{4.29}$$

Using the property 1 of Theorem 2.1, the estimate (4.13) and passing to the supremum with respect to $y \in [0, +\infty)$, we derive

$$\sup_{y\geq 0} \|u_k(y) - u(y)\| \leq \|(I - \gamma B)^k (g_0 - g)\| + \lambda_1^{-1} \|(I - \gamma B)^k (f_0 - f)\|.$$

By virtue of Proposition 4.3 and Theorem 2.4, it follows that

$$\sup_{y\geq 0} \|u_k(y) - u(y)\| \to 0 \quad \text{as } k \to \infty.$$

(b) We have

$$\|u_k(y) - u(y)\| \leq \sqrt{I_1} + \sqrt{I_2}, \tag{4.30}$$

where

$$\begin{aligned} I_1 &= \|(I - \gamma B)^k (g_0 - g)\|^2 \\ &= \sum_{n=1}^{\infty} \left(1 - \gamma \left(\frac{e^{-\sqrt{\lambda_n} T_1} - e^{-\sqrt{\lambda_n} T_2}}{\lambda_n}\right)\right)^{2k} |(g_0 - g, \varphi_n)|^2 \end{aligned}$$

and

$$\begin{aligned} I_2 &= \lambda_1^{-2} \|(I - \gamma B)^k ((f_0 - f)\|^2 \\ &= \lambda_1^{-2} \sum_{n=1}^{\infty} \left(1 - \gamma \left(\frac{e^{-\sqrt{\lambda_n} T_1} - e^{-\sqrt{\lambda_n} T_2}}{\lambda_n}\right)\right)^{2k} |(f_0 - f, \varphi_n)|^2. \end{aligned}$$

We compute

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} \left(1 - \gamma \left(\frac{e^{-\sqrt{\lambda_n} T_1} - e^{-\sqrt{\lambda_n} T_2}}{\lambda_n}\right)\right)^{2k} |(g_0 - g, \varphi_n)|^2 \\ &= \sum_{n=1}^{\infty} \left(1 - \gamma \left(\frac{e^{-\sqrt{\lambda_n} T_1} - e^{-\sqrt{\lambda_n} T_2}}{\lambda_n}\right)\right)^{2k} \lambda_n^{-\alpha_1} \lambda_n^{\alpha_1} |(g_0 - g, \varphi_n)|^2 \\ &= \sum_{n=1}^{\infty} (\phi(\lambda_n))^2 \lambda_n^{\alpha_1} |(g_0 - g, \varphi_n)|^2 \leq \sup_n (\phi(\lambda_n))^2 \sum_{n=1}^{\infty} \lambda_n^{\alpha_1} |(g_0 - g, \varphi_n)|^2, \end{aligned}$$

where $\phi(\lambda_n) = \left(1 - \gamma \left(\frac{e^{-\sqrt{\lambda_n} T_1} - e^{-\sqrt{\lambda_n} T_2}}{\lambda_n}\right)\right)^k \lambda_n^{-\frac{\alpha_1}{2}}$. By virtue of Proposition 4.5, we obtain

$$I_1 \leq (\ln k)^{-2\alpha_1} T_2^{2\alpha_1} E_1^2. \tag{4.31}$$

In the same manner, we get

$$I_2 \leq \lambda_1^{-2} (\ln k)^{-2\alpha_2} T_2^{2\alpha_2} E_2^2. \tag{4.32}$$

Combining (4.30) with (4.31) and (4.32), we have

$$\sup_{y\in[0,+\infty)} \|u_k(y) - u(y)\| \leq T_2^{\beta_2} (1 + \lambda_1^{-1}) (\ln k)^{-\beta_1} (E_1 + E_2). \tag{4.33}$$

Passing to the supremum with respect to $y \in [0, +\infty)$ in (4.33), we obtain (4.25). Since in practice the measured data ψ_1 and ψ_2 are never known exactly, it is our aim to solve the system from the knowledge of the perturbed data functions ψ_1^δ and ψ_2^δ satisfying

$$\|\psi_1 - \psi_1^\delta\| + \|\psi_2 - \psi_2^\delta\| < \delta, \tag{4.34}$$

where $\delta > 0$ denotes a noise level. In the following theorem, we consider the case of inexact data. $\square$

