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Entropy Solutions of the Dirichlet Problem for Some Nonlinear Elliptic Degenerate Second-Order Equations

Yuliya Gorban and Anastasiia Soloviova

In the present paper, we deal with the Dirichlet problem for a model nonlinear elliptic second-order equation with degenerated (with respect to the independent variables) coefficients, lower term, and L^1-right-hand side. The existence of an entropy solution to the problem under consideration is proved.

Key words: degenerate elliptic equations, L^1-right-hand side, Dirichlet problem, entropy solution, existence of solutions

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1. Introduction

Let $n \geqslant 2$ be an arbitrary fixed natural number, Ω be a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ be a boundary of Ω.

We consider a problem on finding a function $u : \overline{\Omega} \to \mathbb{R}$ that satisfies (in some sense) an equation

$$-\sum_{i=1}^{n} D_i \left(\nu(x)|D_i u|^{p-2} D_i u\right) + h(x)g(u) = f(x), \quad x \in \Omega, \tag{1.1}$$

and a boundary condition

$$u\big|_{\partial\Omega} = 0. \tag{1.2}$$

Here and in the sequel we use a notation $D_i := \dfrac{\partial}{\partial x_i}$, $i = 1, \dots, n$, and suppose that $p \in (1, n)$, $\nu : \Omega \to \mathbb{R}$ is a measurable function such that

$$\nu \in L^1_{\mathrm{loc}}(\Omega), \quad \nu > 0 \text{ a.e. in } \Omega, \quad (1/\nu)^{1/(p-1)} \in L^1(\Omega), \tag{1.3}$$

$h \in L^1(\Omega)$, $h \geqslant 0$ in Ω, $g : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function such that $g(0) = 0$, $f \in L^1(\Omega)$.

Our research presents an actual branch of the modern theory of partial differential equations. This area includes studying nonlinear equations and variational inequalities with L^1-data or measures as data. These investigations have been actively carried out since the 80s of the last century. Today the theory of nonlinear isotropic nondegenerate (with respect to the independent variables) elliptic second-order equations with L^1-right-hand sides is built. So, the concepts of the weak, entropy and renormalized solutions were introduced, the theorems on the existence and uniqueness of these types of the solutions were proved, the conditions of their belonging to some Lebesque and Sobolev spaces were obtained. A significant contribution to the development of this theory was made by Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, F. Murat, M. Pierre, J.L. Vázquez, J.-M. Rakotoson, A. Alvino, V. Ferone, A. Mercaldo, L. Orsina, A. Porretta, S. Segura de León, G. Trombetti, A. Kovalevsky and others.

The main difficulty in studying the solvability of elliptic equations with L^1-right-hand sides is that the right-hand side does not generate a linear continuous functional on the corresponding energy Sobolev space. As a result, the using of the well-known theory of monotone operators is impossible. We need to clarify a concept of the solution of such an equation. In the case of sufficiently regular right-hand side, we say about a generalized solution. Otherwise, the natural analogue of the generalized solution is a weak solution (i.e., the solution from $W^{1,1}$ in sense of the integral identity). The existence of weak solutions of the Dirichlet problem for nonlinear elliptic equations with L^1-right-hand sides was investigated by L. Boccardo, T. Gallouët in [5], [6].

An effective approach for studying the solvability of the Dirichlet problem for nonlinear elliptic second-order equations with L^1-right-hand sides was proposed by Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vázquez in [4]. The authors defined an entropy solution to the problem under consideration and introduced new functional classes containing entropy solutions. These classes are a natural extension of the corresponding energy Sobolev spaces. It is found that if the equation coefficients satisfy the standard conditions of the growth, coercitivity and strict monotonicity, then there exists a unique entropy solution for all values of the parameter characterizing the rates of growth of the coefficients with respect to the corresponding derivatives of unknown function. The above-mentioned and other close investigations relate to the L^1-theory of nonlinear isotropic nondegenerate (with respect to the independent variables) second-order equations. As for nonlinear elliptic second-order equations with L^1-data or measures as data with anisotropic or degenerate (with respect to the independent variables) coefficients, we note the following. The existence of a weak solution of the Dirichlet problem for a model elliptic equation with anisotropic and nondegenerate (with respect to the independent variables) coefficients and measure in the right-hand side was established by L. Boccardo, T. Gallouët, P. Marcellini in [7]. The existence of weak solutions for a class of anisotropic and nondegenerate (with respect to the independent variables) equations and locally integrable data was proved by M. Bendahmane and K.H. Karlsen in [3]. The solvability of the Dirichlet problem for elliptic equations with isotropic and degenerate (with respect to the independent variables) coefficients and L^1-data or measures as data was studied by L. Aharouch, E. Azroul, A. Benkirane in [1], Y. Atik, J.-M. Rakotoson in [2], A.C. Cavalheiro in [8], G.R. Cirmi in [9], F.Q. Li in [18].

The questions on the existence and properties of the solutions for anisotropic and degenerate (with respect to the independent variables) second-order equations with L^1-right-hand sides without lower terms were considered by A. Kovalevsky and Yu. Gorban in [16], [17]. Similar questions for the equations with lower terms were later studied by Yu. Gorban in [10] (the existence of an entropy solution), and [11] (the uniqueness of an entropy solution). Namely, in [10], the problem considered is the Dirichlet problem for an equation

$$-\sum_{i=1}^{n} D_i(a_i(x,\nabla u)) = F(x,u), \quad x \in \Omega, \tag{1.4}$$

with a boundary condition (1.2). Here:

- for each $i \in \{1,\dots,n\}$, $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^{n} (1/\nu_i(x))^{1/(q_i-1)} |a_i(x,\xi)|^{q_i/(q_i-1)} \leqslant \widehat{c}_1 \sum_{i=1}^{n} \nu_i(x)|\xi_i|^{q_i} + g_1(x),$$

$$\sum_{i=1}^{n} a_i(x,\xi)\xi_i \geqslant \widehat{c}_2 \sum_{i=1}^{n} \nu_i(x)|\xi_i|^{q_i} - g_2(x),$$

$$\sum_{i=1}^{n} [a_i(x,\xi) - a_i(x,\xi')]\,(\xi_i - \xi_i') > 0,$$

  where $1 < q_i < n$, $\nu_i \in L^1_{\mathrm{loc}}(\Omega)$, $\nu_i > 0$ a.e. in Ω, $(1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega)$, $\widehat{c}_1$, $\widehat{c}_2 > 0$ are constants, g_1, $g_2 \in L^1(\Omega)$, g_1, $g_2 \geqslant 0$ in Ω;

- $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$, $F(x,\cdot)$ is non-increasing on $\mathbb{R}$, and for any $s \in \mathbb{R}$, $F(\cdot,s) \in L^1(\Omega)$.

The main result of [10] is the statement (Theorem 4.1) that under the above conditions there exists the so-called entropy solution of the Dirichlet problem (1.4), (1.2). A general approach of [4] was used for proving it. One of the basic steps of this approach is to get special uniform estimates for the solutions $\{u_l\}$, $l \in \mathbb{N}$, of the approximating Dirichlet problems:

$$\operatorname{meas}\{|u_l| \geqslant k\} \leqslant \widehat{c}_3 k^{-\widehat{q}}, \tag{1.5}$$

$$\operatorname{meas}\{\nu_i^{1/q_i}|D_i u_l| \geqslant k\} \leqslant \widehat{c}_4 k^{-q_i\widehat{q}/(1+\widehat{q})}, \quad i = 1,\dots,n, \tag{1.6}$$

where $k \geqslant 1$ is an arbitrary real number, $\widehat{q} := \dfrac{n(\overline{q}-1)}{(n-1)\overline{q}}$, $\overline{q}$ is the harmonic mean of $q_1,\dots,q_n$, $\widehat{c}_3$, $\widehat{c}_4$ are positive constants depending only on n, $q_1,\dots,q_n$, $\widehat{c}_1$, $\widehat{c}_2$, $\|g_1\|_{L^1(\Omega)}$, $\|g_2\|_{L^1(\Omega)}$, $\|F(\cdot,0)\|_{L^1(\Omega)}$, $\|F(\cdot,-1)\|_{L^1(\Omega)}$, $\|F(\cdot,1)\|_{L^1(\Omega)}$, $\|1/\nu_i\|_{L^{1/(q_i-1)}(\Omega)}$, and $\operatorname{meas}\Omega$.

In the present paper, we deal with a partial case of equation (1.4):

$$q_i = p, \quad \nu_i = \nu, \quad a_i(x,\xi) = \nu(x)|\xi_i|^{p-2}\xi_i, \quad (x,\xi) \in \Omega \times \mathbb{R}^n, \quad i = 1,\dots,n,$$

$$F(x,s) = f(x) - h(x)g(s), \quad (x,s) \in \Omega \times \mathbb{R}.$$

The main result of our paper is the theorem on the existence of entropy solutions to the problem (1.1), (1.2). The approach from [4], mentioned above, was applied. A model case allows us to clarify some results of [10]. It concerns uniform estimates (1.5), (1.6). Such inequalities are used for proving the results on the existence and uniqueness of different types of solutions for equations and variational inequalities with L^1-data or measures as data. Besides, estimates (1.5), (1.6) are used in studying summability properties of solutions (see, for example, [14, § 6]). In our simple case, we can obtain an explicit form of the constants $\widehat{c}_3$, $\widehat{c}_4$ and write down their explicit forms depending on the input parameters. It allows to improve the summability of entropy solutions in a model and more complicated cases.

