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A Kastler–Kalau–Walze Type Theorem for 7-Dimensional Manifolds with Boundary about Witten Deformation

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In the paper, we give a brute-force proof of the Kastler–Kalau–Walze type theorem for 7-dimensional manifolds with boundary about the Witten deformation and give a theoretical explanation of the gravitational action for 7 dimensional manifolds with boundary.

Key words: Witten deformation, noncommutative residue for manifolds with boundary, lower dimensional volumes

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1. Introduction

The noncommutative residue plays a prominent role in noncommutative geometry [8,19]. Connes [5] used the noncommutative residue to derive an analogue of the conformal 4-dimensional Polyakov action. Connes [6] proved that the noncommutative residue on a compact manifold M coincided with the Dixmier trace on pseudodifferential operators of order $-\dim M$. Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein–Hilbert action, which we call the Kastler–Kalau–Walze theorem. Kastler [10] gave a brute-force proof of this theorem. Kalau and Walze [9] proved this theorem in the normal coordinates system simultaneously. Ackermann [1] gave a note on a new proof of this theorem by means of the heat kernel expansion.

Recently, Ponge defined lower dimensional volumes of Riemannian manifolds by the Wodzicki residue [11]. Fedosov et. al. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace [7]. Wang generalized the Connes results to the case of manifolds with boundary in [15, 16] and proved a Kastler–Kalau–Walze type theorem for the Dirac operator and the signature operator for 3-, 4-dimensional manifolds with boundary [17]. Wang also generalized the definition of lower dimensional volumes to manifolds with boundary and found a Kastler–Kalau–Walze type theorem for higher dimensional manifolds with boundary [18]. Weiping Zhang introduced an elliptic differential operator-Witten deformation in [22]. In [2, 3], we proved the Kastler–Kalau–Walze type theorem for the Witten deformation for 4-, 6-dimensional manifolds with boundary [17]. Furthermore, we considered a higher dimensional case. The motivation of this paper is to establish a Kastler–Kalau–Walze type theorem associated with the Witten deformation for 7-dimensional manifolds with boundary, which can give a theoretic explanation to the gravitational action for 7-dimensional manifolds with boundary.

This paper is organized as follows. In Section 2, we recall lower dimensional volumes of compact Riemannian manifolds with boundary. In Section 3, we compute the lower dimensional volume $\mathrm{Vol}_7^{(2,2)}$ associated with the Witten deformation and get a Kastler–Kalau–Walze type theorem for this case, which can give a theoretical explanation of the gravitational action for 7-dimensional manifolds with boundary.

2. Lower dimensional volumes of compact manifolds with boundary about Witten deformation

In this section, we consider an n-dimensional oriented compact Riemannian manifold (M, g^M) with boundary ∂_M equipped with a fixed spin structure. We assume that near the boundary the metric g^M on M has the form

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + \mathtt{d}x_n^2,$$

where $g^{\partial M}$ is the metric on ∂M. Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic $\partial M \times [0,1)$. By the definition of $h(x_n) \in C^\infty([0,1))$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{[0,1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\hat{g}$ on $\hat{M} = M \bigcup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \bigcup_{\partial M} \partial M \times (-\varepsilon, 0]$:

$$\hat{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + \mathtt{d}x_n^2$$

such that $\hat{g}|_M = g$. We fix the metric $\hat{g}$ on $\hat{M}$ such that $\hat{g}|_M = g$.

Firstly, we will recall the expression of the Witten deformation D_T and its square D_T^2 near the boundary [2, 3]. Let ∇^L denote the Levi-Civita connection about g^M. In the local coordinates $\{x_i; 1 \le i \le n\}$ and the fixed orthonormal frame $\{\widetilde{e_1}, \dots, \widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e_1}, \dots, \widetilde{e_n}) = (\widetilde{e_1}, \dots, \widetilde{e_n})(\omega_{s,t}).$$

Let $\epsilon(\widetilde{e_j*}), \iota(\widetilde{e_j*})$ be the exterior and interior multiplications respectively. Write

$$c(\widetilde{e_j}) = \epsilon(\widetilde{e_j*}) - \iota(\widetilde{e_j*}); \quad \bar{c}(\widetilde{e_j}) = \epsilon(\widetilde{e_j*}) + \iota(\widetilde{e_j*}).$$

The Witten deformation is defined by

$$D_T = d + \delta + T\bar{c}(V)$$

$$= \sum_{i=1}^{n} c(\widetilde{e_i}) \left[\widetilde{e_i} + \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})] \right] + T\bar{c}(V),$$

where $d, \delta, V \in \Gamma(TM)$, any $T \in \mathbf{R}$.

By Proposition 4.6 of [22], we have

$$D_T^2 = (d+\delta)^2 + \sum_{i=1}^{n} c(\widetilde{e_i}) \nabla_{\widetilde{e_i}}^{TM} V + T^2 |V|^2.$$

By [20], $(d+\delta)^2$ is expressed by

$$(d+\delta)^2 = -\Delta_0 - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) - \frac{1}{4}s.$$

Let $g^{ij} = g(dx_i, dx_j)$, $\xi = \sum_k \xi_j dx_j$ and $\nabla^L_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$. We denote

$$\sigma_i = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) c(\widetilde{e_s}) c(\widetilde{e_t}), \quad a_i = \frac{1}{4} \sum_{s,t} \omega_{s,t}(\widetilde{e_i}) \bar{c}(\widetilde{e_s}) \bar{c}(\widetilde{e_t})$$

and

$$\xi^j = g^{ij}\xi_i, \quad \Gamma^k = g^{ij}\Gamma^k_{ij}, \quad \sigma^j = g^{ij}\sigma_i, \quad a^j = g^{ij}a_i.$$

Then D_T can be written as

$$D_T = \sum_{i=1}^{n} c(\widetilde{e_i})(\widetilde{e_i} + \sigma_i + a_i) + T\bar{c}(V).$$

By [1, 20], we have

$$-\Delta_0 = \Delta = -g^{ij}(\nabla^L_i \nabla^L_j - \Gamma^k_{ij}\nabla^L_k).$$

Then we have

$$\begin{aligned} D_T^2 = & -\sum_{i,j} g^{i,j} \Big[\partial_i\partial_j + 2\sigma_i\partial_j + 2a_i\partial_j - \Gamma^k_{i,j}\partial_k + (\partial_i\sigma_j) + (\partial_i a_j) + \sigma_i\sigma_j + \sigma_i a_j \\ & + a_i\sigma_j + a_i a_j - \Gamma^k_{i,j}\sigma_k - \Gamma^k_{i,j}a_k \Big] - \frac{1}{8} \sum_{ijkl} R_{ijkl} \bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) + \frac{1}{4}s \\ & + \sum_{i}^{n} c(\widetilde{e_i})\bar{c}(\nabla^{TM}_{\widetilde{e_i}} V) + T^2|V|^2. \end{aligned}$$

To define the lower dimensional volume, some basic facts and formulae about Boutet de Monvel's calculus, which can be found in Section 2 in [17], are needed. Let

$$F : L^2(\mathbf{R}_t) \to L^2(\mathbf{R}_v), \quad F(u)(v) = \int e^{-ivt} u(t) \mathrm{d}t$$

denote the Fourier transform and $\Phi(\overline{\mathbf{R}^+}) = r^+\Phi(\mathbf{R})$ (similarly define $\Phi(\overline{\mathbf{R}^-})$), where $\Phi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \to C^\infty(\overline{\mathbf{R}^+}), \quad f \to f|\overline{\mathbf{R}^+}; \quad \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}.$$

We define $H^+ = F(\Phi(\overline{\mathbf{R}^+}))$, $H_0^- = F(\Phi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+$ (H_0^-) iff $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\mathrm{Im}\xi < 0\}$ ($\{\mathrm{Im}\xi > 0\}$) such that for all nonnegative integer l,

$$\frac{\mathrm{d}^l h}{\mathrm{d}\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\left(\frac{c_k}{\xi^k}\right)$$

as $|\xi| \to +\infty$, $\mathrm{Im}\xi \leq 0$ $(\mathrm{Im}\xi \geq 0)$.