Theorem 4.7. *Let f_0, g_0 be arbitrary data elements for the iterative procedure proposed above such that (4.24) holds and let u_k (respectively, u_k^δ) be the k-th approximate solution corresponding to the exact data ψ_1, ψ_2 (respectively, to the inexact data ψ_1^δ, ψ_2^δ) such that (4.34) holds. Then we have the following estimate:*

$$\sup_{y\in[0,+\infty)} \|u_k(y) - u(y)\| \leq (1 + \lambda_1^{-1})(\delta\gamma k + T_2^{\beta_2}(\ln k)^{-\beta_1}(E_1 + E_2)). \tag{4.35}$$

Proof. Let

$$g_k = (I - \gamma B)^k g_0 + \gamma \sum_{j=0}^{k-1} (I - \gamma B)^j \eta_1,$$

$$f_k = (I - \gamma B)^k f_0 + \gamma \sum_{j=0}^{k-1} (I - \gamma B)^j \eta_2,$$

$$u_k(y) = S(y) g_k - K(y) f_k,$$

$$g_k^\delta = (I - \gamma B)^k g_0 + \gamma \sum_{j=0}^{k-1} (I - \gamma B)^j \eta_1^\delta, \tag{4.36}$$

$$f_k^\delta = (I - \gamma B)^k f_0 + \gamma \sum_{j=0}^{k-1} (I - \gamma B)^j \eta_2^\delta, \tag{4.37}$$

$$u_k^\delta(y) = S(y) g_k^\delta - K(y) f_k^\delta,$$

with

$$\eta_1^\delta = K(T_2)\psi_1^\delta - K(T_1)\psi_2^\delta \quad \text{and} \quad \eta_2^\delta = S(T_2)\psi_1^\delta - S(T_1)\psi_2^\delta.$$

Then we have

$$\begin{aligned} \|\eta_1 - \eta_1^\delta\| &= \|K(T_2)(\psi_1 - \psi_1^\delta) - K(T_1)(\psi_2 - \psi_2^\delta)\| \\ &\leq \lambda_1^{-1}(\|(\psi_1 - \psi_1^\delta)\| + \|\psi_2 - \psi_2^\delta\|) \leq \lambda_1^{-1}\delta, \end{aligned} \tag{4.38}$$

$$\begin{aligned} \|\eta_2 - \eta_2^\delta\| &= \|S(T_2)(\psi_1 - \psi_1^\delta) - S(T_1)(\psi_2 - \psi_2^\delta)\| \\ &\leq \|\psi_1 - \psi_1^\delta\| + \|\psi_2 - \psi_2^\delta\| \leq \delta. \end{aligned} \tag{4.39}$$

Using the triangle inequality, we have

$$\|u_k^\delta - u\| \leq \|u_k^\delta - u_k\| + \|u_k - u\|. \tag{4.40}$$

We compute

$$\|u_k^\delta(y)-u_k(y)\| \le \|S(y)(g_k^\delta-g_k)\|+\|K(y)(f_k^\delta-f_k)\|$$
$$\le \gamma\left\|\sum_{j=0}^{k-1}(I-\gamma B)^j(\eta_1^\delta-\eta_1)\right\|+\lambda_1^{-1}\gamma\left\|\sum_{j=0}^{k-1}(I-\gamma B)^j(\eta_2^\delta-\eta_2)\right\|.$$

Using (4.38) and (4.39), we obtain

$$\|u_k^\delta(y)-u_k(y)\| \le \delta\gamma\left(\sum_{j=0}^{k-1}\|(I-\gamma B)\|^j+\lambda_1^{-1}\sum_{j=0}^{k-1}\|(I-\gamma B)\|^j\right).$$

Since $\|I-\gamma B\|\le 1$, it follows that

$$\sup_{y\in[0,+\infty)}\|u_k^\delta(y)-u_k(y)\| \le (1+\lambda_1^{-1})\delta\gamma k. \tag{4.41}$$

Combining (4.40) with (4.25) and (4.41), then passing to the supremum with respect to $y\in[0,+\infty)$, we obtain the estimate (4.35). □

Remark 4.8. If we choose the number of iterations $k(\delta)$ such that $k(\delta)\to 0$ as $\delta\to 0$, we obtain

$$\sup_{y\in[0,+\infty)}\|u_k^\delta(y)-u(y)\|\to 0 \quad \text{as} \quad k\to+\infty.$$