2. Preliminaries

In this section, we introduce some concepts and present the results similar to those from [15] which will be used in the sequel.

We set

$$\widehat{p} := \frac{n(p-1)}{(n-1)p}, \quad c_{p,\nu} := 1 + \|(1/\nu)^{1/(p-1)}\|_{L^1(\Omega)}.$$

Let $W^{1,p}(\nu,\Omega)$ be a linear space of all functions $u \in W^{1,1}(\Omega)$ such that $\nu|D_i u|^p \in L^1(\Omega)$. The mapping

$$\|u\|_{1,p,\nu} = \int_\Omega |u|\,dx + \sum_{i=1}^n \left(\int_\Omega \nu|D_i u|^p\,dx\right)^{1/p}$$

is a norm in $W^{1,p}(\nu,\Omega)$, and, in view of the second inclusion of (1.3), $W^{1,p}(\nu,\Omega)$ is a Banach space. Moreover, by virtue of the first inclusion of (1.3), we have $C_0^\infty(\Omega) \subset W^{1,p}(\nu,\Omega)$.

We denote by $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in the space $W^{1,p}(\nu,\Omega)$.

Proposition 2.1. *The space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ has the following properties:*

(i) $\overset{\circ}{W}{}^{1,p}(\nu,\Omega) \subset \overset{\circ}{W}{}^{1,1}(\Omega)$*, and for every function* $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$*, we have*

$$\|u\|_{W^{1,1}(\Omega)} \leqslant c_{p,\nu}\|u\|_{1,p,\nu};$$

(ii) *if* $u_j \to u$ *weakly in* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$*, then* $u_j \to u$ *strongly in* $L^1(\Omega)$*;*

(iii) $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *is a reflexive space.*

Proof. Let $u \in W^{1,p}(\nu,\Omega)$. Using an inclusion $u \in W^{1,1}(\Omega)$ and the Hölder inequality, we establish

$$\|u\|_{W^{1,1}(\Omega)} = \int_\Omega |u|\,dx + \sum_{i=1}^n \int_\Omega (1/\nu)^{1/p}\,\nu^{1/p}\,|D_i u|\,dx \leqslant \int_\Omega |u|\,dx$$
$$+\sum_{i=1}^n \left(\int_\Omega (1/\nu)^{1/(p-1)} dx\right)^{(p-1)/p} \left(\int_\Omega \nu |D_i u|^p\,dx\right)^{1/p} \leqslant c_{p,\nu}\|u\|_{1,p,\nu}.$$

So, assertion (i) is true. From this fact and the compactness of the embedding $\overset{\circ}{W}{}^{1,1}(\Omega)$ in $L^1(\Omega)$ we deduce (ii). Finally, the proof of (iii) can be found in [14]. □

Further, for every $k>0$, let $T_k : \mathbb{R} \to \mathbb{R}$ be a function such that

$$T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k \operatorname{sign} s & \text{if } |s| > k. \end{cases}$$

By analogy with known results for nonweighted Sobolev spaces (see, for instance, [12, Chapter 2]), we have: if $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $k>0$, then $T_k(u) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and for every $i \in \{1,\dots,n\}$,

$$D_i T_k(u) = D_i u \, 1_{\{|u|<k\}} \quad \text{a.e. in } \Omega. \tag{2.1}$$

By $\overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$, we denote the set of all functions $u : \Omega \to \mathbb{R}$ such that for every $k>0$ we have $T_k(u) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$.

Clearly,

$$\overset{\circ}{W}{}^{1,p}(\nu,\Omega) \subset \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega). \tag{2.2}$$

Definition 2.2. For a function $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$ we take

$$\delta u(x) := (\delta_1 u(x), \dots, \delta_n u(x)), \quad x \in \Omega,$$

where for every $i \in \{1,\dots,n\}$ we put

$$\delta_i u(x) := \lim_{k\to\infty} D_i T_k(u)(x) \quad \text{for a. e. } x \in \Omega.$$

Proposition 2.3. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$. Then for every $k>0$ we have $D_i T_k(u) = \delta_i u\, 1_{\{|u|<k\}}$ a. e. in Ω, $i=1,\dots,n$.*

The proof of this proposition is in [13].

From (2.1), (2.2) and Proposition 2.3, it follows that for $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ we have $\delta_i u = D_i u$ a.e. in Ω, $i=1,\dots,n$.

Proposition 2.4. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$ and $v \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$. Then $u - v \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$, and for every $k>0$,*

$$D_i T_k(u-v) = \delta_i u - D_i v \quad \text{a.e. in } \{|u-v|<k\}, \quad i=1,\dots,n.$$

The proof of this proposition can be found in [14].

Proposition 2.5. *Let* $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$ *and* $w \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega) \cap L^\infty(\Omega)$*. Then for every* $k > 0$ *we have* $\nu\,|\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w) \in L^1(\Omega)$*,* $i = 1,\dots,n$*.*

Proof. First of all, we note that there exists a measure zero set $E \subset \Omega$ such that

$$\forall x \in \Omega \setminus E \quad |w(x)| \leqslant \|w\|_{L^\infty(\Omega)}.$$

Fix $k > 0$, $i \in \{1,\dots,n\}$. From the definition of the truncated function T_k it follows that

$$\nu\,|\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w) = 0 \quad \text{a.e. in } \{|u-w| \geqslant k\}. \tag{2.3}$$

Put $k_1 = k + \|w\|_{L^\infty(\Omega)}$. Using an inclusion $\{|u-w| < k\} \setminus E \subset \{|u| < k_1\}$ and Proposition 2.3, we obtain

$$D_i T_{k_1}(u) = \delta_i u \quad \text{a.e. in } \{|u-w| < k\}.$$

From the latter quality and Proposition 2.4, we deduce

$$\begin{aligned}\nu|\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w) &= \nu|D_i T_{k_1}(u)|^{p-2} D_i T_{k_1}(u)(D_i T_{k_1}(u) - D_i w)\\ &= \nu|D_i T_{k_1}(u)|^p - \nu|D_i T_{k_1}(u)|^{p-2} D_i T_{k_1}(u) D_i w \quad \text{a.e. in } \{|u-w| < k\},\end{aligned}$$

and thus

$$\begin{aligned}\big|\nu\,|\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w)\big| \leqslant \nu\,|D_i T_{k_1}(u)|^p + \nu\,|D_i T_{k_1}(u)|^{p-1}|D_i w|&\\ \text{a.e. in } \{|u-w| < k\}.&\end{aligned} \tag{2.4}$$

We apply the Young inequality to estimate the second term on the right-hand side of (2.4):

$$\begin{aligned}\nu\,|D_i T_{k_1}(u)|^{p-1}|D_i w| &= \nu^{1/p}\nu^{(p-1)/p}\,|D_i T_{k_1}(u)|^{p-1}|D_i w|\\ &\leqslant \nu\,|D_i T_{k_1}(u)|^p + \nu\,|D_i w|^p \quad \text{a.e. in } \{|u-w| < k\}.\end{aligned} \tag{2.5}$$

Taking into account (2.4), (2.5) and the summability of the functions $\nu\,|D_i T_{k_1}(u)|^p$ and $\nu\,|D_i w|^p$ in Ω, we infer that a function $\nu\,|\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w)$ is summable a.e. in $\{|u-w| < k\}$. This fact and (2.3) provide the required result. □

Proposition 2.6. *There exists a positive constant* c_0 *depending on* n*,* p*, and* $\|(1/\nu)^{1/(p-1)}\|_{L^1(\Omega)}$ *such that for every function* $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$*,*

$$\left(\int_\Omega |u|^{n/(n-1)} dx\right)^{(n-1)/n} \leqslant c_0 \prod_{i=1}^{n}\left(\int_\Omega \nu|D_i u|^p dx\right)^{1/np}.$$

The proof of this proposition is in [14].

3. Existence of the entropy solution to the Dirichlet problem (1.1), (1.2)

In this section, we define an entropy solution of the problem (1.1), (1.2) and prove its existence.

Definition 3.1. An entropy solution of the Dirichlet problem (1.1), (1.2) is a function $u \in \mathring{\mathcal{T}}^{1,p}(\nu,\Omega)$ such that:

(i) $hg(u) \in L^1(\Omega)$;

(ii) for every $w \in \mathring{W}^{1,p}(\nu,\Omega) \cap L^\infty,(\Omega)$ and $k \geqslant 1$,

$$\int_\Omega \sum_{i=1}^n \nu\,|\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w)\,dx + \\ + \int_\Omega hg(u)\,T_k(u-w)\,dx \leqslant \int_\Omega f\,T_k(u-w)\,dx. \quad (3.1)$$

Note that Definition 3.1 is well-defined. From Proposition 2.5, it follows that the first left-hand integral in (3.1) is finite. Using (i) and the boundedness of $T_k(u-w)$, we obtain that the second left-hand integral in (3.1) is also finite. Finally, by virtue of the inclusion $f \in L^1(\Omega)$ and the boundedness of $T_k(u-w)$, we deduce that the right-hand integral in (3.1) is finite too.

Theorem 3.2. *Under the above assumptions, there exists an entropy solution of the Dirichlet problem* (1.1), (1.2).