Let H' be the space of all polynomials and $H^- = H_0^- \bigoplus H'$; $H = H^+ \bigoplus H^-$. Denote by π^+ (π^-) the projection on H^+ (H^-), respectively. For calculations, we take $H = \widetilde{H} = \{$rational functions having no poles on the real axis$\}$ ($\tilde{H}$ is a dense set in the topology of H). Then on $\tilde{H}$,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} \mathrm{d}\xi, \tag{2.1}$$

where Γ^+ is a Jordan close curve included in $\mathrm{Im}(\xi) > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on $\tilde{H}$,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi)\mathrm{d}\xi.$$

So, $\pi'(H^-) = 0$. For $h \in H \bigcap L^1(R)$, $\pi' h = \frac{1}{2\pi}\int_R h(v)dv$ and for $h \in H^+ \bigcap L^1(R)$, $\pi' h = 0$.

Denote by $\mathcal{B}$ Boutet de Monvel's algebra and recall the main theorem in [7].

Theorem 2.1 (Fedosov–Golse–Leichtnam–Schrohe)**.** *Let X and ∂X be connected,* $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$. *Denote by p,s and b the local symbols of pseudo-differential operators P, S and a singular Green operator G respectively, T is a trace operator. Define:*

$$\begin{aligned} \widetilde{\mathrm{Wres}}(A) = &\int_X \int_{\mathbf{S}} \mathrm{tr}_E\left[p_{-n}(x,\xi)\right]\sigma(\xi)\mathrm{d}x \\ &+ 2\pi \int_{\partial X}\int_{\mathbf{S}'} \left\{\mathrm{trace}_E\left[(\mathrm{tr} b_{-n})(x',\xi')\right] + \mathrm{tr}_F\left[s_{1-n}(x',\xi')\right]\right\}\sigma(\xi')\mathrm{d}x'. \end{aligned} \tag{2.2}$$

Then

a) $\widetilde{\mathrm{Wres}}([A,B]) = 0$ *for any $A, B \in \mathcal{B}$;*

b) *it is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.*

Let p_1, p_2 be nonnegative integers and $p_1 + p_2 \leq n$. From Section 2 of [2], we have the following definition.

Definition 2.2 ([2]). Lower dimensional volumes of compact spin manifolds with boundary about the Witten deformation D_T are defined by

$$\mathrm{Vol}_n^{(p_1,p_2)}M := \widetilde{\mathrm{Wres}}[\pi^+D_T^{-p_1}\circ\pi^+D_T^{-p_2}].$$

Denote by $\sigma_l(A)$ the l-order symbol of an operator A. An application of (2.1.4) from [15] shows that

$$\widetilde{\mathrm{Wres}}[\pi^+D_T^{-p_1}\circ\pi^+D_T^{-p_2}] = \int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)}[\sigma_{-n}(D_T^{-p_1}\circ D_T^{-p_2})]\sigma(\xi)\,\mathrm{d}x + \int_{\partial M}\Phi,$$

$$\Phi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\,\mathrm{trace}_{S(TM)}\Big[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+D_T^{-p_1}(x',0,\xi',\xi_n)$$
$$\times\,\partial_{x'}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l D_T^{-p_2}(x',0,\xi',\xi_n)\Big]\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x', \qquad (2.3)$$

and the sum is taken over $r-k+|\alpha|+\ell-j-1=-n$, $r\le -p_1$, $\ell\le -p_2$.

The following proposition plays a key role in the computation of lower dimensional volumes of compact spin manifolds with boundary.

Proposition 2.3 ([18]). *The following identity holds:*

1) *if* $n-p_1-p_2=0$, *then* $\mathrm{Vol}_n^{(p_1,p_2)}M = c_0\mathrm{Vol}_M$;

2) *if* $n-p_1-p_2$ *is odd, then* $\mathrm{Vol}_n^{(p_1,p_2)}M = \int_{\partial M}\Phi$.

3. A Kastler–Kalau–Walze type theorem for 7-dimensional spin manifolds with boundary

In this section, we compute the lower dimensional volume $\mathrm{Vol}_7^{(2,2)}$ for 7-dimensional spin compact manifolds with boundary and prove a Kastler–Kalau–Walze type theorem for this case. By Proposition 2.3, we have

$$\widetilde{\mathrm{Wres}}[\pi^+D_T^{-2}\circ\pi^+D_T^{-2}] = \int_{\partial M}\Phi. \qquad (3.1)$$

So, the only thing we need to do is to compute $\int_{\partial M}\Phi$.

Firstly, we will compute some symbols of the Witten deformation. Let

$$\Gamma^k = \sum_{i,j<n}\sum_{l<n} g^{ij}g^{lk}\langle\nabla_{\partial_i}^L\partial_j,\partial_l\rangle + \sum_{l<n}g^{lk}\langle\nabla_{\partial_n}^L\partial_n,\partial_l\rangle.$$

By the conclusions of [2, 3], we have

Lemma 3.1 ([2, 3]). *The symbols of the Witten deformation are:*

$$\sigma_{-1}(D_T^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2},$$
$$\sigma_{-2}(D_T^{-1}) = \frac{c(\xi)\sigma_0(D_T)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)\Big],$$

$$\sigma_{-2}(D_T^{-2}) = |\xi|^{-2},$$
$$\sigma_{-3}(D_T^{-2}) = -\sqrt{-1}|\xi|^{-4}\xi_k(\Gamma^k - 2a^k - 2\sigma^k) - \sqrt{-1}|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_j g^{\alpha\beta},$$

where

$$\sigma_0(D_T) = \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e}_i)c(\widetilde{e}_i)\bar{c}(\widetilde{e}_s)\bar{c}(\widetilde{e}_t) - \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e}_i)c(\widetilde{e}_i)c(\widetilde{e}_s)c(\widetilde{e}_t) + T\bar{c}(V).$$

