Journal of Mathematical Physics, Analysis, Geometry
2020, Vol. 16, No. 3, pp. 364–371
doi: https://doi.org/10.15407/mag16.032.364

On Projective Classification of Points of a Submanifold in the Euclidean Space

Alexander Yampolsky

We propose the classification of points of a submanifold in the Euclidean space in terms of the indicatrix of normal curvature up to projective transformation and give a necessary condition for finiteness of number of such classes. We apply the condition to the case of three-dimensional submanifold in six-dimensional Euclidean space and prove that there are 10 types of projectively equivalent points.

Key words: normal curvature indicatrix, submanifold point type, projective transformation

Mathematical Subject Classification 2010: 53A07, 53B20, 53B25

1. Introduction

The affine classification of points of a submanifold $F^l \subset E^{l+p}$ was given by A. Borisenko [1–3], A. Borisenko and Yu. Nikolayevskii [4], and in complex setting by A. Borisenko and O. Lejbina [6]. By definition, two *points of a submanifold are said to be affinely equivalent if there is an affine transformation $G = GL(l, \mathbb{R}) \times GL(p, \mathbb{R})$ in the ambient space under which the osculating paraboloids at one point map onto the corresponding osculating paraboloids at the other point.* The osculating paraboloids are completely defined by the vector-valued second fundamental form. At a given point, the subgroup $GL(l, \mathbb{R}) \subset G$ action means the change of parameterization in a neighborhood of the point and the subgroup $G(p, \mathbb{R}) \subset G$ action means the change of normal frame in normal subspace of the submanifold. In these terms, *two points of a submanifold are affinely equivalent if there are local parameterizations and normal framing such that the vector-valued second fundamental forms at these points coincide.* Informally speaking, one can "recultivate" the osculating paraboloids from one point to the other by affine transformation. The orbits of such a "recultivation" define affine classes of points. In general, the number of classes could be infinite. As it was proved by A. Borisenko, for the number of affine classes to be finite it is necessary that dimension and codimension of the submanifold satisfy the inequality

$$p\left(\frac{l(l+1)}{2} - p\right) \leq l^2 - 1.$$

Detailed analysis of the inequality one can find in [1, 2]. As it was shown, in case $l = 3$ there is a finite number of classes of affinely equivalent points if $p = 2$ or $p = 4$. In this paper, we prove that one can distinguish a *finite number* of classes of points in case $(l = 3, p = 3)$ too, but with respect to a wider (projective) group of transformations. The classification is based on the notion of indicatrix of normal curvature and extended point-wise codimension. It would be interesting to analyze the types of points in special case of submanifolds of revolution with the metric of revolution [5] in order to understand what types of normal curvature indicatrix one can get.

2. Steiner surfaces and their projective classes

Definition 2.1. Let p_0, p_1, p_2, p_3 be second order polynomials of two variables (u_1, u_2). Suppose one of them is non-zero, say $p_0 \not\equiv 0$. A surface in E^3, parameterized by the vector-function

$$\mathbf{S} = \left\{ \frac{p_1(u_1,u_2)}{p_0(u_1,u_2)}, \frac{p_2(u_1,u_2)}{p_0(u_1,u_2)}, \frac{p_3(u_1,u_2)}{p_0(u_1,u_2)} \right\},$$

is called Steiner surface provided that the polynomials p_0, p_1, p_2 and p_3 are linearly independent and have no base points (i.e., common roots).

The Steiner surface has natural projective representation. To do this, pass to the homogeneous coordinates $(u_0 : u_1 : u_2)$ in the domain of definition and to the homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ in the target space. Then the polynomials (p_0, p_1, p_2, p_3) take the form of *homogeneous* second order polynomials of homogeneous coordinates in the domain of definition, and

$$\mathbf{PS} = \{p_0(u_0 : u_1 : u_2) : p_1(u_0 : u_1 : u_2) : p_2(u_0 : u_1 : u_2) : p_3(u_0 : u_1 : u_2)\}$$

defines the mapping

$$\mathbf{PS} : \mathbb{R}P^2 \to \mathbb{R}P^3. \tag{2.1}$$

The image of the mapping $\mathbf{PS}$ is called by the *projective* Steiner surface.