Proof. We will use the approach proposed in [4] to study the solvability of the nondegenerate (with respect to the independent variables) isotropic second-order equations with L^1-right-hand sides. The proof is in 9 steps.

Step 1. For every $l \in \mathbb{N}$, we put

$$g_l(s) := T_l(g(s)),\ \ s\in\mathbb{R};\ \ h_l(x) = T_l(h(x)),\ \ f_l(x) := T_l(f(x)),\ x\in\Omega;$$
$$F_l(v)(x) := h_l(x)g_l(v(x)),\quad x\in\Omega,\quad v:\Omega\to\mathbb{R}\ \text{ is an arbitrary function.}$$

Clearly, $\{h_l\} \subset L^\infty(\Omega)$, $\{f_l\} \subset L^\infty(\Omega)$,

$$\forall\, l\in\mathbb{N}\qquad \|h_l\|_{L^1(\Omega)} \leqslant \|h\|_{L^1(\Omega)},\qquad \|f_l\|_{L^1(\Omega)} \leqslant \|f\|_{L^1(\Omega)}, \quad (3.2)$$

$$\lim_{l\to\infty} \|h_l - h\|_{L^1(\Omega)} = 0,\qquad \lim_{l\to\infty} \|f_l - f\|_{L^1(\Omega)} = 0, \quad (3.3)$$

$$F_l(v)(x)v(x) \geqslant 0\quad \text{for a.e. } x\in\Omega,\ v:\Omega\to\mathbb{R} \text{ is an arbitrary function.} \quad (3.4)$$

From the properties of the higher part of equation (1.1), assertion (3.4) and the results from [19] on the solvability of equations with monotone operators, we deduce that for all $l \in \mathbb{N}$ there exists a unique function $u_l \in \mathring{W}^{1,p}(\nu,\Omega)$ such that

$$\int_\Omega \left\{ \sum_{i=1}^n \nu\,|D_i u_l|^{p-2} D_i u_l\, D_i w + F_l(u_l)\,w \right\} dx = \int_\Omega f_l\, w\, dx \quad (3.5)$$

for every function $w \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$.

By c_i, $i = 1, 2, \ldots$, we denote the positive constants depending only on n, p, $\|f\|_{L^1(\Omega)}$, $\|h\|_{L^1(\Omega)}$, $\|(1/\nu)^{1/(p-1)}\|_{L^1(\Omega)}$ and $\operatorname{meas}\Omega$.

Let us show that for every k, $l \in \mathbb{N}$ the following inequalities hold:

$$\int_{\{|u_l|<k\}} \sum_{i=1}^{n} \nu |D_i u_l|^p \, dx \leqslant c_1 k, \tag{3.6}$$

$$\int_{\{|u_l|\geqslant k\}} |F_l(u_l)| \, dx \leqslant c_2. \tag{3.7}$$

In fact, let k, $l \in \mathbb{N}$. As $u_l \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, we have $T_k(u_l) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Choosing $w = T_k(u_l)$ as a test function in (3.5) and taking into account (2.1), we get

$$\int_{\{|u_l|<k\}} \sum_{i=1}^{n} \nu \, |D_i u_l|^p \, dx + \int_\Omega F_l(u_l) \, T_k(u_l) \, dx = \int_\Omega f_l T_k(u_l) \, dx.$$

In view of (3.2), from the latter equality we obtain

$$\int_{\{|u_l|<k\}} \sum_{i=1}^{n} \nu \, |D_i u_l|^p \, dx + \int_\Omega F_l(u_l) \, T_k(u_l) \, dx \leqslant k \|f\|_{L^1(\Omega)}. \tag{3.8}$$

Assertion (3.4) and the properties of the function T_k imply that

$$F_l(u_l) T_k(u_l) \geqslant 0 \text{ a.e. in } \Omega, \tag{3.9}$$

$$F_l(u_l) T_k(u_l) = k |F_l(u_l)| \text{ a.e. in } \{|u_l| \geqslant k\}. \tag{3.10}$$

The estimate (3.6) follows from (3.9) and (3.8). Finally, inequality (3.7) follows from (3.10) and (3.8).

Step 2. Now we show that for every k, $l \in \mathbb{N}$,

$$\operatorname{meas}\{|u_l| \geqslant k\} \leqslant c_3 k^{-\widehat{p}}, \tag{3.11}$$

$$\operatorname{meas}\{\nu^{1/p} |D_i u_l| \geqslant k\} \leqslant c_4 k^{-p\widehat{p}/(1+\widehat{p})}, \quad i = 1, \ldots, n. \tag{3.12}$$

In fact, let k, $l \in \mathbb{N}$. We have $|T_k(u_l)| = k$ on $\{|u_l| \geqslant k\}$; then

$$k^{n/(n-1)} \operatorname{meas}\{|u_l| \geqslant k\} \leqslant \int_\Omega |T_k(u_l)|^{n/(n-1)} dx. \tag{3.13}$$

Using Proposition 2.6, (2.1) and (3.6), we obtain

$$\left(\int_\Omega |T_k(u_l)|^{n/(n-1)} dx \right)^{(n-1)/n} \leqslant c_0 \prod_{i=1}^{n} \left(\int_{\{|u_l|<k\}} \nu |D_i u_l|^p dx \right)^{1/np} \leqslant c_0 (c_1 k)^{1/p}.$$

Inequality (3.11) follows from the latter estimate and (3.13).

Next, we fix $i \in \{1, \ldots, n\}$ and set

$$k_* := k^{p/(1+\widehat{p})}, \quad G := \{|u_l| < k_*, \, \nu^{1/p} \, |D_i u_l| \geqslant k\}.$$

We have

$$\operatorname{meas}\{\nu^{1/p}\,|D_i u_l| \geqslant k\} \leqslant \operatorname{meas}\{|u_l| \geqslant k_*\} + \operatorname{meas} G. \tag{3.14}$$

From (3.11), it follows that

$$\operatorname{meas}\{|u_l| \geqslant k_*\} \leqslant c_3 k_*^{-\widehat{p}}. \tag{3.15}$$

Moreover, in view of the definition of the set G and (3.6), we get

$$k^p \operatorname{meas} G \leqslant \int_{\{|u_l|<k_*\}} \nu\,|D_i u_l|^p\,dx \leqslant c_1 k_*.$$

Inequality (3.12) follows from the latter estimate and (3.14), (3.15).

Step 3. Assertions (2.1) and (3.6) imply that for every $k \geqslant 1$ the sequence $\{T_k(u_l)\}$ is bounded in $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. As the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ is reflexive, then there exists a sequence $\{z_k\} \subset \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and a subsequence of the sequence $\{u_l\}$ (we denote it by $\{u_l\}$) such that

$$\forall\, k \in \mathbb{N} \quad T_k(u_l) \to z_k \quad \text{weakly in} \quad \overset{\circ}{W}{}^{1,p}(\nu,\Omega). \tag{3.16}$$

Step 4. Let us show that the sequence $\{u_l\}$ is fundamental on measure.

Indeed, let $k, l, j \in \mathbb{N}$. We fix $t > 0$ and set

$$G' = \{|u_l| < k, |u_j| < k, |u_l - u_j| \geqslant t\}.$$

It is clear that

$$\operatorname{meas}\{|u_l - u_j| \geqslant t\} \leqslant \operatorname{meas}\{|u_l| \geqslant k\} + \operatorname{meas}\{|u_j| \geqslant k\} + \operatorname{meas} G'. \tag{3.17}$$

As $t \leqslant |T_k(u_l) - T_k(u_j)|$ on G', we obtain

$$t \operatorname{meas} G' \leqslant \int_\Omega |T_k(u_l) - T_k(u_j)|\,dx.$$

This inequality, (3.11) and (3.17) imply that for every $k, l, j \in \mathbb{N}$,

$$\operatorname{meas}\{|u_l - u_j| \geqslant t\} \leqslant 2c_3 k^{-\widehat{p}} + t^{-1} \int_\Omega |T_k(u_l) - T_k(u_j)|\,dx. \tag{3.18}$$

Let $\varepsilon > 0$. We fix $k \in \mathbb{N}$ such that

$$2c_3 k^{-\widehat{p}} \leqslant \varepsilon/2. \tag{3.19}$$

Taking into account (3.16) and Proposition 2.1, we infer a strong convergence $T_k(u_l) \to z_k$ in $L^1(\Omega)$. Then there exists $N \in \mathbb{N}$ such that for every $l, j \in \mathbb{N}$, $l, j \geqslant N$,

$$\int_\Omega |T_k(u_l) - T_k(u_j)|\,dx \leqslant \varepsilon t/2.$$

From this inequality, (3.18) and (3.19), we deduce that for every $l, j \in \mathbb{N}$, $l, j \geqslant N$,

$$\text{meas}\{|u_l - u_j| \geqslant t\} \leqslant \varepsilon.$$

This means that the sequence $\{u_l\}$ is fundamental on measure.

Step 5. Now we show that for every $i \in \{1, \dots, n\}$ the sequence $\{\nu^{1/p} D_i u_l\}$ is fundamental on measure.