Now we will compute the symbol $\sigma_{-4}(D_T^{-2})$. Since the equation (4.16) of [9] is

$$\sigma_{-4}^{\tilde{\Delta}^{-1}}(x,\xi) = \sigma_2^{-1}(\gamma_{-1}^2 + \gamma_{-2}) + i\sigma_2^{-2}\partial_{\xi_\mu}\sigma_2\partial_{x^\mu}\gamma_{-1},$$

where

$$\begin{aligned}
\sigma^{\tilde{\Delta}^1}(x,\xi) &= \sigma_2 + \sigma_1 + \sigma_0,\\
\gamma_{-1}(x,\xi) &= -\sigma_2^{-1}\sigma_1 - i\sigma_2^{-2}\partial_{\xi_\mu}\sigma_2\partial_{x^\mu}\sigma_2,\\
\gamma_{-2}(x,\xi) &= -\sigma_2^{-1}\sigma_0 - \sigma_2^{-2}(i\partial_{\xi_\mu}\sigma_1\partial_{x^\mu}\sigma_2 + \frac{1}{2}\partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2\partial_{x^\mu}\partial_{x^\nu}\sigma_2)\\
&\quad + \sigma_2^{-3}\partial_{\xi_\mu}\partial_{\xi_\nu}\sigma_2\partial_{x^\mu}\sigma_2\partial_{x^\nu}\sigma_2,
\end{aligned}$$

then, by Lemma 3.1 and some calculations, we get

Lemma 3.2. *We have*

$$\begin{aligned}
\sigma_{-4}(D_T^{-2}) &= \sigma_{-4}(D^{-2}) - 4i|\xi|^{-6}(\Gamma^k - 2\sigma^k)\xi_k a^l\xi_l - 4|\xi|^{-6}(a^k\xi_k)^2\\
&\quad + 4|\xi|^{-8}a^k\xi_k\partial_{\xi_\mu}(|\xi|^2)\partial_{x^\mu}(|\xi|^2) + |\xi|^{-4}\big[\partial^i a_j + \sigma^i a_j + a^i\sigma_j + a^i a_j\\
&\quad - \Gamma^k a_k + \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e}_i)\bar{c}(\widetilde{e}_j)c(\widetilde{e}_k)c(\widetilde{e}_l) - \sum_i^n c(\widetilde{e}_i)\bar{c}(\nabla^{TM}_{\widetilde{e}_i}V)\\
&\quad - T^2|V|^2\big] - 2|\xi|^{-6}\partial_{\xi_\mu}(a^k\xi_k)\partial_{x^\mu}(|\xi|^2)\\
&\quad - 2|\xi|^{-2}\partial_{\xi_\mu}(|\xi|^2)\partial_{x^\mu}(|\xi|^2)a^k\xi_k - 2|\xi|^{-4}\partial_{\xi_\mu}(|\xi|^2)\partial_{x^\mu}(a^k\xi_k),
\end{aligned}$$

where $\sigma_{-4}(D^{-2})$ has the following expression by the equation (115) *in* [13]*:*

$$\begin{aligned}
\sigma_{-4}(D^{-2}) &= -|\xi|^{-6}\xi_k\xi_l(\Gamma^k - 2\sigma^k)(\Gamma^l - 2\sigma^l) + 2|\xi|^{-8}\xi^k\xi_l\xi_\alpha\xi_\beta(\Gamma^l - 2\sigma^l)\partial_\mu^x g^{\alpha\beta}\\
&\quad + |\xi|^{-4}(\partial^{x_k}\sigma_k + \sigma^k\sigma_k - \Gamma^k\sigma_k) - 2|\xi|^{-6}\xi^k\xi_l\partial_k^x(\Gamma^l - 2\sigma^l)\\
&\quad + 12|\xi|^{-10}\xi^k\xi_l\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial_k^x g^{\alpha\beta}\partial_l^x g^{\gamma\delta} - 4|\xi|^{-8}\xi^k\xi_\alpha\xi_\gamma\xi_\delta\partial_k^x g^{l\alpha}\partial_l^x g^{\gamma\delta}\\
&\quad - 4|\xi|^{-8}\xi^k\xi^l\xi_\gamma\xi_\delta\partial_{kl}^x g^{\gamma\delta} + |\xi|^{-6}\xi_\alpha\xi_\beta(\Gamma^k - 2\sigma^k)\partial_k^x g^{\alpha\beta} - \frac{1}{4}|\xi|^{-4}s(x)\\
&\quad + 2|\xi|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta g^{kl}\partial_k^x g^{\alpha\beta}\partial_l^x g^{\gamma\delta} - |\xi|^{-6}\xi_\alpha\xi_\beta g^{kl}\partial_{kl}^x g^{\alpha\beta}.
\end{aligned}$$

Since Φ is a global form on ∂M, then, for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates U of x_0 in ∂M(not in M) and compute $\Phi(x_0)$ in the coordinates $\widetilde{U} = U \times [0,1)$ and the metric $\frac{1}{h(x_n)}g^{\partial M} + \mathrm{d}x_n^2$. The dual metric of g^M on $\widetilde{U}$ is $h(x_n)g^{\partial M}+\mathrm{d}x_n^2$. Write $g^M_{ij}=g^M\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)$; $g^{ij}_M=g^M(\mathrm{d}x_i,\mathrm{d}x_j)$. Then

$$\left[g^M_{i,j}\right]=\begin{bmatrix}\frac{1}{h(x_n)}\left[g^{\partial M}_{i,j}\right] & 0\\ 0 & 1\end{bmatrix};\quad \left[g^{i,j}_M\right]=\begin{bmatrix}h(x_n)\left[g^{i,j}_{\partial M}\right] & 0\\ 0 & 1\end{bmatrix} \tag{3.2}$$

and

$$\partial_{x_s}g^{\partial M}_{ij}(x_0)=0,\quad 1\le i,j\le n-1;\quad g^M_{i,j}(x_0)=\delta_{ij}. \tag{3.3}$$

Let $\{E_1,\cdots,E_{n-1}\}$ be an orthonormal frame field in U about $g^{\partial M}$, which is parallel along geodesics, and $E_i=\frac{\partial}{\partial x_i}(x_0)$. Then

$$\left\{\widetilde{E_1}=\sqrt{h(x_n)}E_1,\cdots,\widetilde{E_{n-1}}=\sqrt{h(x_n)}E_{n-1},\widetilde{E_n}=\mathrm{d}x_n\right\}$$

is the orthonormal frame field in $\widetilde{U}$ about g^M. Locally, $S(TM)|\widetilde{U}\cong\widetilde{U}\times\wedge^*_C(\frac{n}{2})$. Let $\{f_1,\cdots,f_n\}$ be the orthonormal basis of $\wedge^*_C(\frac{n}{2})$. If we take a spin frame field $\sigma:\widetilde{U}\to\mathrm{Spin}(M)$ such that $\pi\sigma=\left\{\widetilde{E_1},\cdots,\widetilde{E_n}\right\}$, where $\pi:\mathrm{Spin}(M)\to O(M)$ is a double covering, then $\{[\sigma,f_i],1\le i\le 6\}$ is an orthonormal frame of $S(TM)|_{\widetilde{U}}$. In the following, since the global form Φ is independent of the choice of the local frame, we can compute $\mathrm{trace}_{S(TM)}$ (we will shorten it to trace) in the frame $\{[\sigma,f_i],1\le i\le 6\}$. Let $\left\{\hat{E}_1,\cdots,\hat{E}_n\right\}$ be the canonical basis of R^n and let $c(\hat{E}_i)\in cl_C(n)\cong\mathrm{Hom}(\wedge^*_C(\frac{n}{2}),\wedge^*_C(\frac{n}{2}))$ be the Clifford action. By [17],

$$c(\widetilde{E_i})=\left[(\sigma,c(\hat{E}_i))\right],\quad c(\widetilde{E_i})\left[(\sigma,f_i)\right]=\left[\sigma,(c(\hat{E}_i))f_i\right],\quad \frac{\partial}{\partial x_i}=\left[(\sigma,\frac{\partial}{\partial x_i})\right]. \tag{3.4}$$

Then we have $\frac{\partial}{\partial x_i}c(\widetilde{E_i})=0$ in the above frame.