5. Numerical implementation

In this section, an example is devised for verifying the effectiveness of the proposed method. Consider the problem of finding the functions $f(x)$, $g(x)$ and $u(x,y)$ in the system

$$\begin{cases} \dfrac{\partial^2}{\partial y^2}u(x,y)+\dfrac{\partial^2}{\partial x^2}u(x,y)=f(x), & (x,y)\in(0,1)\times(0,+\infty),\\ u(0,y)=u(1,y)=0, & y\in[0,+\infty),\\ \|u(x,+\infty)\|<+\infty, & 0\le x\le 1,\\ u(x,0)=g(x), & 0\le x\le 1,\\ u(x,1/2)=\psi_1(x),\ u(x,1)=\psi_2(x), & 0\le x\le 1. \end{cases} \tag{5.1}$$

Let $A=-\frac{\partial^2}{\partial x^2}$ be the differential operator with $\mathcal{D}(A)=H_0^1(0,1)\cap H^2(0,1)\subset H=L^2(0,1)$. Then

$$\lambda_n=n^2\pi^2,\quad \varphi_n=\sqrt{2}\sin(n\pi x),\quad n=1,2,\ldots,$$

are its eigenvalues and orthonormal eigenfunctions, which form a basis for H. The solution of the above problem is given by

$$u(x,y)=\sum_{n=1}^{\infty}\left(e^{-(n\pi)y}(g,\varphi_n)-\frac{1-e^{-(n\pi)y}}{(n\pi)^2}(f,\varphi_n)\right)\varphi_n,$$

where for $b \in H$, $b_n = (b, \varphi_n) = \sqrt{2}\int_0^1 b(s)\sin(n\pi s)\,ds$, $n = 1, 2, \ldots$. From (4.36) and (4.37), by choosing $f_0 = g_0 \equiv 0$, we get

$$g_k^\delta(x) = 2\gamma \sum_{j=0}^{k-1}\sum_{n=1}^{\infty}(1-\gamma\sigma_n)^j \int_0^1 \Big(\beta_{2n}\psi_1^\delta(s) - \beta_{1n}\psi_2^\delta(s)\Big)\sin(n\pi s)\sin(n\pi x)\,ds$$

and

$$f_k^\delta(x) = 2\gamma \sum_{j=0}^{k-1}\sum_{n=1}^{\infty}(1-\gamma\sigma_n)^j \int_0^1 \Big(\alpha_{2n}\psi_1^\delta(s) - \alpha_{1n}\psi_2^\delta(s)\Big)\sin(n\pi s)\sin(n\pi x)\,ds,$$

where $\sigma_n = \frac{(e^{-\frac{n\pi}{2}} - e^{-n\pi})}{(n\pi)^2}$, $\alpha_{1n} = \frac{(1-e^{-\frac{n\pi}{2}})}{(n\pi)^2}$, $\alpha_{2n} = \frac{(1-e^{-n\pi})}{(n\pi)^2}$, $\beta_{1n} = e^{-\frac{n\pi}{2}}$, and $\beta_{2n} = e^{-n\pi}$.

We use the trapezoidal rule to approach the integral and do an approximate truncation for the series by choosing the sum of the front $M+1$ terms. After considering an equidistant grid

$$0 = x_1 < x_2 < \cdots < x_{M+1} = 1, \quad x_i = \frac{i-1}{M},\ i = 1, \ldots, M+1,$$

we get the discrete approximations

$$g_k^\delta = (g_k^\delta(x_1), g_k^\delta(x_2), \ldots, g_k^\delta(x_{M+1})) \text{ and } f_k^\delta = (f_k^\delta(x_1), f_k^\delta(x_2), \ldots, f_k^\delta(x_{M+1}))$$

of (4.36) and (4.37), respectively, given by

$$g_k^\delta(x_l) = 2h\gamma \sum_{j=0}^{k-1}\sum_{i=1}^{M+1}\sum_{n=1}^{N}(1-\gamma\sigma_n)^j \Big(\alpha_{2n}\psi_1^\delta(x_i) - \alpha_{1n}\psi_2^\delta(x_i)\Big)\sin(n\pi x_i)\sin(n\pi x_l),$$

$$f_k^\delta(x_l) = 2h\gamma \sum_{j=0}^{k-1}\sum_{i=1}^{M+1}\sum_{n=1}^{N}(1-\gamma\sigma_n)^j \Big(\beta_{2n}\psi_1^\delta(x_i) - \beta_{1n}\psi_2^\delta(x_i)\Big)\sin(n\pi x_i)\sin(n\pi x_l),$$

where $h = 1/M$ and the inexact data are obtained by adding a random distributed perturbation to each data function. Hence,

$$\psi^\delta = \psi + \varepsilon\,\mathrm{randn}(\mathrm{size}(\psi)),$$

here ε indicates the noise level of the measurements data, and the function randn$(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$ and the standard deviation $\sigma = 1$. The function randn(size(g)) returns an array of random entries that is of the same size as ψ. The bound on the measurement error δ can be measured in the sense of root mean square error (RMSE) according to

$$\delta = \|\psi^\delta - \psi\|_{l^2} = \left(\frac{1}{M+1}\sum_{i=1}^{M+1}\Big(\psi(x_i) - \psi^\delta(x_i)\Big)^2\right)^{1/2}.$$