For every $t > 0$ and $l, j \in \mathbb{N}$, we put

$$N_t(l, j) = \text{meas}\left\{ \sum_{i=1}^{n} \nu^{1/p} \left| D_i u_l - D_i u_j \right| \geqslant t \right\}.$$

Besides, for every $t > 0$, $q, k, l, j \in \mathbb{N}$, we set

$$E_{t,q,k}(l, j) = \left\{ \sum_{i=1}^{n} \nu^{1/p} |D_i u_l - D_i u_j| \geqslant t, \right.$$
$$\left. \sum_{i=1}^{n} \nu^{1/p} |D_i u_l| \leqslant q, \sum_{i=1}^{n} \nu^{1/p} |D_i u_j| \leqslant q, |u_l - u_j| < \frac{1}{k} \right\}.$$

Using (3.12), we establish that for every $t > 0$, q, k, l, $j \in \mathbb{N}$,

$$N_t(l, j) \leqslant 2c_4 n^{n+1} q^{-p\widehat{p}/(1+\widehat{p})} + \text{meas}\{|u_l - u_j| \geqslant 1/k\} + \text{meas}\, E_{t,q,k}(l, j). \quad (3.20)$$

Further, we get one estimate for some integrals over $E_{t,q,k}(l, j)$. So we introduce now auxiliary functions and sets.

Let for every $x \in \Omega$, $\Phi_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function such that for every pair $(\xi, \xi') \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\Phi_x(\xi, \xi') = \sum_{i=1}^{n} \nu(x) \left[|\xi_i|^{p-2} \xi_i - |\xi_i{}'|^{p-2} \xi_i') \right] (\xi_i - \xi_i').$$

Note that from the definition of Φ_x it follows that

(i) for every $x \in \Omega \setminus E$, the function Φ_x is continuous on $\mathbb{R}^n \times \mathbb{R}^n$;

(ii) for every $x \in \Omega \setminus E$ and $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, we have $\Phi_x(\xi, \xi') > 0$.

Put for every $t > 0$, $q > t$, and $x \in \Omega$,

$$G_{t,q}(x) = \left\{ (\xi, \xi') \in \mathbb{R}^n \times \mathbb{R}^n : \sum_{i=1}^{n} \nu^{1/p}(x) |\xi_i| \leqslant q, \right.$$
$$\left. \sum_{i=1}^{n} \nu^{1/p}(x) |\xi_i'| \leqslant q, \quad \sum_{i=1}^{n} \nu^{1/p}(x) |\xi_i - \xi_i'| \geqslant t \right\}.$$

As $\nu > 0$ a.e. in Ω, then there exists a set $\tilde{E} \subset \Omega$, meas $\tilde{E} = 0$, such that the set $G_{t,q}(x)$ is nonempty for every $t > 0$, $q > t$, and $x \in \Omega \setminus \tilde{E}$.

Let for every $t > 0$ and $q > t$, $\mu_{t,q} : \Omega \to \mathbb{R}$ be a function such that

$$\mu_{t,q}(x) = \begin{cases} \min\limits_{G_{t,q}(x)} \Phi_x & \text{if } x \in \Omega \setminus \tilde{E} \\ 0 & \text{if } x \in \tilde{E} \end{cases}.$$

For every $t > 0$ and $q > t$, we have $\mu_{t,q} > 0$ a.e. in Ω, and $\mu_{t,q} \in L^1(\Omega)$.

Let $t > 0$, $q \geqslant t + 1$, and $k, l, j \in \mathbb{N}$. We fix $x \in E_{t,q,k}(l,j) \setminus \tilde{E}$, and set $\xi = \nabla u_l(x)$, $\xi' = \nabla u_j(x)$. As $(\xi, \xi') \in G_{t,q}(x)$, then $\mu_{t,q}(x) \leqslant \Phi_x(\xi, \xi')$. This inequality and the definition of the function Φ_x imply that

$$\mu_{t,q}(x) \leqslant \sum_{i=1}^{n} \nu \left[|D_i u_l(x)|^{p-2} D_i u_l(x) - |D_i u_j(x)|^{p-2} D_i u_j(x) \right] (D_i u_l(x) - D_i u_j(x)).$$

Then, taking into account the nonnegativity of the right-hand side of the latter inequality and (2.1), we obtain

$$\int_{E_{t,q,k}(l,j)} \mu_{t,q} \, dx \leqslant \int_\Omega \left\{ \sum_{i=1}^{n} \nu \big[|D_i u_l|^{p-2} D_i u_l - |D_i u_j|^{p-2} D_i u_j \big] \, D_i T_{1/k}(u_l - u_j) \right\} dx. \quad (3.21)$$

In view of (3.5), we have

$$\int_\Omega \left\{ \sum_{i=1}^{n} \nu |D_i u_l|^{p-2} D_i u_l D_i T_{1/k}(u_l - u_j) \right\} dx = \int_\Omega f_l \, T_{1/k}(u_l - u_j) \, dx - \int_\Omega F_l(u_l) T_{1/k}(u_l - u_j) \, dx,$$

$$\int_\Omega \left\{ \sum_{i=1}^{n} \nu |D_i u_j|^{p-2} D_i u_j D_i T_{1/k}(u_j - u_l) \right\} dx = \int_\Omega f_j \, T_{1/k}(u_j - u_l) \, dx - \int_\Omega F_j(u_j) T_{1/k}(u_j - u_l) \, dx.$$

From these equalities and (3.21), it follows that

$$\int_{E_{t,q,k}(l,j)} \mu_{t,q} \, dx \leqslant \frac{1}{k} \int_\Omega |f_l - f_j| \, dx + \frac{1}{k} \int_\Omega |F_l(u_l) - F_j(u_j)| \, dx. \quad (3.22)$$

As $h \geqslant 0$ in Ω, from the definitions of F_l and F_j, and (3.7), we infer that for every $l, j \in \mathbb{N}$,

$$\int_\Omega |F_l(u_l) - F_j(u_j)| \, dx \leqslant 2c_2 + 4(g(1) - g(-1)) \|h\|_{L^1(\Omega)}.$$

Using the latter estimate and (3.22), we find that for every $t > 0$, $q \geqslant t + 1$, $k, l, j \in \mathbb{N}$, the following inequality holds:

$$\int_{E_{t,q,k}(l,j)} \mu_{t,q} \, dx \leqslant \frac{1}{k} \int_\Omega |f_l - f_j| \, dx + \frac{2}{k} \Big(c_2 + 2g(1) - 2g(-1) \Big) \|h\|_{L^1(\Omega)}. \quad (3.23)$$

The sequence $\{u_l\}$ is fundamental on measure. Then there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that for every $k, l, j \in \mathbb{N}$, $l, j \geqslant n_k$, we have

$$\text{meas}\{|u_l - u_j| \geqslant 1/k\} \leqslant 1/k. \tag{3.24}$$

Let $t > 0$ and $\varepsilon > 0$. We fix $q \geqslant t + n$ such that

$$2c_4 n^{n+1} q^{-p\widehat{p}/(1+\widehat{p})} \leqslant \varepsilon/4. \tag{3.25}$$

Put for every $k \in \mathbb{N}$,

$$\alpha_k = \sup_{l,j \geqslant n_k} \text{meas}\, E_{t,q,k}(l,j).$$

Let us show that $\alpha_k \to 0$. Assume the converse. Then there exists $\tau > 0$, an increasing sequence $\{k_s\} \subset \mathbb{N}$ and the sequences $\{l_s\}, \{j_s\} \subset \mathbb{N}$ such that for every $s \in \mathbb{N}$ we have $l_s, j_s \geqslant n_{k_s}$ and

$$\text{meas}\, E_{t,q,k_s}(l_s, j_s) \geqslant \tau. \tag{3.26}$$

Assume $G_s = E_{t,q,k_s}(l_s, j_s)$, $s \in \mathbb{N}$. In view of (3.23) and (3.2), for every $s \in \mathbb{N}$ we get

$$\int_{G_s} \mu_{t,q}\, dx \leqslant \frac{2}{k_s}\Big(c_5 + 2g(1) - 2g(-1)\Big)\|h\|_{L^1(\Omega)}.$$

It means that

$$\lim_{s\to\infty} \int_{G_s} \mu_{t,q}\, dx = 0.$$

From this assertion, taking into account $\mu_{t,q} \in L^1(\Omega)$ and $\mu_{t,q} > 0$ a.e. in Ω, we infer that $\text{meas}\, G_s \to 0$. This fact contradicts to (3.26). Hence we conclude that $\alpha_k \to 0$.

Finally, we fix $k \in \mathbb{N}$ such that the inequalities hold:

$$1/k \leqslant \varepsilon/4, \qquad \alpha_k \leqslant \varepsilon/2. \tag{3.27}$$

Let $l, j \in \mathbb{N}$, $l, j \geqslant n_k$. From (3.20), (3.24), (3.25) and (3.27), it follows that $N_t(l,j) \leqslant \varepsilon$. It means that for every $i \in \{1, \ldots, n\}$ the sequence $\{\nu^{1/p} D_i u_l\}$ is fundamental on measure.