Nextly, we will give some conclusions as our computing tools. By [13], we have

Lemma 3.3 ([13]). *With the metric g^M on M near the boundary:*

$$\partial_{x_j}(|\xi|^2_{g^M})(x_0)=\begin{cases}0 & \text{if } j<n\\ h'(0)|\xi'|^2_{g^{\partial M}} & \text{if } j=n\end{cases},$$

$$\partial_{x_j}(c(\xi))(x_0)=\begin{cases}0 & \text{if } j<n\\ \partial_{x_n}(c(\xi'))(x_0) & \text{if } j=n\end{cases}.$$

where $\xi=\xi'+\xi_n\mathrm{d}x_n$.

Lemma 3.4 ([13]). *With the metric g^M on M near the boundary:*

$$\partial_{x_i}\partial_{x_j}(|\xi|^2_{g^M})(x_0)\Big|_{|\xi'|=1}=0\quad \text{if } i<n,j=n,\ \text{or } i=n,\ j<n,$$

$$\partial_{x_i}\partial_{x_j}(|\xi|^2_{g^M})(x_0)\Big|_{|\xi'|=1}=-\frac{1}{3}\sum_{\alpha,\beta<n}\left(R^{\partial_M}_{i\alpha j\beta}(x_0)+R^{\partial_M}_{i\beta j\alpha}(x_0)\right)\xi_\alpha\xi_\beta\quad \text{if } i,j<n,$$

$$\partial_{x_i}\partial_{x_j}(|\xi|^2_{g^M})(x_0)\Big|_{|\xi'|=1}=h''(0)\quad \text{if } i=j=n;$$

$$\partial_{x_i}\partial_{x_j}[c(\xi)](x_0)\Big|_{|\xi'|=1} = 0 \quad \textit{if } i<n, j=n, \ \textit{or } i=n, \ j<n,$$

$$\partial_{x_i}\partial_{x_j}[c(\xi)](x_0)\Big|_{|\xi'|=1} = \frac{1}{6}\sum_{l,t<n}\xi_l\Big(R^{\partial_M}_{i\alpha j\beta}(x_0)+R^{\partial_M}_{i\beta j\alpha}(x_0)\Big)c(\widetilde{e_t}) \quad \textit{if } i,j<n,$$

$$\partial_{x_i}\partial_{x_j}[c(\xi)](x_0)\Big|_{|\xi'|=1} = \left(\frac{3}{4}(h'(0))^2-\frac{1}{2}h''(0)\right)\sum_{j<n}\xi_j c(\widetilde{e_j}) \quad \textit{if } j=n,$$

where $\xi=\xi'+\xi_n\mathrm{d}x_n$.

Similarly to the conclusions from [13], we get

Lemma 3.5. *With the metric* g^M *on* M *near the boundary:*

$$\Gamma^k(x_0)=\begin{cases}0 & \textit{if } k<n\\ 3h'(0) & \textit{if } k=n\end{cases},$$

$$\sigma^k(x_0)=\begin{cases}-\frac{1}{4}h'(0)c(\widetilde{e_k})c(\widetilde{e_n}) & \textit{if } k<n,\\ 0 & \textit{if } k=n\end{cases},$$

$$a^k(x_0)=\begin{cases}\frac{1}{4}h'(0)\overline{c}(\widetilde{e_k})\overline{c}(\widetilde{e_n}) & \textit{if } k<n\\ 0 & \textit{if } k=n\end{cases}.$$

Similarly to the conclusions from [13], we also get

Lemma 3.6. *With the metric* g^M *on* M *near the boundary:*

$$\partial_{x_\gamma}\Gamma^k(x_0)\Big|_{|\xi'|=1} = \frac{5}{6}\sum_{i<n}R^{\partial_M}_{i\gamma ik}(x_0) \quad \textit{if } \gamma<n, \ k<n,$$

$$\partial_{x_\gamma}\Gamma^k(x_0)\Big|_{|\xi'|=1} = 0 \quad \textit{if } \gamma<n, \ k=n,$$

$$\partial_{x_\gamma}\Gamma^k(x_0)\Big|_{|\xi'|=1} = 0 \quad \textit{if } \gamma=n, \ k<n,$$

$$\partial_{x_\gamma}\Gamma^k(x_0)\Big|_{|\xi'|=1} = 3h''(0)-\frac{9}{2}\big(h'(0)\big)^2 \quad \textit{if } \gamma=n, k=n;$$

$$\partial_{x_\gamma}\sigma^k(x_0)\Big|_{|\xi'|=1} = -\frac{1}{8}\sum_{s\neq t<n}R^{\partial_M}_{k\gamma st}(x_0)c(\widetilde{e_s})c(\widetilde{e_t}) \quad \textit{if } \gamma<n, \ k<n,$$

$$\partial_{x_\gamma}\sigma^k(x_0)\Big|_{|\xi'|=1} = 0 \quad \textit{if } \gamma<n, \ k=n,$$

$$\partial_{x_\gamma}\sigma^k(x_0)\Big|_{|\xi'|=1} = -\sum_{t<n}\Big(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\Big)c(\widetilde{e_n})c(\widetilde{e_t}) \quad \textit{if } \gamma=n, \ k<n,$$

$$\partial_{x_\gamma}\sigma^k(x_0)\Big|_{|\xi'|=1} = -\frac{1}{8}\sum_{t<n}\Big(\big(h'(0)\big)^2-h''(0)\Big)c(\widetilde{e_s})c(\widetilde{e_t}) \quad \textit{if } \gamma=n, \ k=n;$$

$$\partial_{x_\gamma}a^k(x_0)\Big|_{|\xi'|=1} = \frac{1}{8}\sum_{s\neq t<n}R^{\partial_M}_{k\gamma st}(x_0)\overline{c}(\widetilde{e_s})\overline{c}(\widetilde{e_t}) \quad \textit{if } \gamma<n, \ k<n,$$

$$\partial_{x_\gamma} a^k(x_0)\Big|_{|\xi'|=1} = 0 \qquad \textit{if } \gamma < n,\ k = n;$$

$$\partial_{x_\gamma} a^k(x_0)\Big|_{|\xi'|=1} = \sum_{t<n}\Big(\frac{3}{8}(h'(0))^2 - \frac{1}{4}h''(0)\Big)\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_t}) \qquad \textit{if } \gamma = n,\ k < n,$$

$$\partial_{x_\gamma} a^k(x_0)\Big|_{|\xi'|=1} = \frac{1}{8}\sum_{t<n}\Big((h'(0))^2 - h''(0)\Big)\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) \qquad \textit{if } \gamma = n,\ k = n.$$

Now we will compute Φ (see formula (2.3) for the definition of Φ). Since the sum is taken over $-r-\ell+1+k+j+|\alpha| = 7,\ r,\ell \le -2$, then we have that $\int_{\partial_M}\Phi$ is the sum of the fifteen cases below.

Case (1): $r=-2,\ \ell=-2,\ k=0,\ j=1,\ |\alpha|=1$. From (2.3), we have

$$\text{Case (1)} = \frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[\partial_{x_n}\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2}) \times \partial^{\alpha}_{x'}\partial^2_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.$$

By Lemmas 3.1 and 3.3, for $i<n$, we have

$$\partial_{x_i}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \partial_{x_i}\big(|\xi|^{-2}\big)(x_0)\Big|_{|\xi'|=1} = 0. \tag{3.5}$$

So, Case (1) vanishes.