The relative error Rer(f) is given by

$$\mathrm{Rer}(f) = \frac{\| f_k^\delta - f \|_2}{\| f \|_2}.$$

Example 5.1. It is easy to see that if $f(x) = -2\pi^2 \sin(\pi x)$ and $g(x) = \sin(\pi x)$, then $u(x, y) = (2 - e^{-\pi y}) \sin(\pi x)$ is the exact solution to problem (5.1). Consequently, $\psi_1(x) = (2 - e^{-\frac{1}{2}\pi}) \sin(\pi x)$ and $\psi_2(x) = (2 - e^{-\pi}) \sin(\pi x)$.

Table 5.1: The relative errors $\mathrm{Rer}(f)$ and $\mathrm{Rer}(g)$ with $M = 300$, $k = 6$, $\epsilon = 0.1$ and $\omega = 53.9435$.

N	2	5	7	10	15	20	50	70
$\mathrm{Rer}(f)$	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031	0.0031
$\mathrm{Rer}(g)$	0.0713	0.0776	0.0782	0.0784	0.0786	0.0786	0.0786	0.0786

Table 5.2: The relative errors $\mathrm{Rer}(f)$ and $\mathrm{Rer}(g)$ with $N = 10$, $k = 6$, $\epsilon = 0.1$ and $\omega = 53.9435$.

M	10	50	100	150	200	250	300	350
$\mathrm{Rer}(f)$	0.0554	0.0014	0.0140	0.0043	0.0067	0.0051	0.0031	0.0035
$\mathrm{Rer}(g)$	0.4846	0.4094	0.1918	0.1018	0.1026	0.0946	0.0784	0.0797

Table 5.3: The relative errors $\mathrm{Rer}(f)$ and $\mathrm{Rer}(g)$ with $M = 300$, $N = 10$, $k = 6$ and $\omega = 53.9435$.

ϵ	0.5	0.1	0.01	0.001	0.0001
$\mathrm{Rer}(f)$	0.0165	0.0031	$3.0911e-004$	$3.0366e-005$	$9.7455e-006$
$\mathrm{Rer}(g)$	0.3921	0.0784	0.0078	$7.8955e-004$	$8.4171e-005$

Table 5.4: The relative errors $\mathrm{Rer}(f)$ and $\mathrm{Rer}(g)$ with $M = 300$, $N = 10$, $k = 7$ and $\omega = 53.9435$.

ϵ	0.5	0.1	0.01	0.001	0.0001
$\mathrm{Rer}(f)$	0.0100	0.0036	$3.6057e-004$	$3.5887e-005$	$3.5418e-006$
$\mathrm{Rer}(g)$	0.4423	0.0885	0.0088	$8.8504e-004$	$8.8934e-005$

Tables 5.1 and 5.2 show the influence of N and M respectively on the relative errors. From Table 5.1, we find that N has a small influence on the relative error when it becomes larger. From Table 5.2, we see that the degree of ill-posedness of the numerical problem does not increase with the refinement of the mesh used.