Step 6. From the results of Steps 4, 5 and by F. Riesz's theorem, we get the following facts: there exist measurable functions $u : \Omega \to \mathbb{R}$ and $v_i : \Omega \to \mathbb{R}$, $i = \overline{1, n}$, such that the sequence $\{u_l\}$ converges to u on measure, and for every $i \in \{1, \ldots, n\}$ the sequence $\{\nu^{1/p} D_i u_l\}$ converges to v_i on measure. As is generally known, we can extract the subsequences converging almost everywhere in Ω to the corresponding functions. We may assume without loss of generality that

$$u_l \to u \quad \text{a.e. in } \Omega, \tag{3.28}$$

$$\forall\, i \in \{1, \ldots, n\} \quad \nu^{1/p} D_i u_l \to v_i \quad \text{a.e. in } \Omega. \tag{3.29}$$

From (3.28), (3.16) and Proposition 2.1, we deduce that for every $k \in \mathbb{N}$,

$$T_k(u) \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega), \tag{3.30}$$

$$T_k(u_l) \to T_k(u) \quad \text{weakly in } \overset{\circ}{W}{}^{1,p}(\nu,\Omega). \tag{3.31}$$

Let us show that $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$. Indeed, let $k > 0$. Take $r \in \mathbb{N}$, $r > k$. In view of (3.30), we have $T_r(u) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Hence, by inclusion (2.2), we obtain $T_k(T_r(u)) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. This fact and the equality $T_k(u) = T_k(T_r(u))$ imply that $T_k(u) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Therefore, $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$.

Step 7. Now we show that

$$\forall\, i \in \{1,\dots,n\} \quad D_i u_l \to \delta_i u \quad \text{a.e. in } \Omega. \tag{3.32}$$

In fact, let $i \in \{1,\dots,n\}$. In view of (3.28), there exists a set $E' \subset \Omega$, $\operatorname{meas} E' = 0$, such that

$$\forall\, x \in \Omega \setminus E' : \quad u_l(x) \to u(x), \tag{3.33}$$

and in view of (3.29), there exists a set $E'' \subset \Omega$, $\operatorname{meas} E'' = 0$, such that

$$\forall\, x \in \Omega \setminus E'' \quad \nu^{1/p}(x) D_i u_l(x) \to v_i(x). \tag{3.34}$$

Fix $k \in \mathbb{N}$. By (2.1), if $l \in \mathbb{N}$, then there exists a set $E^{(l)} \subset \Omega$, $\operatorname{meas} E^{(l)} = 0$, such that

$$\forall\, x \in \{|u_l| < k\} \setminus E^{(l)} : \quad D_i T_k(u_l)(x) = D_i u_l(x). \tag{3.35}$$

We denote by $\widehat{E}$ a union of sets E', E'' and $E^{(l)}$, $l \in \mathbb{N}$. Clearly, $\operatorname{meas} \widehat{E} = 0$. Let $x \in \{|u| < k\} \setminus \widehat{E}$. In view of (3.33), there exists $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$, we have $|u_l(x)| < k$. Let $l \in \mathbb{N}$, $l \geqslant l_0$. Then $x \in \{|u_l| < k\} \setminus E^{(l)}$ and, according to (3.35), we get

$$\nu^{1/p}(x) D_i T_k(u_l)(x) = \nu^{1/p}(x) D_i u_l(x).$$

From this equality and (3.34), we deduce that $\nu^{1/p} D_i T_k(u_l)(x) \to v_i(x)$. Thus,

$$\nu^{1/p} D_i T_k(u_l) \to v_i \quad \text{a.e. in } \{|u| < k\}. \tag{3.36}$$

Besides, in view of (2.1) and (3.6), for every $l \in \mathbb{N}$,

$$\int_\Omega \nu |D_i T_k(u_l)|^p \, dx \leqslant c_1 k. \tag{3.37}$$

Using Fatou's lemma, from (3.36) and (3.37) we infer that the function $|v_i|^p$ is summable on $\{|u| < k\}$.

Further, let $\varphi : \Omega \to \mathbb{R}$ be a measurable function such that $|\varphi| \leqslant 1$ in Ω, and let $\varepsilon > 0$. As the function $|v_i|$ is summable on $\{|u| < k\}$, then there exists $\varepsilon_1 \in (0,\varepsilon)$ such that for every measurable set $G \subset \{|u| < k\}$, $\operatorname{meas} G \leqslant \varepsilon_1$, we have

$$\int_G |v_i| \, dx \leqslant \varepsilon. \tag{3.38}$$

Moreover, in view of (3.36) and Egorov's theorem, there exists a measurable set $\Omega' \subset \{|u| < k\}$ such that

$$\operatorname{meas}(\{|u| < k\} \setminus \Omega') \leqslant \varepsilon_1, \tag{3.39}$$

$$\nu^{1/p} D_i T_k(u_l) \to v_i \quad \text{uniformly in } \ \Omega'. \tag{3.40}$$

From (3.38) and (3.39), we infer that

$$\int_{\{|u|<k\}\setminus\Omega'} |v_i|\, dx \leqslant \varepsilon. \tag{3.41}$$

From (3.40), we deduce that there exists $l_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $\ l \geqslant l_1$, the following inequality holds:

$$\int_{\Omega'} |\nu^{1/p} D_i T_k(u_l) - v_i|\, dx \leqslant \varepsilon. \tag{3.42}$$

Let $l \in \mathbb{N}$, $l \geqslant l_1$. Using (3.41), (3.42), the Hölder inequality, (3.39) and (3.37), we get

$$\begin{aligned}\left| \int_{\{|u|<k\}} [\,\nu^{1/p} D_i T_k(u_l) - v_i]\, \varphi\, dx \right| &\leqslant 2\varepsilon + \int_{\{|u|<k\}\setminus\Omega'} \nu^{1/p} |D_i T_k(u_l)|\, dx \\ &\leqslant 2\varepsilon + \varepsilon^{(p-1)/p} \left(\int_\Omega \nu |D_i T_k(u_l)|^p dx \right)^{1/p} \leqslant 2\varepsilon + \varepsilon^{(p-1)/p} (c_1 k)^{1/p}.\end{aligned}$$

As ε is an arbitrary constant, from the latter estimate it follows that

$$\lim_{l\to\infty} \int_{\{|u|<k\}} [\,\nu^{1/p} D_i T_k(u_l) - v_i]\, \varphi\, dx = 0. \tag{3.43}$$

On the other hand, let $F : \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \to \mathbb{R}$ be a functional such that for every function $v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$,

$$\langle F, v \rangle = \int_{\{|u|<k\}} \left(\nu^{1/p} D_i v \right) \varphi\, dx.$$

It is easy to see that $F \in (\overset{\circ}{W}{}^{1,p}(\nu, \Omega))^*$. Hence, by virtue of (3.31), we have

$$\langle F, T_k(u_l) \rangle \to \langle F, T_k(u) \rangle.$$

This fact and the definition of the functional F imply that

$$\lim_{l\to\infty} \int_{\{|u|<k\}} \left(\nu^{1/p} D_i T_k(u_l) \right) \varphi\, dx = \int_{\{|u|<k\}} \left(\nu^{1/p} D_i T_k(u) \right) \varphi\, dx. \tag{3.44}$$

From (3.43) and (3.44), we deduce that

$$\int_{\{|u|<k\}} \left[v_i - \nu^{1/p} D_i T_k(u) \right] \varphi\, dx = 0.$$

In turn, from this equality and Proposition 2.1, we infer that

$$v_i = \nu^{1/p}\delta_i u \quad \text{a.e. in } \ \{|u| < k\}.$$

Since $k \in \mathbb{N}$ is an arbitrary number, from the latter assertion it follows that

$$v_i = \nu^{1/p}\delta_i u \quad \text{a.e. in } \ \Omega. \tag{3.45}$$

Taking into account that $\nu > 0$ a.e. in Ω, from (3.29) and (3.45), we obtain that $D_i u_l \to \delta_i u$ a.e. in Ω. Thus, (3.32) is proved. Then

$$\forall\, i \in \{1, \dots, n\} \quad \nu\, |D_i u_l|^{p-2} D_i u_l \to \nu\, |\delta_i u|^{p-2}\delta_i u \quad \text{a.e. in } \ \Omega. \tag{3.46}$$

Step 8. Let us show that the following assertions are fulfilled:

$$hg(u) \in L^1(\Omega), \tag{3.47}$$

$$F_l(u_l) \to hg(u) \text{ strongly in } L^1(\Omega). \tag{3.48}$$

Indeed, in view of (3.28), we have

$$F_l(u_l) \to hg(u) \qquad \text{a.e. in } \Omega. \tag{3.49}$$

Moreover, using (3.7) and the conditions on the functions h, g, we get for every $l \in \mathbb{N}$,

$$\int_\Omega |F_l(u_l)|\, dx \leqslant c_6.$$

From this fact, (3.49), an inclusion $h \in L^1(\Omega)$ and Fatou's lemma, we obtain (3.47).

Now let us prove (3.48). Firstly, we establish that for every k, $l \in \mathbb{N}$ the following estimate holds:

$$\int_{\{|u_l| \geqslant 2k\}} |F_l(u_l)|\, dx \leqslant \int_{\{|u_l| \geqslant k\}} |f|\, dx + \|f_l - f\|_{L^1(\Omega)}. \tag{3.50}$$

Let $z \in C^1(\mathbb{R})$ be a function such that $0 \leqslant z \leqslant 1$ on $\mathbb{R}$, $z = 0$ on $[-1; 1]$, $z = 1$ on $(-\infty; -2] \cup [2; +\infty)$, and for every $s \in \mathbb{R}$, $z'(s)\,\mathrm{sign}\, s \geqslant 0$.