Case (2): $r=-2,\ \ell=-2,\ k=0,\ j=2,\ |\alpha|=0$. From (2.2), we have that

$$\text{Case (2)} = \frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j=2}\operatorname{trace}\Big[\partial^2_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2}) \times \partial^3_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.$$

By Lemmas 3.1, 3.3 and a simple calculation, we get

$$\partial^3_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{24\xi_n - 24\xi_n^3}{(1+\xi_n^2)^4} \tag{3.6}$$

and

$$\partial^2_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{2(h'(0))^2}{(1+\xi_n^2)^3} - \frac{h''(0)}{(1+\xi_n^2)^2}.$$

Since $\partial^2_{x_n}$ and $\pi^+_{\xi_n}$ can be exchanged, by (2.1), we have

$$\partial^2_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{-3i\xi_n^2 - 9\xi_n + 8i}{8(\xi_n - i)^3}(h'(0))^2 + \frac{2+i\xi_n}{4(\xi_n - i)^2}h''(0). \tag{3.7}$$

Note that for 7-dimensional spin compact manifolds with boundary we have $\operatorname{trace}[\mathtt{id}] = 8$. Then, by (3.6), (3.7) and some direct computations, we obtain

$$\operatorname{trace}\Big[\partial^2_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\partial^3_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)$$

$$= \big(h'(0)\big)^2 \frac{(-3i\xi_n^2 - 9\xi_n + 8i)(24\xi_n - 24\xi_n^3)}{(\xi_n - i)^3(1+\xi_n^2)^4} + h''(0)\frac{(4+2i\xi_n)(24\xi_n - 24\xi_n^3)}{(\xi_n - i)^2(1+\xi_n^2)^4}.$$

Therefore,

$$\begin{aligned}
\text{Case (2)}&\\
&= \frac{i}{6}\big(h'(0)\big)^2 \frac{2\pi i}{6!}\left[\frac{72i\xi_n^5 + 216\xi_n^4 - 264i\xi_n^3 - 216\xi_n^2 + 192i\xi_n}{(\xi_n+i)^4}\right]^{(6)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x' \\
&\quad + \frac{i}{6}h''(0)\frac{2\pi i}{5!}\left[\frac{-48i\xi_n^4 - 96\xi_n^3 + 48i\xi_n^2 + 96\xi_n}{(\xi_n^2+i)^4}\right]^{(5)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x' \\
&= \Big(\frac{7}{8}\big(h'(0)\big)^2 - \frac{3}{8}h''(0)\Big)\pi\Omega_5\,\mathrm{d}x',
\end{aligned}$$

where Ω_5 is the canonical volume of S^5.

Case (3): $r=-2$, $\ell=-2$, $k=0$, $j=0$, $|\alpha|=2$. From (2.3), we have that

$$\begin{aligned}
\text{Case (3)} = \frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=2}\operatorname{trace}\Big[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-2}(D_T^{-2}) \\
\times \partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned}$$

By Lemmas 3.1, 3.5 and a simple calculation, we have

$$\begin{aligned}
\partial_{\xi'}^{\alpha}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} &= \sum_{i,j<n}\partial_{\xi_j}\partial_{\xi_i}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} \\
&= \sum_{i,j<n}\frac{-2\delta_i^j}{(1+\xi_n^2)^2} + \sum_{i,j<n}\frac{8}{(1+\xi_n^2)^3}\xi_i\xi_j. \qquad (3.8)
\end{aligned}$$

By (3.8) and (2.1), we obtain

$$\pi_{\xi_n}^{+}\partial_{\xi'}^{\alpha}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \sum_{i,j<n}\frac{(2+i\xi_n)\delta_i^j}{2(\xi_n-i)^2} + \sum_{i,j<n}\frac{-3i\xi_n^2 - 9\xi_n + 8i}{2(\xi_n-i)^3}\xi_i\xi_j. \quad (3.9)$$

On the other hand, by Lemmas 3.1, 3.3, and 3.4, we obtain

$$\begin{aligned}
&\partial_{x'}^{\alpha}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} \\
&= \sum_{i,j<n}\frac{-1}{(1+\xi_n^2)^2}\partial_{x_i}\partial_{x_j}(|\xi|^2)(x_0) + \sum_{i,j<n}\frac{2}{(1+\xi_n^2)^3}\partial_{x_j}(|\xi|^2)\partial_{x_i}(|\xi|^2)(x_0) \\
&= \frac{1}{3(1+\xi_n^2)^2}\sum_{i,j,\alpha,\beta<n}\Big(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0)\Big)\xi_\alpha\xi_\beta + \frac{2\big(h'(0)\big)^2}{(1+\xi_n^2)^3}.
\end{aligned}$$

Hence, in this case,

$$\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{-4\xi_n}{3(1+\xi_n^2)^3}\sum_{i,j,\alpha,\beta<n}\Big(R_{i\alpha j\beta}^{\partial_M}(x_0) + R_{i\beta j\alpha}^{\partial_M}(x_0)\Big)\xi_\alpha\xi_\beta$$

$$+\frac{-12\xi_n\big(h'(0)\big)^2}{(1+\xi_n^2)^4}. \tag{3.10}$$

By (3.9), (3.10) and some direct computations, we obtain

$$\begin{aligned}
&\operatorname{trace}\Big[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-2}(D_T^{-2})\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\\
&\quad=\frac{-4\xi_n-2i\xi_n^2}{3(\xi_n-i)^2(1+\xi_n^2)^3}\sum_{i,j,\alpha,\beta<n}\Big(R^{\partial_M}_{i\alpha j\beta}(x_0)+R^{\partial_M}_{i\beta j\alpha}(x_0)\Big)\xi_\alpha\xi_\beta\\
&\qquad+\frac{-16i\xi_n+18\xi_n^2+6i\xi_n^3}{3(\xi_n-i)^3(1+\xi_n^2)^3}\sum_{i,j,\alpha,\beta<n}\Big(R^{\partial_M}_{i\alpha j\beta}(x_0)+R^{\partial_M}_{i\beta j\alpha}(x_0)\Big)\xi_i\xi_j\xi_\alpha\xi_\beta\\
&\qquad+\big(h'(0)\big)^2\frac{-12\xi_n-6i\xi_n^2}{(\xi_n-i)^2(1+\xi_n^2)^4}+\big(h'(0)\big)^2\frac{-48i\xi_n+54\xi_n^2+18i\xi_n^3}{(\xi_n-i)^3(1+\xi_n^2)^4}.
\end{aligned}$$

Similarly to (16) from [10], we have

$$\int\xi^\mu\xi^\nu=\frac{1}{6}[^{\mu\nu}],\qquad\int\xi^\mu\xi^\nu\xi^\alpha\xi^\beta=c_0[^{\mu\nu\alpha\beta}], \tag{3.11}$$

where $[^{\mu\nu\alpha\beta}]$ stands for the sum of products of $g^{\alpha\beta}$ determined by all "pairings" of $\mu\nu\alpha\beta$, and c_0 is a constant. Using the integration over S^5 and the shorthand $\int=\frac{1}{\pi^3}\int_{S^5}d^5\nu$, we obtain $\Omega_5=\pi^3$. Let s_{∂_M} be the scalar curvature of ∂_M, then

$$\begin{aligned}
&\sum_{i,\alpha,j,\beta<n}R^{\partial_M}_{i\alpha j\beta}(x_0)\int_{|\xi'|=1}\xi_\alpha\xi_\beta\xi_i\xi_j\sigma(\xi')\\
&\qquad=c\pi^3\sum_{i,\alpha,j,\beta<n}R^{\partial_M}_{i\alpha j\beta}(x_0)\Big(\delta_\alpha^\beta\delta_i^j+\delta_\alpha^i\delta_\beta^j+\delta_\alpha^j\delta_\beta^i\Big)=0,
\end{aligned}$$