Definition 2.2 ([7]). Two Steiner surfaces $(u, \mathbf{PS}_1)$ and $(v, \mathbf{PS}_2)$ are said to be projectively equivalent if there are affine transformations $g_1 \in GL(3, \mathbb{R})$ and $g_2 \in GL(4, \mathbb{R})$ such that $v = g_1(u)$ and $\mathbf{PS}_2 = g_2(\mathbf{PS}_1)$.

In accordance with Definition 2.2, A. Coffman et el. [7] have found the simplest forms of projective Steiner surfaces.

Theorem 2.3 ([7]). *A projective Steiner surface is projectively equivalent to one of the following ones:*

1. *Roman surface* $[u_0^2 + u_1^2 + u_2^2 : u_1u_2 : u_0u_2 : u_0u_1]$;
2. *Steiner's parabolic surface* $[u_0u_2 : u_1u_2 : u_0^2 - u_1^2 + u_2^2 : u_0u_1]$;
3. *Whitney's cross-cap* $[u_0^2 + u_1^2 + u_2^2 : u_1u_2 : 2u_0u_1 : u_0^2 - u_1^2]$;

4. *Steiner's butterfly* $[u_0^2 - u_1^2 + u_2^2 : u_2^2 - u_1^2 : u_1u_2 : u_0u_1]$;

5. *Steiner's T-surface* $[u_0^2 + u_1^2 + u_2^2 : 2u_0u_2 : 2u_0u_1 : u_0^2 - u_1^2 + u_2^2]$;

6. *Steiner's cross-cup* $[u_0^2 + 2u_1^2 + u_2^2 : 2u_1^2 + u_2^2 : u_2^2 + 2u_0u_2 : u_1u_2 + u_0u_1]$;

7. *Zindler's conoid* $[u_1^2 - u_2^2 : u_1u_2 : u_0u_1 : u_0u_2]$;

8. *Whitney's umbrella* $[u_2^2 : u_0u_1 : u_0u_2 : u_1^2]$;

9. *Caylay's ruled cubic* $[u_0u_1 : u_0u_2 - u_1^2 : u_1u_2 : u_2^2]$.

A detailed description of these classes together with the images one can find on A. Coffman web-page [8].

3. Indicatrix of normal curvature and affine-projective classes of points

Definition 3.1. The normal curvature of a submanifold $(F^l, g) \subset E^{l+p}$ at $q \in F^l$ in a direction $X \in T_qF^l$ with respect to $\xi \in T_q^\perp F^l$ is a number

$$k_\xi(q, X) = \left.\frac{B_\xi(X, X)}{g(X, X)}\right|_q, \tag{3.1}$$

where $B_\xi(X, X)$ is the second fundamental form of the submanifold with respect to unit normal ξ and $g(X, X)$ is the first fundamental form of F^l.

If X varies all over the unit sphere $S^{l-1} \subset T_qF^l$, then (3.1) defines the affine mapping

$$k_\xi(q, X) : S^{l-1} \to T_q^\perp F^l \tag{3.2}$$

for each fixed $\xi \in T_q^\perp F^l$.

Definition 3.2. The image of the mapping (3.2) is called by indicatrix of the normal curvature at $q \in F^l$ with respect to $\xi \in T_q^\perp F^l$.

Remark that the indicatrix of normal curvature is always compact. Due to the evident property $k_\xi(q, \lambda X) = k_\xi(q, X)$, the indicatrix can be considered as the *affine projection* to the first homogeneous coordinate of a projective immersion $\mathrm{ind} : RP^{l-1} \to RP^p$ given by

$$\mathrm{Ind}(X^1 : X^2 : \ldots : X^l) = (g(X, X) : B_1(X, X) : \ldots : B_p(X, X)),$$

where $B_1, \ldots, B_p$ are the second fundamental forms with respect to some normal frame $n_1, \ldots, n_p$.

Definition 3.3. The points of a regular submanifold $F^l \subset E^{l+p}$ are said to be projectively equivalent if their normal curvature indicatrices are the same up to projective transformations $GL(l, \mathbb{R}) \times GL(p + 1, \mathbb{R})$ of $RP^{l-1} \times RP^p$ acting over their projective images.

The following lemma is similar to the one in [3] and gives necessary condition for the number of projectively equivalent points to be finite.

Lemma 3.4. *Let F^l be a submanifold in the Euclidean space E^{l+p}. If*

$$(p+1)\Big(\frac{l(l+1)}{2}-(p+1)\Big)\leq l^2-1,$$

then the number of classes of projectively equivalent points is finite.