Fix arbitrary k, $l \in \mathbb{N}$, and denote by $z_k : \mathbb{R} \to \mathbb{R}$ a function such that for every $s \in \mathbb{R}$,

$$z_k(s) = T_1\left(\frac{s}{k}\right) z\left(\frac{s}{k}\right). \tag{3.51}$$

From the properties of the functions T_1 and z, it follows that for every $s \in \mathbb{R}$,

$$|z_k(s)| \leqslant 1. \tag{3.52}$$

Besides,

$$\forall s \in \mathbb{R} \quad |s| \leqslant k \Rightarrow z_k(s) = 0; \tag{3.53}$$

$$\forall s \in \mathbb{R} \quad |s| \geqslant 2k \Rightarrow |z_k(s)| = 1. \tag{3.54}$$

Definition (3.51) implies that $z_k(u_l) \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, and

$$D_i z_k(u_l) = k^{-1} z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) D_i u_l \quad \text{a.e. in } \ \Omega, \ i = 1,\dots,n. \tag{3.55}$$

Substituting $w = z_k(u_l)$ into (3.5) and taking into account (3.52), (3.53), we get

$$\int_\Omega \left\{ \sum_{i=1}^n \nu \, |D_i u_l|^{p-2} D_i u_l \, D_i z_k(u_l) \right\} dx + \int_\Omega F_l(u_l)\, z_k(u_l)\, dx \leqslant \int_{\{|u_l| \geqslant k\}} |f|\, dx + \|f_l - f\|_{L^1(\Omega)}. \tag{3.56}$$

We denote by $I'_{k,l}$ the first integral in the left-hand side of (3.56). In view of the definition of the function z

$$\forall s \in \mathbb{R} \quad \big(|s| \leqslant k \ \text{ or } \ |s| \geqslant 2k\big) \Rightarrow |z'(s)| = 0. \tag{3.57}$$

Using (3.55) and (3.57), we establish that

$$I'_{k,l} = \frac{1}{k} \int_{\{k \leqslant |u_l| \leqslant 2k\}} \left[z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) \left\{ \sum_{i=1}^n \nu \, |D_i u_l|^p \right\} \right] dx. \tag{3.58}$$

From the property of the truncated function and our condition $z'(s)\,\mathrm{sign}\, s \geqslant 0$, $\forall s \in \mathbb{R}$, it follows that almost everywhere in $\{k \leqslant |u_l| \leqslant 2k\}$,

$$z'\left(\frac{u_l}{k}\right) T_1\left(\frac{u_l}{k}\right) = z'\left(\frac{u_l}{k}\right) \mathrm{sign}\left(\frac{u_l}{k}\right) \geqslant 0.$$

Taking into account this fact and our condition (3.58), we deduce that

$$I'_{k,l} \geqslant 0.$$

This and (3.56) imply

$$\int_\Omega F_l(u_l) z_k(u_l)\, dx \leqslant \int_{\{|u_l| \geqslant k\}} |f|\, dx + \|f_l - f\|_{L^1(\Omega)}. \tag{3.59}$$

Note that in view of (3.4) and the definition of the function z_k, we have

$$F_l(u_l) z_k(u_l) \geqslant 0 \ \text{a.e. in } \Omega,$$

and in view of (3.54), we get

$$F_l(u_l) z_k(u_l) = |F_l(u_l)| \quad \text{a.e. in } \{|u_l| \geqslant 2k\}.$$

Then

$$\int_\Omega F_l(u_l) z_k(u_l)\, dx \geqslant \int_{\{|u_l| \geqslant 2k\}} |F_l(u_l)|\, dx.$$

Finally, assertion (3.50) is derived from the latter inequality and (3.59).

Next, we fix an arbitrary $\varepsilon > 0$. It is clear that there exists $\varepsilon_1 > 0$ such that for every measurable set $G \subset \Omega$, $\operatorname{meas} G \leqslant \varepsilon_1$,

$$\int_G |f|\,dx \leqslant \varepsilon, \quad \int_G |h||g(u)|\,dx \leqslant \varepsilon.$$

We fix $k \in \mathbb{N}$ such that the following inequalities hold:

$$c_3 k^{-p} \leqslant \varepsilon_1. \tag{3.60}$$

As $h \in L^1(\Omega)$, we infer that the function $h\,(g(2k) - g(-2k))$ belongs to $L^1(\Omega)$. Hence, there exists $\varepsilon_2 > 0$ such that for every measurable set $G \subset \Omega$, $\operatorname{meas} G \leqslant \varepsilon_2$,

$$\int_G |h|\,(g(2k) - g(-2k))\,dx \leqslant \varepsilon.$$

In view of (3.49), there exists a measurable set $\Omega_1 \subset \Omega$ such that

$$\operatorname{meas}(\Omega \setminus \Omega_1) \leqslant \min(\varepsilon_1, \varepsilon_2), \tag{3.61}$$

and $F_l(u_l) \to hg(u)$ uniformly in Ω_1. Then there exists $L_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant L_1$,

$$\int_{\Omega_1} |F_l(u_l) - hg(u)|\,dx \leqslant \varepsilon. \tag{3.62}$$

Besides, in view of (3.3), there exists $L_2 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant L_2$,

$$\|h_l - h\|_{L^1(\Omega)} \leqslant \varepsilon, \quad \|f_l - f\|_{L^1(\Omega)} \leqslant \varepsilon. \tag{3.63}$$

Now fix $l \in \mathbb{N}$, $l \geqslant \max(L_1, L_2)$. Using (3.61) and (3.62), we obtain

$$\|F_l(u_l) - hg(u)\|_{L^1(\Omega)} \leqslant \int_{\{|u_l| \geqslant 2k\}} |F_l(u_l)|\,dx + \int_{(\Omega\setminus\Omega_1)\cap\{|u_l|<2k\}} |F_l(u_l)|\,dx + 2\varepsilon. \tag{3.64}$$

By virtue of (3.11) and (3.60), we get $\operatorname{meas}\{|u_l| \geqslant k\} \leqslant \varepsilon_1$. Then

$$\int_{\{|u_l| \geqslant k\}} |f|\,dx \leqslant \varepsilon. \tag{3.65}$$

From (3.50), (3.63) and (3.65), we deduce

$$\int_{\{|u_l| \geqslant 2k\}} |F_l(u_l)|\,dx \leqslant 3\varepsilon. \tag{3.66}$$

In view of the definition of the function F_l, we have

$$|F_l(u_l)| \leqslant |h|\,(g(2k) - g(-2k)) \ \text{ a.e. in } \{|u_l| < 2k\}. \tag{3.67}$$

According to the (3.61), we find

$$\int_{\Omega\setminus\Omega_1} |h|\,(g(2k) - g(-2k))\,dx \leqslant \varepsilon. \tag{3.68}$$

From (3.67) and (3.68), it follows that

$$\int_{(\Omega\setminus\Omega_1)\cap\{|u_l|<2k\}} |F_l(u_l)|\, dx \leqslant \varepsilon. \tag{3.69}$$

Using (3.64), (3.66) and (3.69), we infer

$$\|F_l(u_l) - hg(u)\|_{L^1(\Omega)} \leqslant 6\varepsilon.$$

Therefore, $\|F_l(u_l) - hg(u)\|_{L^1(\Omega)} \to 0$. Assertion (3.48) is proved.

Step 9. Let $w \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)\cap L^\infty(\Omega)$, $k \geqslant 1$. Now we show that

$$\int_\Omega \sum_{i=1}^n \nu\, |\delta_i u|^{p-2}\delta_i u\, D_i T_k(u-w)\, dx + \int_\Omega hg(u)\, T_k(u-w)dx \leqslant \int_\Omega f\, T_k(u-w)dx. \tag{3.70}$$

Put

$$H = \{|u-w|<k\}, \quad H_0 = \{|u-w|=k\},$$

and let for every $l\in\mathbb{N}$,

$$H_l = \{|u_l - w|<k\}\setminus H_0, \quad E_l = \{|u_l-w|<k\}\cap H_0.$$

First of all, we prove that for every function $\varphi\in L^1(\Omega)$,

$$\int_{H_l} \varphi\, dx \to \int_H \varphi\, dx. \tag{3.71}$$

Indeed, let $\varphi\in L^1(\Omega)$. For every $j\in\mathbb{N}$ put

$$H^{(j)} = \{|u-w|<k-1/j\}, \qquad \tilde{H}^{(j)} = \{|u-w|>k+1/j\}.$$

We have

$$\text{meas}\left(H\setminus H^{(j)}\right)\to 0, \quad \text{meas}\left(\{|u-w|>k\}\setminus \tilde{H}^{(j)}\right)\to 0. \tag{3.72}$$

We fix an arbitrary $\varepsilon>0$. In view of the property of Lebesgue integral absolute continuity and (3.72), there exists $j\in\mathbb{N}$ such that

$$\int_{H\setminus H^{(j)}} |\varphi|\, dx \leqslant \varepsilon/4, \quad \int_{\{|u-w|>k\}\setminus\tilde{H}^{(j)}} |\varphi|\, dx \leqslant \varepsilon/4. \tag{3.73}$$

Moreover, in view of absolute continuity of Lebesgue integral, (3.28) and Egorov's theorem, there exists a measurable set $\Omega'\subset\Omega$ such that

$$\int_{\Omega\setminus\Omega'} |\varphi|\, dx \leqslant \varepsilon/4, \tag{3.74}$$

$$u_l \to u \quad \text{uniformly in } \ \Omega'. \tag{3.75}$$

Assertion (3.75) means that we can find $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$, and $x \in \Omega'$,

$$|u_l(x) - u(x)| < 1/j. \tag{3.76}$$

Let $l \in \mathbb{N}$, $l \geqslant l_0$. From (3.76), it follows that

$$\left(H^{(j)} \setminus H_l\right) \cap \Omega' = \emptyset, \qquad \{|u_l - w| < k\} \cap \tilde{H}^{(j)} \cap \Omega' = \emptyset.$$

Then

$$H \setminus H_l \subset \left(H \setminus H^{(j)}\right) \cup (\Omega \setminus \Omega'), \quad H_l \setminus H \subset \left(\{|u-w| > k\} \setminus \widetilde{H}^{(j)}\right) \cup (\Omega \setminus \Omega').$$

These inclusions, (3.73) and (3.74) imply that

$$\int_{H \setminus H_l} |\varphi| \, dx \leqslant \varepsilon/2, \quad \int_{H_l \setminus H} |\varphi| \, dx \leqslant \varepsilon/2.$$

Hence,

$$\left| \int_{H_l} \varphi \, dx - \int_H \varphi \, dx \right| \leqslant \varepsilon.$$

The latter estimate means that (3.71) is true.