where c is a constant. Therefore,

$$\begin{aligned}
\text{Case (3)}&=\frac{i}{2}\Omega_5\Big(s_{\partial_M}\int_{-\infty}^{+\infty}\frac{-4\xi_n-2i\xi_n^2}{9(\xi_n-i)^2(1+\xi_n^2)^3}\,\mathrm{d}\xi_n\\
&\quad+\big(h'(0)\big)^2\int_{-\infty}^{+\infty}\frac{4i\xi_n-9\xi_n^2-3i\xi_n^3}{(\xi_n-i)^3(1+\xi_n^2)^4}\mathrm{d}\xi_n\Big)\,\mathrm{d}x'\\
&=\frac{i}{2}\Omega_5\left(s_{\partial_M}\frac{2\pi i}{4!}\left[\frac{-4\xi_n-2i\xi_n^2}{9(\xi_n+i)^3}\right]^{(4)}\Bigg|_{\xi_n=i}\right.\\
&\quad\left.+\big(h'(0)\big)^2\frac{2\pi i}{6!}\left[\frac{4i\xi_n-9\xi_n^2-3i\xi_n^3}{(\xi_n+i)^4}\right]^{(6)}\Bigg|_{\xi_n=i}\right)\mathrm{d}x'\\
&=\frac{\pi}{6}s_{\partial_M}\Omega_5\mathrm{d}x'+\frac{11\pi}{128}\big(h'(0)\big)^2\Omega_5\mathrm{d}x',
\end{aligned}$$

where $\sum_{t,l<n}R^{\partial_M}_{tltl}(x_0)$ is the scalar curvature s_{∂_M}.

Case (4): $r=-2$, $\ell=-2$, $k=1$, $j=1$, $|\alpha|=0$. From (2.3) and the Leibniz rule,

$$\text{Case (4)}=\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{x_n}\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-2}(D_T^{-2})$$

$$\times\, \partial^2_{\xi_n}\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.$$

By Lemmas 3.1, 3.3, (2.1), and some calculations, we obtain

$$\partial_{x_n}\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = h'(0)\frac{-3-i\xi_n}{4(\xi_n-i)^3}, \tag{3.12}$$

$$\partial^2_{\xi_n}\partial_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = h'(0)\frac{4-20\xi_n^2}{(1+\xi_n^2)^4}. \tag{3.13}$$

Note that trace[id] $= 8$. Then, by (3.12), (3.13), and some direct computations, we obtain

$$\begin{aligned}\text{Case (4)} &= \frac{i}{6}\big(h'(0)\big)^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{-24-8i\xi_n+120\xi_n^2+40i\xi_n^3}{(\xi_n-i)^3(1+\xi_n^2)^4}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x' \\ &= \frac{i}{6}\big(h'(0)\big)^2\frac{2\pi i}{6!}\left[\frac{-24-8i\xi_n+120\xi_n^2+40i\xi_n^3}{(\xi_n+i)^4}\right]^{(6)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x' \\ &= -\frac{5}{8}\big(h'(0)\big)^2\pi\Omega_5\,\mathrm{d}x'.\end{aligned}$$

Case (5): $r=-2$, $\ell=-2$, $k=1$, $j=0$, $|\alpha|=1$. From (2.3), we have

$$\begin{aligned}\text{Case (5)} = \frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[&\partial^\alpha_{\xi'}\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2}) \\ &\times \partial^\alpha_{x'}\partial_{\xi_n}\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.\end{aligned} \tag{3.14}$$

By Lemmas 3.1 and 3.3, for $i<n$, we obtain

$$\begin{aligned}\partial_{x'}\partial_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} &= \frac{-1}{(1+\xi_n^2)^2}\partial_{x_i}\partial_{x_n}(|\xi|^2)(x_0) \\ &\quad+\frac{2}{(1+\xi_n^2)^3}\partial_{x_n}(|\xi|^2)\partial_{x_i}(|\xi|^2)(x_0)=0.\end{aligned}$$

Hence, Case (5) vanishes.

Case (6): $r=-2$, $\ell=-2$, $k=2$, $j=0$, $|\alpha|=0$. From (2.3), we have

$$\begin{aligned}\text{Case (6)} = \frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{k=2}\operatorname{trace}\Big[&\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2}) \\ &\times \partial_{\xi_n}\partial^2_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.\end{aligned}$$

By Lemma 3.1, (2.1), and some calculations, we have

$$\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{-i}{(\xi_n-i)^3}, \tag{3.15}$$

$$\partial_{\xi_n}\partial^2_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{4\xi_n h''(x_0)}{(1+\xi_n^2)^3}+\frac{-12\xi_n(h'(0))^2}{(1+\xi_n^2)^4}. \tag{3.16}$$

Note that trace[id] $= 8$. Then, by (3.15), (3.16), and some direct computations, we obtain

$$\begin{aligned}
\text{Case (6)} &= \frac{i}{6}h''(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{-32i\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^3}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x' \\
&\quad+\frac{i}{6}\big(h'(0)\big)^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{96i\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^4}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x' \\
&= \frac{i}{6}h''(0)\frac{2\pi i}{5!}\left[\frac{-32i\xi_n}{(\xi_n+i)^3}\right]^{(5)}\bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x' \\
&\quad+\frac{i}{6}\big(h'(0)\big)^2\frac{2\pi i}{6!}\left[\frac{96i\xi_n}{(\xi_n+i)^4}\right]^{(6)}\bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x' \\
&= \Big(-\frac{3}{8}h''(0)+\frac{7}{8}\big(h'(0)\big)^2\Big)\pi\Omega_5\,\mathrm{d}x'.
\end{aligned}$$

Case (7): $r=-2$, $\ell=-3$, $k=0$, $j=1$, $|\alpha|=0$. From (2.3) and the Leibniz rule, we obtain

$$\begin{aligned}
\text{Case (7)} &= \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\partial_{\xi_n}\sigma_{-3}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x' \\
&= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\sigma_{-3}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.
\end{aligned}$$

By Lemmas 3.1, 3.3, (2.1), and some calculations, we have

$$\pi^+_{\xi_n}\partial_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\big|_{|\xi'|=1}=h'(0)\frac{2+i\xi_n}{4(\xi_n-i)^2}. \tag{3.17}$$

Then

$$\partial^2_{\xi_n}\pi^+_{\xi_n}\partial_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\big|_{|\xi'|=1}=h'(0)\frac{4+i\xi_n}{2(\xi_n-i)^4}. \tag{3.18}$$

In the normal coordinate we have

$$\begin{aligned}
g^{ij}(x_0)&=\delta_i^j, \\
\partial_{x_j}(g^{\alpha\beta})(x_0)&=\begin{cases}0 & \text{if } j<n \\ h'(0)\delta^\alpha_\beta & \text{if } j=n.\end{cases}
\end{aligned}$$

By Lemmas 3.1, 3.5 and some calculations, we obtain

$$\begin{aligned}
\sigma_{-3}(D_T^{-2})(x_0)\big|_{|\xi'|=1} &= -i|\xi|^{-4}\xi_k(\Gamma^k-2\delta^k)(x_0)|_{|\xi'|=1} \\
&\quad-i|\xi|^{-6}2\xi^j\xi_\alpha\xi_\beta\partial_j g^{\alpha\beta}(x_0)|_{|\xi'|=1}-2i|\xi|^{-4}a^k\xi_k \\
&= \frac{ih'(0)\sum_{k<n}\xi_k c(\widetilde{e_k})c(\widetilde{e_n})}{2(1+\xi_n^2)^2}+\frac{i3h'(0)\xi_n}{(1+\xi_n^2)^2} \\
&\quad-\frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3}-\frac{ih'(0)\sum_{k<n}\xi_k\bar{c}(\widetilde{e_k})\bar{c}(\widetilde{e_n})}{2(1+\xi_n^2)^2}.
\end{aligned} \tag{3.19}$$