In case $l=3$ the inequality implies $p\leq 3$. Therefore, the unique non-trivial case is $(l=3,p=3)$.

Definition 3.5. Denote by $\nu = \dim(\mathrm{span}(B_1,\dots,B_p)$ the point-wise codimension and define extended point-wise codimension by $\mu = \dim(\mathrm{span}(g,B_1,\dots,B_p))$. Define a point-wise extended codimension index as the pair (ν,μ).

The definition is correct since the index ν does not depend on the choice of the normal frame and the index μ does not depend on the choice of parameterization of a submanifold. The index μ is the rank of mapping $\mathrm{Ind} : \mathbb{R}P^{l-1}\to\mathbb{R}P^p$ acting as

$$\mathrm{Ind}(X^1:X^2:\ldots:X^l)=(g(X,X):B_1(X,X):\ldots:B_p(X,X).$$

The image of Ind defines a projective image of the normal curvature indicatrix. The affine projection to the first homogeneous coordinate defines the affine indicatrix of normal curvature, namely

$$\mathrm{ind}:T_qF^l\to T^\perp F^l_q,$$

acting as

$$\mathrm{ind}(X)=\left\{\frac{B_1(X,X)}{g(X,X)},\dots,\frac{B_p(X,X)}{g(X,X)}\right\}.$$

The main result consists of the following theorem.

Theorem 3.6. *There are 10 projective classes of points of a submanifold $F^3\subset E^6$ in accordance to the values of extended point-wise codimension index (ν,μ) and the type of normal curvature indicatrix, namely*

Index	Type of indicatrix
(3,4)	• Roman surface • Cross-cap • Cross-cup • T-surface
(3,3)	a compact part of a plane that does not pass trough the origin
(2,3)	a compact part of a plane that passes trough the origin
(2,2)	a segment on a straight line that does not pass trough the origin
(1,2)	a segment on a straight line that passes trough the origin
(1,1)	a point that does not coincide with the origin
(0,1)	a point that coincides with the origin

Proof. Let q_1 and q_2 be two different points of a submanifold. By using an affine transformation $G_1 \in GL(3,\mathbb{R})\times GL(3,\mathbb{R})$, one can achieve $T_{q_2}F^3\times T^{\perp}_{q_2}F^3 = G_1(T_{q_1}F^3\times T^{\perp}_{q_1}F^3)$. By using another transformation $G_2 \in GL(3,\mathbb{R})\times GL(3,\mathbb{R})$, one can achieve the coincidence of the G_1-transformed tangent and normal frames at q_1 with the tangent and normal frames at q_2. The G_2 tangent transformation means the local parameterization change and G_2-normal transformation means the change of normal frame. The $G_2 \circ G_1$ transformation leads to "joint" framing at q_2. As a consequence, we get $g(q_2) = (G_2 \circ G_1)(g(q_1))$ while for the second fundamental forms $B_\alpha(q_2) \neq (G_2 \circ G_1)(B_\alpha(q_1))$ in general.

So the problem is reduced to the following one: *at a given point, find the best tangent and normal frames that reduce the first fundamental form and the second fundamental forms to the simplest/canonical forms simultaneously*. The problem can not be solved by using the affine transformations only, except codimension 1. To overcome this obstruction, consider the projective mapping

$$\mathrm{Ind}(X^1 : X^2 : X^3) = (g(X,X) : B_1(X,X) : B_2(X,X) : B_3(X,X))$$

that assigns to the tangent direction $X = \{X^1, X^2, X^3\}$ a point in $\mathbb{R}P^3$ and allows the actions of $GL(3,\mathbb{R})\times GL(4,\mathbb{R})$ over points of $\mathbb{R}P^2\times\mathbb{R}P^3$. In generic case the formes $g(X,X)$, $B_1(X,X)$, $B_2(X,X)$ and $B_3(X,X)$ are homogeneous second order linearly independent polynomials and hence define the projective Steiner surface. *This is exactly the case of index* $(3,4)$. As it was proved in [7], the number of orbits of group action $GL(3,\mathbb{R})\times GL(4,\mathbb{R})$ is finite and each orbit can be represented by one of the surfaces listed in Theorem 2.3. Excluding the non-compact forms, we come to 4 surfaces from A. Coffman's list. Namely, the Roman surface (Fig. 3.1), the T-surface (Fig. 3.2), the Cross-cap (Fig. 3.3) and the Cross-cup (Fig. 3.4).