Further, put

$$k_1 = k + \|w\|_{L^\infty(\Omega)}, \quad \varphi_1 = \sum_{i=1}^n \nu \, |D_i T_{k_1+1}(u)|^{p-2} D_i T_{k_1+1}(u) \, D_i w,$$

and for every $l \in \mathbb{N}$,

$$\psi_l = \sum_{i=1}^n \nu \, |D_i u_l|^p,$$

$$S'_l = \int_{H_l} \left\{ \sum_{i=1}^n \nu \left[|D_i u_l|^{p-2} D_i u_l - |D_i T_{k_1+1}(u)|^{p-2} D_i T_{k_1+1}(u) \right] \, D_i w \right\} dx,$$

$$S''_l = \int_{E_l} \left\{ \sum_{i=1}^n \nu |D_i w|^{p-2} D_i w \, [D_i u_l - D_i w] \right\} dx.$$

We fix an arbitrary $l \in \mathbb{N}$. In view of (3.5), we have

$$\int_\Omega \sum_{i=1}^n \nu \, |D_i u_l|^{p-2} D_i u_l \, D_i T_k(u_l - w) \, dx = \int_\Omega (f_l - F_l(u_l)) \, T_k(u_l - w) \, dx. \tag{3.77}$$

Taking into account (2.1) and the fact that for almost every $x \in \Omega$ and every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^n \nu(x) \left[|\xi_i|^{p-2} \xi_i - |\xi'_i|^{p-2} \xi'_i \right] \, (\xi_i - \xi'_i) > 0,$$

we infer

$$\int_{\Omega}\left\{\sum_{i=1}^{n}\nu\,|D_iu_l|^{p-2}D_iu_l\,D_iT_k(u_l-w)\right\}dx$$
$$\geqslant\int_{H_l}\left\{\sum_{i=1}^{n}\nu\,|D_iu_l|^{p-2}D_iu_l\,[D_iu_l-D_iw]\right\}dx+S_l''.$$

From this inequality and (3.77) we obtain

$$\int_{H_l}\left\{\sum_{i=1}^{n}\nu\,|D_iu_l|^{p}\right\}dx\leqslant\int_{H_l}\left\{\sum_{i=1}^{n}\nu\,|D_iu_l|^{p-2}D_iu_l\,D_iw\right\}dx$$
$$+\int_{\Omega}(f_l-F_l(u_l))\,T_k(u_l-w)\,dx-S_l''.$$

Hence, for every $l\in\mathbb{N}$,

$$\int_{H_l}\psi_l\,dx\leqslant\int_{\Omega}(f_l-F_l(u_l))\,T_k(u_l-w)\,dx+\int_{H_l}\varphi_1\,dx+S_l'-S_l''. \tag{3.78}$$

Note that by virtue of (3.3) and (3.28), we get $f_l\,T_k(u_l-w)\to f\,T_k(u-w)$ strongly in $L^1(\Omega)$. Therefore,

$$\int_{\Omega}f_l\,T_k(u_l-w)\,dx\to\int_{\Omega}f\,T_k(u-w)\,dx. \tag{3.79}$$

Besides, in view of (3.48) and (3.28), we obtain

$$F_l(x,u_l)\,T_k(u_l-w)\to(f-F(x,u))\,T_k(u-w)\text{ strongly in }L^1(\Omega).$$

Hence,

$$\int_{\Omega}F_l(u_l)\,T_k(u_l-w)\,dx\to\int_{\Omega}hg(u)\,T_k(u-w)\,dx. \tag{3.80}$$

As $u\in\overset{\circ}{\mathcal{T}}{}^{1,p}(\nu,\Omega)$, then we have $T_k(u)\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Besides, $w\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. From two latter inclusions and the Young inequality, we imply $\varphi_1\in L^1(\Omega)$. Thus, using (3.71), we deduce that

$$\int_{H_l}\varphi_1\,dx\to\int_{H}\varphi_1\,dx. \tag{3.81}$$

Now we prove that

$$S_l'\to 0. \tag{3.82}$$

Indeed, let $\varepsilon\in(0,1)$. In view of the property of Lebesgue integral absolute continuity, (3.28), (3.46) and Egorov's theorem, there exists a measurable set $\Omega_1\subset\Omega$ such that

$$\int_{\Omega\setminus\Omega_1}|\varphi_1|\,dx\leqslant\varepsilon, \tag{3.83}$$

$$u_l \to u \quad \text{uniformly in} \;\; \Omega_1, \tag{3.84}$$

$$\sum_{i=1}^{n} \nu \, |D_i u_l|^{p-2} D_i u_l \, D_i w \to \sum_{i=1}^{n} \nu \, |\delta_i u|^{p-2} \delta_i u \, D_i w \quad \text{uniformly in} \;\; \Omega_1. \tag{3.85}$$

Assertion (3.84) means that we can find $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$, and $x \in \Omega_1$,

$$|u_l(x) - u(x)| \leqslant \varepsilon. \tag{3.86}$$

Moreover, in view of (3.85), there exists $l_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_1$,

$$\int_{\Omega_1} \left| \sum_{i=1}^{n} \nu \, |D_i u_l|^{p-2} D_i u_l \, D_i w - \sum_{i=1}^{n} \nu \, |\delta_i u|^{p-2} \delta_i u \, D_i w \right| dx \leqslant \varepsilon. \tag{3.87}$$

Let $l \in \mathbb{N}$, $l \geqslant \max(l_0, l_1)$. As $w \in L^\infty(\Omega)$, there exists a set $\widehat{E} \subset \Omega$, meas $\widehat{E} = 0$, such that for every $x \in \Omega \setminus \widehat{E}$ we have $|w(x)| \leqslant \|w\|_{L^\infty(\Omega)}$. From this fact and (3.86), it follows that $(H_l \cap \Omega_1) \setminus \widehat{E} \subset \{|u| < k_1 + 1\}$. Using this inclusion, Proposition 2.3, and (3.87), we obtain

$$\int_{H_l \cap \Omega_1} \left| \sum_{i=1}^{n} \nu \, |D_i u_l|^{p-2} D_i u_l \, D_i w - \varphi_1 \right| dx \leqslant \varepsilon.$$

The latter inequality and (3.83) imply that

$$|S'_l| \leqslant 2\varepsilon + \sum_{i=1}^{n} \int_{H_l \setminus \Omega_1} \nu \, |D_i u_l|^{p-1} \, |D_i w| \, dx. \tag{3.88}$$

Taking into account the Hölder inequality, an inclusion $H_l \setminus \widehat{E} \subset \{|u_l| < k_1\}$, (3.6) and (3.83), we establish that for every $i \in \{1, \ldots, n\}$,

$$\int_{H_l \setminus \Omega_1} \nu \, |D_i u_l|^{p-1} \, |D_i w| \, dx \leqslant (c_1 k_1 + 1)\varepsilon.$$

From this and (3.88) we deduce

$$|S'_l| \leqslant 2\varepsilon + n(c_1 k_1 + 1)\varepsilon.$$

Thus, (3.82) is true.

Further, we show that

$$S''_l \to 0. \tag{3.89}$$

It suffices to take meas $H_0 > 0$. Let $i \in \{1, \ldots, n\}$. As $u \in \mathring{\mathcal{T}}^{1,p}(\nu, \Omega)$ and $w \in \mathring{W}^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)$, by virtue of Proposition 2.4, we have $u - w \in \mathring{\mathcal{T}}^{1,p}(\nu, \Omega)$. Hence, from Proposition 2.3 it follows that

$$D_i T_k(u - w) = 0 \quad \text{a.e. in} \;\; H_0. \tag{3.90}$$

On the other hand, for almost every $x \in H_0$, the inequality $|u(x)| < k_1 + 1$ holds. So, $T_k(u-w) = T_{k_1+1}(u) - w$ a.e. in H_0. Therefore,

$$D_i T_k(u-w) = D_i T_{k_1+1}(u) - D_i w \quad \text{a.e. in } \ H_0.$$

Then, taking into account (3.90), we get $D_i T_{k_1+1}(u) = D_i w$ a.e. in H_0. This and Proposition 2.3 imply $\delta_i u = D_i w$ a.e. in H_0. From this result and (3.32) we infer that for every $i \in \{1, \dots, n\}$ $\ D_i u_l \to D_i w$ a.e. in H_0. Hence,

$$\sum_{i=1}^{n} \nu \, |D_i w|^{p-2} D_i w \, [D_i u_l - D_i w] \to 0 \quad \text{a.e. in } \ H_0. \tag{3.91}$$

Next, we put

$$\varphi_2 = \sum_{i=1}^{n} \nu \, |D_i w|^p.$$

As $w \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$, the function φ_2 is summable on Ω.