We note that
$$\int_{|\xi'|=1} \xi_1 \cdots \xi_{2q+1}\sigma(\xi') = 0. \tag{3.20}$$
So, the first term and the last term in (3.19) have no contribution for computing case (7), which we will omit in the following equation. Combining (3.18), (3.19), and some direct computations, we obtain

$$\begin{aligned}\operatorname{trace}\Big[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\sigma_{-3}(D_T^{-2})\Big](x_0)\\ = (h'(0))^2\frac{-80i\xi_n + 20\xi_n^2 - 48i\xi_n^3 + 12\xi_n^4}{(\xi_n - i)^4(1+\xi_n^2)^3}.\end{aligned}$$

Note that trace[id] = 8. By some direct computations, we obtain

$$\begin{aligned}\text{Case (7)} &= -\frac{1}{2}\big(h'(0)\big)^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{-80i\xi_n + 20\xi_n^2 - 48i\xi_n^3 + 12\xi_n^4}{(\xi_n - i)^4(1+\xi_n^2)^3}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x' \\ &= -\frac{1}{2}\big(h'(0)\big)^2\frac{2\pi i}{6!}\left[\frac{-80i\xi_n + 20\xi_n^2 - 48i\xi_n^3 + 12\xi_n^4}{(\xi_n + i)^3}\right]^{(6)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x' \\ &= \frac{21}{8}\big(h'(0)\big)^2\pi\Omega_5\,\mathrm{d}x'.\end{aligned}$$

Case (8): $r = -2$, $\ell = -3$, $k = 0$, $j = 0$, $|\alpha| = 1$. From (2.3) and the Leibniz rule, we obtain that

$$\begin{aligned}\text{Case (8)} &= -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2}) \\ &\qquad\times \partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x' \\ &= \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2}) \\ &\qquad\times \partial^\alpha_{x'}\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.\end{aligned}$$

By Lemma 3.1 and some calculations, we get

$$\partial^\alpha_{\xi'}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \sum_{k<n}\partial_{\xi_k}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \sum_{k<n}\frac{-2\xi_k}{(1+\xi_n^2)^2}. \tag{3.21}$$

By some calculations, we obtain

$$\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \sum_{k<n}\frac{-3-i\xi_n}{2(\xi_n - i)^3}\xi_k. \tag{3.22}$$

By Lemma 3.1, 3.6 and some direct computations, we obtain

$$\partial_{x'}\sigma_{-3}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{-5i\xi_k}{6(1+\xi_n^2)^2}\sum_{i<n}R^{\partial_M}_{i\gamma ik}(x_0)$$

$$+\frac{i\xi_k}{4(1+\xi_n^2)^2}\sum_{s\neq t<n}R^{\partial_M}_{k\gamma st}(x_0)c(\widetilde{e_s})c(\widetilde{e_t})$$
$$+\frac{2i}{3(1+\xi_n^2)^3}\sum_{\alpha,\beta<n}\Big(R^{\partial_M}_{i\alpha j\beta}(x_0)+R^{\partial_M}_{i\beta j\alpha}(x_0)\Big)\xi_j\xi_\alpha\xi_\beta$$
$$+\frac{i\sum_k\xi_k}{4(1+\xi_n^2)^2}\sum_{s\neq t<n}R^{\partial_M}_{k\gamma st}(x_0)\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}). \qquad (3.23)$$

By the relation of the Clifford action and trace $(AB)=\operatorname{trace}(BA)$, we have

$$\sum_{s\neq t}\operatorname{trace}[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})](x_0)=0.$$

So, by (3.22) and (3.23), we have

$$\operatorname{trace}\Big[\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\partial^\alpha_{x'}\sigma_{-3}(D_T^{-2})\Big](x_0)$$
$$=\frac{8\xi_n-24i}{3(\xi_n-i)^3(1+\xi_n^2)^3}\sum_{\alpha,\beta<n}\Big(R^{\partial_M}_{i\alpha j\beta}(x_0)+R^{\partial_M}_{i\beta j\alpha}(x_0)\Big)\xi_i\xi_j\xi_\alpha\xi_\beta$$
$$+\frac{30i-10\xi_n}{3(\xi_n-i)^3(1+\xi_n^2)^2}\sum_{i<n}R^{\partial_M}_{i\gamma ik}(x_0)\xi_\gamma\xi_k. \qquad (3.24)$$

Since $R^{\partial_M}_{i\alpha j\beta}(x_0)=-R^{\partial_M}_{i\beta j\alpha}(x_0)$, then, by (3.24) and some calculations, we obtain

$$\text{Case (8)}=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{30i-10\xi_n}{3(\xi_n-i)^3(1+\xi_n^2)^2}\sum_{i<n}R^{\partial_M}_{i\gamma ik}(x_0)\xi_\gamma\xi_k\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'$$
$$=\frac{1}{9}s_{\partial_M}\frac{2\pi i}{4!}\left[\frac{15i-5\xi_n}{(\xi_n+i)^2}\right]^{(4)}\bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'=\frac{5}{16}s_{\partial_M}\pi\Omega_5\,\mathrm{d}x'.$$

Case (9): $r=-2$, $\ell=-3$, $k=1$, $j=0$, $|\alpha|=0$. From (2.3) and the Leibniz rule, we obtain that

$$\text{Case (9)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})$$
$$\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'$$
$$=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})$$
$$\times\partial_{x_n}\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.$$

From (3.15), we have

$$\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1}=\frac{-i}{(\xi_n-i)^3}. \qquad (3.25)$$

By Lemmas 3.1, 3.3, 3.5 and some calculations, we obtain

$$\partial_{x_n}\sigma_{-3}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \partial_{x_n}\sigma_{-3}(D^{-2})(x_0)\Big|_{|\xi'|=1} + \partial_{x_n}(2i|\xi|^{-4}a^n\xi_n)(x_0)\Big|_{|\xi'|=1} + \partial_{x_n}(2i|\xi|^{-4}a^k\xi_k)(x_0)\Big|_{|\xi'|=1}.$$

Furthermore, we have

$$\begin{aligned}\partial_{x_n}\sigma_{-3}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} &= \frac{2ih'(0)}{(1+\xi_n^2)^3}\Big(-\frac{1}{2}h'(0)\sum_{k<n}\xi_k c(\widetilde{e_k})c(\widetilde{e_n}) + 3h'(0)\xi_n\Big)\\ &- \frac{i}{(1+\xi_n^2)^2}\Big(\xi_n\big(3h''(0)-\frac{9}{2}(h'(0))^2\big) - 2\xi_k\big(\frac{3}{8}(h'(0))^2-\frac{1}{4}h''(0)\big)\sum_{t<n}c(\widetilde{e_n})c(\widetilde{e_t})\\ &- \frac{1}{4}\xi_n\big((h'(0))^2-h''(0)\big)\sum_{s\neq t<n}c(\widetilde{e_s})c(\widetilde{e_t})\Big) + \frac{\big(h'(0)\big)^2}{4(1+\xi_n^2)^2}\sum_{s\neq t,k<n}\xi_k\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})\\ &+ \frac{2i}{(1+\xi_n^2)^2}\sum_{k,t<n}\xi_k\Big(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\Big)\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_t})\\ &+ \frac{i\xi_n}{4(1+\xi_n^2)^2}\sum_{t<n}\Big(\big(h'(0)\big)^2-h''(0)\Big)\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}). \end{aligned} \tag{3.26}$$