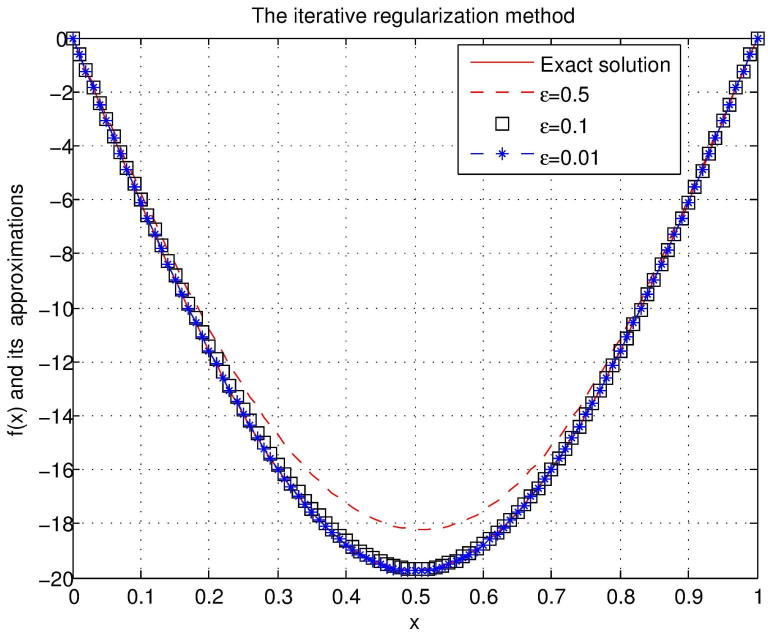

Fig. 5.1: The comparison between f and its computed approximations for $k = 6$, $M = 100$, $N = 10$, with different noise level.

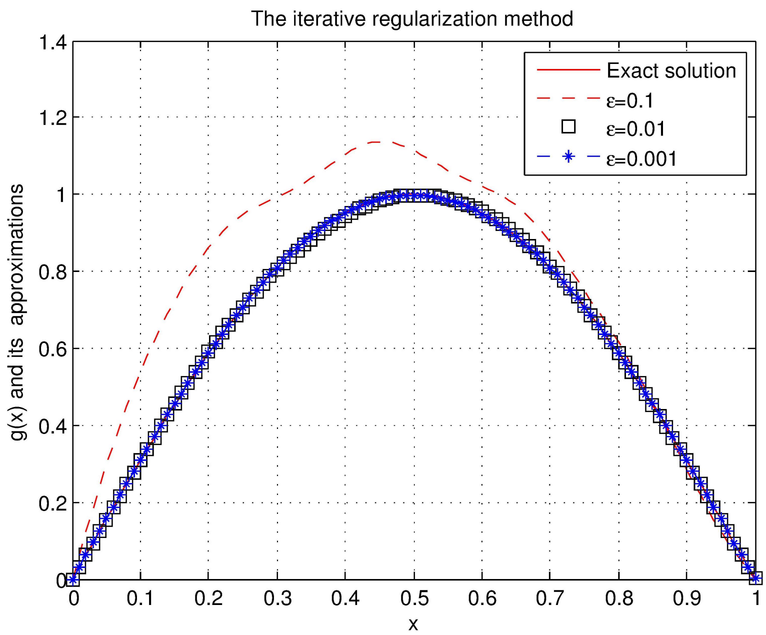

Fig. 5.2: The comparison between f and its computed approximations for $k = 6$, $M = 100$, $N = 10$, with different noise level.

Tables 5.3 and 5.4 give the relative errors with different amounts of noise added into the data for $k = 6$ and $k = 7$ respectively.

Figure 5.1 (respectively, Figure 5.2) compares the function f (respectively, g) and its computed approximations with different noise level. It can be seen that as the amount of noise ϵ decreases, the regularized solutions approximate better the exact solution, and for the function f, even with the noise level $\varepsilon = 0.1$, the approximate solutions are still in good agreement with the corresponding exact solution.

6. Conclusion

In this paper, we have extended the iterative method for identifying an unknown source term and an unknown boundary condition in a class of inverse boundary value problems of elliptic type. The convergence results were established, and the error estimates were obtained under an apriori bound of the exact solution. The presented numerical examples justified the efficiency and accuracy of the method.

Acknowledgment. This work is supported by the CNEPRU Projet of Algeria, code C00L03UN230120150012.

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Received October 15, 2018, revised November 16, 2018.

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Ітеративний метод регуляризації для класу обернених крайових задач еліптичного типу

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У даній роботі розглядається проблема визначення невідомого джерела та невідомої граничної умови $u(0)$ для крайової задачі еліптичного типу за даними додаткових вимірювань у внутрішніх точках. Задача є некоректною в тому сенсі, що її розв'язок (якщо він існує) не залежить неперервно від даних задачі. Для розв'язання цієї задачі запропоновано ітеративний метод. За допомогою цього методу побудовано регуляризований розв'язок і одержано апріорну оцінку похибки між точним розв'язком та його регуляризацією. Крім того, представлено числові результати для ілюстрації точності та ефективності цього методу.

Ключові слова: обернена задача, некоректна задача, еліптичні задачі, метод регуляризації.