We fix an arbitrary $\varepsilon > 0$. In view of the property of Lebesgue integral absolute continuity, (3.91) and Egorov's theorem, there exists a measurable set $\Omega_2 \subset H_0$ such that

$$\int_{H_0 \setminus \Omega_2} \varphi_2 \, dx \leqslant \varepsilon, \tag{3.92}$$

$$\sum_{i=1}^{n} \nu \, |D_i w|^{p-2} D_i w \, [D_i u_l - D_i w] \to 0 \quad \text{uniformly in } \ \Omega_2.$$

The latter property means that me can find $l_0 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_0$,

$$\int_{\Omega_2} \left| \sum_{i=1}^{n} \nu \, |D_i w|^{p-2} D_i w \, [D_i u_l - D_i w] \right| dx \leqslant \varepsilon. \tag{3.93}$$

Let $l \in \mathbb{N}$, $l \geqslant l_0$. Using (3.92) and (3.93), we infer that

$$|S_l''| \leqslant 2\varepsilon + \sum_{i=1}^{n} \int_{E_l \setminus \Omega_2} \nu \, |D_i w|^{p-1} \, |D_i u_l| \, dx. \tag{3.94}$$

By the virtue of the Hölder inequality, (3.92) and (3.6), we deduce that for every $i \in \{1, \dots, n\}$,

$$\int_{E_l \setminus \Omega_2} \nu \, |D_i w|^{p-1} \, |D_i u_l| \, dx$$
$$\leqslant \left(\int_{E_l \setminus \Omega_2} \varphi_2 \, dx \right)^{(p-1)/p} \left(\int_{\{|u_l| < k_1\}} \nu |D_i u_l|^p dx \right)^{1/p} \leqslant \varepsilon^{(p-1)/p} (c_1 k_1)^{1/p}.$$

This fact along with (3.94) and an arbitrariness of ε implies that (3.89) is true.

Further, let $\chi : \Omega \to \mathbb{R}$ be a characteristic function of the set H, and let for every $l \in \mathbb{N}$, $\chi_l : \Omega \to \mathbb{R}$ be a characteristic function of the set H_l. We have

$$\varliminf_{l\to\infty} \chi_l \geqslant \chi \quad \text{a.e. in} \quad \Omega. \tag{3.95}$$

Indeed, in view of (3.28) there exists a set $E_0 \subset \Omega$, $\operatorname{meas} E_0 = 0$, such that for every $x \in \Omega \setminus E_0$ $\quad u_l(x) \to u(x)$. Let $x \in \Omega \setminus E_0$. If $x \notin H$, then $\chi(x) = 0$. Hence, $\chi(x) \leqslant \chi_l(x)$, for all $l \in \mathbb{N}$. Let $x \in H$. As $u_l(x) \to u(x)$, there exists $l_1 \in \mathbb{N}$ such that for every $l \in \mathbb{N}$, $l \geqslant l_1$, we have $|u_l(x) - u(x)| < k - |u(x) - w(x)|$. Then, for arbitrary $l \in \mathbb{N}$, $l \geqslant l_1$, we get $|u_l(x) - w(x)| < k$. Therefore, $x \in H_l$ and $\chi_l(x) = 1 = \chi(x)$. Thus, in any case we have $\chi(x) \leqslant \varliminf_{l\to\infty} \chi_l(x)$ and assertion (3.95) holds.

From (3.95) and (3.32) it follows that

$$\varliminf_{l\to\infty} (\psi_l \chi_l) \geqslant \sum_{i=1}^n \nu |\delta_i u|^p \chi \quad \text{a.e. in} \quad \Omega. \tag{3.96}$$

Using (3.78)–(3.82), (3.89), Fatou's lemma and (3.96), we establish that the function $\left(\sum_{i=1}^n \nu |\delta_i u|^p\right) \chi$ is summable in Ω and

$$\int_\Omega \left\{ \sum_{i=1}^n \nu |\delta_i u|^p \right\} \chi \, dx \leqslant \int_\Omega (f - hg(u))\, T_k(u - w)\, dx + \int_H \varphi_1 \, dx.$$

From the latter inequality and Propositions 2.3 and 2.4 we obtain (3.70). From this fact, an inclusion $u \in \overset{\circ}{\mathcal{T}}{}^{1,p}(\nu, \Omega)$, and when the conditions (i), (ii) of Definition 3.1 are satisfied, we deduce that u is an entropy solution to the Dirichlet problem (1.1), (1.2). $\square$

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References

[1] L. Aharouch, E. Azroul, and A. Benkirane, *Quasilinear degenerated equations with L^1 datum and without coercivity in perturbation terms,* Electron. J. Qual. Theory Differ. Equ. (2006), No. 19, 1–18.

[2] Y. Atik and J.-M. Rakotoson, *Local T-sets and degenerate variational problems. I,* Appl. Math. Lett. **7** (1994), No. 4, 49–53.

[3] M. Bendahmane and K.H. Karlsen, *Nonlinear anisotropic elliptic and parabolic equations in R^N with advection and lower order terms and locally integrable data,* Potential Anal. **22** (2005), No. 3, 207–227.

[4] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J.L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), No. 2, 241–273.

[5] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data,* J. Funct. Anal. **87** (1989), No. 1, 149–169.

[6] L. Boccardo and T. Gallouët, *Nonlinear elliptic equations with right hand side measures,* Comm. Partial Differential Equations **17** (1992), No. 3–4, 641–655.

[7] L. Boccardo, T. Gallouët, and P. Marcellini, *Anisotropic equations in L^1*, Differential Integral Equations **9** (1996), No. 1, 209–212.

[8] A.C. Cavalheiro, *Existence of entropy solutions for degenerate quasilinear elliptic equations,* Complex Var. Elliptic Equ. **53** (2008), No. 10, 945–956.

[9] G. R. Cirmi, *On the existence of solutions to non-linear degenerate elliptic equations with measures data,* Ricerche Mat. **42** (1993), No. 2, 315–329.

[10] Yu. Gorban, *Existence of entropy solutions for nonlinear elliptic degenerate anisotropic equations,* Open Math. **15** (2017), 768–786.

[11] Yu. Gorban, *On uniqueness of entropy solutions for nonlinear elliptic degenerate anisotropic equations,* Mat. Stud. **47** (2017), No. 1, 59–70.

[12] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications,* United Kingdom Edition, Academic Press, New York-London, 1980.

[13] A.A. Kovalevsky, *On a sharp condition of limit summability of solutions of nonlinear elliptic equations with L^1-right-hand sides,* Ukr. Math. Bull. **2** 2005, No. 4, 507–545.

[14] A.A. Kovalevsky and Yu.S. Gorban, *Degenerate anisotropic variational inequalities with L^1-data,* Donetsk, IAMM NAS of Ukraine, 2007.

[15] A.A. Kovalevsky and Yu.S. Gorban, *On T-solutions of degenerate anisotropic elliptic variational inequalities with L^1-data,* Izv. Math. **75** (2011), No. 1, 101–160 (Russian).

[16] A.A. Kovalevsky and Yu.S. Gorban, *Solvability of degenerate anisotropic elliptic second-order equations with L^1-data,* Electron. J. Differential Equations (2013), No. 167, 1–17.

[17] A. Kovalevsky and Yu. Gorban, *Conditions of solvability of the Dirichlet problem for degenerate anisotropic elliptic second-order equations with L^1-data,* Tr. Inst. Prikl. Mat. Mekh. **26** (2013), 76–94.

[18] F.Q. Li, *Nonlinear degenerate elliptic equations with measure data,* Comment. Math. Univ. Carolin. **48** (2007), No. 4, 647–658.

[19] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Paris, 1969 (French).

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Yuliya Gorban,
Vasyl' Stus Donetsk National University, 21 600-richchia Str., Vinnytsia, 21021, Ukraine,
E-mail: yuliyagorban77@gmail.com

Anastasiia Soloviova,
Vasyl' Stus Donetsk National University, 21 600-richchia Str., Vinnytsia, 21021, Ukraine,
E-mail: nastiasoloviova1997@gmail.com

Ентропійні розв'язки задачі Діріхле для деяких нелінійних еліптичних вироджених рівнянь другого порядку

Yuliya Gorban and Anastasiia Soloviova

У роботі досліджено розв'язність задачі Діріхле для модельного нелінійного еліптичного рівняння другого порядку з ізотропними і виродженими (за незалежними змінними) коефіцієнтами, молодшим членом та L^1-правою частиною. Встановлено умови існування ентропійного розв'язку розглянутої задачі.

Ключові слова: вироджені еліптичні рівняння, L^1-права частина, задача Діріхле, ентропійний розв'язок, існування розв'язків