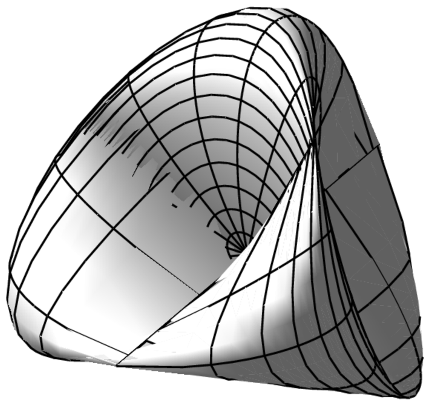

Fig. 3.1: Roman surface

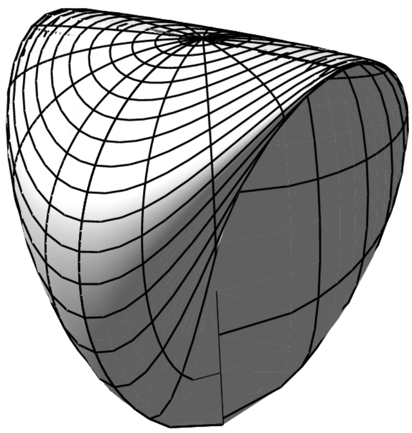

Fig. 3.2: T-surface

In the case of index $(3,3)$, the second fundamental forms $B_1(X,X)$, $B_2(X,X)$ and $B_3(X,X)$ are linearly independent, while 4 forms $g(X,X)$, $B_1(X,X)$, $B_2(X,X)$ and $B_3(X,X)$ are linearly dependent at a point $q \in F^3$

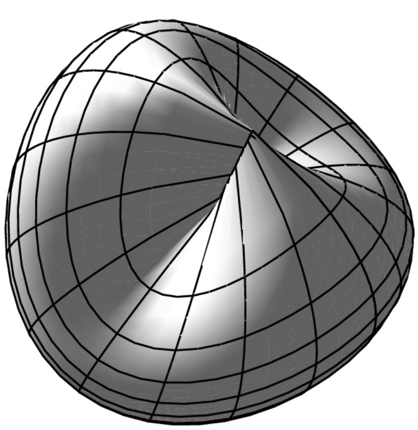

Fig. 3.3: Cross-cap

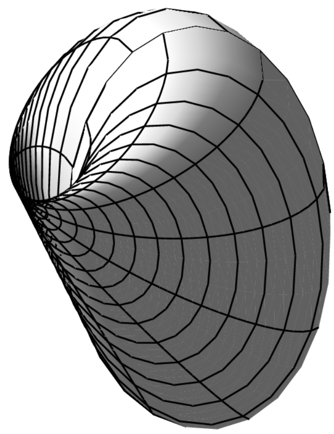

Fig. 3.4: Cross-cup

we are interested in. Therefore,

$$g(X,X) = \xi^1 B_1(X,X) + \xi^2 B_2(X,X) + \xi^3 B_3(X,X) = B_\xi(X,X)$$

for some $\xi \in T_q^\perp F^3$ which means that q is umbilical point with respect to ξ. By making additional affine transformation in $T_q^\perp F^3$ one can achieve $n_3 = \xi$, and hence the normal curvature indicatrix parameterization takes the form

$$\operatorname{ind}(X) = \left\{ \frac{B_1(X,X)}{g(X,X)}, \frac{B_2(X,X)}{g(X,X)}, 1 \right\}$$

and degenerates into some compact part of a plane that does not pass through the origin.

The case of index $(2,3)$ means that the second fundamental formes are linearly dependent, and hence

$$\xi^1 B_1(X,X) + \xi^2 B_2(X,X) + \xi^3 B_3(X,X) = 0$$

for some $\xi \in T_q^\perp F^3$ which means that there is a normal such that the second fundamental form with respect to this normal is zero. By making additional affine transformation in $T_q^\perp F^3$ one can achieve $n_3 = \xi$, and hence the normal curvature indicatrix parameterization takes the form

$$\operatorname{ind}(X) = \left\{ \frac{B_1(X,X)}{g(X,X)}, \frac{B_2(X,X)}{g(X,X)}, 0 \right\}$$

and degenerates into some compact part of a plane that passes through the origin.

The case of index $(2,2)$ means that there are two specific normals $\xi_2, \xi_3 \in T_q^\perp F^3$ such that $B_{\xi_2}(X,X) \equiv g(X,X)$ and $B_{\xi_3}(X,X) \equiv 0$. So, we have

$$\operatorname{ind}(X) = \left\{ \frac{B_1(X,X)}{g(X,X)}, 1, 0 \right\},$$

and the indicatrix degenerates into a segment of a line that does not pass trough the origin.

The case of index $(1,2)$ means that there are two specific normals $\xi_2, \xi_3 \in T_q^{\perp}F^3$ such that $B_{\xi_2}(X,X) \equiv 0$ and $B_{\xi_3}(X,X) \equiv 0$. So, we have

$$\mathrm{ind}(X) = \left\{ \frac{B_1(X,X)}{g(X,X)}, 0, 0 \right\},$$

and the indicatrix degenerates into a segment of a line that passes trough the origin.

The case of index $(1,1)$ means that there are three specific normals $\xi_1, \xi_2, \xi_3 \in T_q^{\perp}F^3$ such that $B_{\xi_1}(X,X) \equiv g(X,X)$ and $B_{\xi_2}(X,X) \equiv 0, B_{\xi_3}(X,X) \equiv 0$. So, we have

$$\mathrm{ind}(X) = \{1, 0, 0\},$$

and the indicatrix degenerates into a point that does not coincide with the origin.

Finally, **the case of index** $(1,1)$ means that at a given point $B_1(X,X) \equiv 0, B_2(X,X) \equiv 0, B_3(X,X) \equiv 0$. So, we have

$$\mathrm{ind}(X) = \{0, 0, 0\},$$

and the indicatrix degenerates into the origin. The proof is complete. □

References

[1] A. Borisenko, *Intrinsic and exrtinsic geometry of multidimensional submanifolds*, Examen, Moskow, 2003 (Russian).

[2] A. Borisenko, *Affine classification of points of multidimensional surfaces*, Sibirsk. Mat. Zh. **31** (1990), No. 3, 19–29 (Russian); Eng. transl.: Sib. Math. J. **31** (1990), 379–387.

[3] A. Borisenko, *Extrinsic geometry of stronglyt parabolic multidimensional submanifolds*, Uspekhi Mat. Nauk **52** (1997), No. 6, 3–52; Eng. transl.: Russian Math. Surveys **52** (1997), No. 6, 1141–1190.

[4] A. Borisenko and Yu. Nikolayevskii, *Classification of the points of three-dimensional surfacesaccording to the Grassmann image*, Ukraiskii Geometricheskii Sbornik **22** (1989), No. 32, pp. 11–27; Eng. transl.: J. Sov. Math. **59** (1992), No. 2, 697–709.

[5] A. Borisenko, *On the Structure of Multidimensional Submanifolds with Metric of Revolution in Euclidean Space*, Zh. Mat. Fiz. Anal. Geom. **15** (2019), No. 2, 192–202.

[6] A. Borisenko and O. Lejbina, *Classification of points of two- and tree- dimensional complex surfaces by Grassmann image*, Mat. Fiz. Anal. Geom. **9** (2002), No. 4, 572–594 (Russian).

[7] A. Coffman and A. Schwartz, and C. Stanton, The algebra and geometry of Steiner and other quadratically parametrizable surfaces, Comput. Aided Geom. Design. **13**, No. 3, 257–286.

[8] A. Coffman, https://users.pfw.edu/CoffmanA/steinersurface.html.

Received June 1, 2020.

Alexander Yampolsky,
V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: a.yampolsky@karazin.ua

Про проєктивну класифікацію точок підмноговидів у евклідовому просторі

Alexander Yampolsky

Ми пропонуємо класифікацію точок підмноговидів у евклідовому просторі за типом індикатриси нормальної кривини з точністю до проєктивного перетворення і даємо необхідну умову для існування скінченного числа таких класів. Ми застосовуємо цю умову до випадку тривимірного підмноговиду у шестивимірному евклідовому просторі та доводимо, що існує 10 типів проєктивно еквівалентних точок.

Ключові слова: індикатриса нормальної кривини, тип точки підмноговиду, проєктивне перетворення