By the relation of the Clifford action and trace$(AB) = $ trace(BA), we have

$$\sum_{t<n}\mathrm{trace}[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})](x_0) = \sum_{t<n}\mathrm{trace}[\bar{c}(\widetilde{e_t})\bar{c}(\widetilde{e_t})](x_0) = 48.$$

By (3.25) and (3.26), we obtain

$$\mathrm{trace}\Big[\partial_{\xi_n}^2\pi_{\xi_n}^+\sigma_{-2}(D_T^{-2})\partial_{x_n}\sigma_{-3}(D_T^{-2})\Big](x_0) = \frac{12\xi_n\Big(\big(h'(0)\big)^2-h''(0)\Big)}{(\xi_n-i)^3(1+\xi_n^2)^2} + \frac{\big(h'(0)\big)^2(84\xi_n+36\xi_n^3)}{(\xi_n-i)^3(1+\xi_n^2)^3} + \frac{-24\xi_n h''(0)}{(\xi_n-i)^3(1+\xi_n^2)^2}.$$

Since

$$\partial_{x_n}c(\xi')(x_0)\big|_{|\xi'|=1} = \sum_{j<n}\partial_{x_n}\xi_j c(dx_j) = \sum_{\substack{j<n\\ 1\leq l\leq n-1}}\xi_j\partial_{x_n}(\sqrt{h}<dx^j,\widetilde{e}^l>_{\partial M})c(\widetilde{e}^l)$$

and

$$\int_{|\xi'|=1}\xi_1\cdots\xi_{2q+1}\sigma(\xi') = 0, \tag{3.27}$$

we get

$$\begin{aligned}\text{Case (9)} &= \frac{1}{2}\big(h'(0)\big)^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{(84\xi_n+36\xi_n^3)}{(\xi_n-i)^3(1+\xi_n^2)^3}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\\ &+ \frac{1}{2}h''(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{-24\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^2}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\end{aligned}$$

$$+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{12\xi_n\Big(\big(h'(0)\big)^2-h''(0)\Big)}{(\xi_n-i)^3(1+\xi_n^2)^2}\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'$$
$$=\frac{1}{2}\big(h'(0)\big)^2\frac{2\pi i}{5!}\left[\frac{84\xi_n+36\xi_n^3}{(\xi_n+i)^3}\right]^{(5)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'$$
$$+\frac{1}{2}h''(0)\frac{2\pi i}{4!}\left[\frac{-24\xi_n}{(\xi_n+i)^2}\right]^{(4)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'$$
$$+6\Big(\big(h'(0)\big)^2-h''(0)\Big)\frac{2\pi i}{4!}\left[\frac{\xi_n}{(\xi_n+i)^2}\right]^{(4)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'$$
$$=\Big(\frac{9}{16}h''(0)-\frac{45}{16}\big(h'(0)\big)^2\Big)\pi\Omega_5\,\mathrm{d}x'.$$

Case (10): $r=-3$, $\ell=-2$, $k=0$, $j=1$, $|\alpha|=0$. From (2.3), we have

$$\text{Case (10)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-3}(D_T^{-2})\times\partial^2_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\sigma(\xi')\mathrm{d}x'.$$

By the Leibniz rule, the trace property and "++" and "− −" vanishing after the integration over ξ_n in [7], we arrive at

$$\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-3}(D_T^{-2})\partial^2_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n$$
$$=\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{x_n}\sigma_{-3}(D_T^{-2})\partial^2_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n$$
$$-\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{x_n}\sigma_{-3}(D_T^{-2})\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n.$$

By Case(9), we obtain

$$\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\partial_{x_n}\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'$$
$$=\Big(\frac{9}{16}h''(0)-\frac{45}{16}\big(h'(0)\big)^2\Big)\pi\Omega_5\,\mathrm{d}x'.$$

By Lemma 3.1 and a simple computation, we obtain

$$\partial^2_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1}=\frac{6\xi_n^2-2}{(1+\xi_n^2)^3}. \tag{3.28}$$

By (3.26) and (3.28), we obtain

$$\operatorname{trace}\Big[\partial_{x_n}\sigma_{-3}(D_T^{-2})\partial^2_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)=(h'(0))^2\frac{(84i\xi_n+36i\xi_n^3)(-2+6\xi_n^2)}{(1+\xi_n^2)^6}$$
$$+h''(0)\frac{24i\xi_n(2-6\xi_n^2)}{(1+\xi_n^2)^5}+\frac{12i(6\xi_n^3-2\xi_n)\Big(\big(h'(0)\big)^2-h''(0)\Big)}{(1+\xi_n^2)^5}.$$

By a simple calculation, we have

$$-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\partial_{x_n}\sigma_{-3}(D_T^{-2})\partial_{\xi_n}^2\sigma_{-2}(D_T^{-2})\right](x_0)\mathrm{d}\xi_n = \frac{45i}{32}\Big(\big(h'(0)\big)^2 - h''(0)\Big)\pi\Omega_5\mathrm{d}x'.$$

Therefore,

$$\text{Case (10)} = \left(\left(\frac{9}{16}-\frac{45i}{32}\right)h''(0) + \left(\frac{45i}{32}-\frac{45}{16}\right)\big(h'(0)\big)^2\right)\pi\Omega_5\,\mathrm{d}x'.$$

Case (11): $r=-3$, $\ell=-2$, $k=0$, $j=0$, $|\alpha|=1$. From (2.3), we have

$$\text{Case (11)} = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\Big[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-3}(D_T^{-2}) \times \partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.$$

By Lemmas 3.1 and 3.3, for $i<n$, we have

$$\partial_{x_i}\sigma_{-2}(D_T^{-2})(x_0)\big|_{|\xi'|=1} = \partial_{x_i}\big(|\xi|^{-2}\big)(x_0)\big|_{|\xi'|=1} = 0.$$

Hence, Case (11) vanishes.

Case (12): $r=-3$, $\ell=-2$, $k=1$, $j=0$, $|\alpha|=0$. From (2.3) and the Leibniz rule, we have that

$$\begin{aligned}\text{Case (12)} &= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-3}(D_T^{-2}) \times \partial_{\xi_n}\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\\ &= \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi_{\xi_n}^{+}\sigma_{-3}(D_T^{-2}) \times \partial_{\xi_n}^2\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.\end{aligned}$$

By the Leibniz rule, the trace property and "$++$" and "$--$" vanishing after the integration over ξ_n in [7], we have

$$\begin{aligned}&\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi_{\xi_n}^{+}\sigma_{-3}(D_T^{-2})\partial_{\xi_n}^2\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\\ &\quad= \int_{-\infty}^{+\infty}\operatorname{trace}\Big[\sigma_{-3}(D_T^{-2})\partial_{\xi_n}^2\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n\\ &\quad\quad- \int_{-\infty}^{+\infty}\operatorname{trace}\Big[\sigma_{-3}(D_T^{-2})\partial_{\xi_n}^2\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-2}(D_T^{-2})\Big](x_0)\mathrm{d}\xi_n. \end{aligned}\tag{3.29}$$

Similarly to Case(7), we get the second term

$$-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\sigma_{-3}(D_T^{-2})\partial_{\xi_n}^2\partial_{x_n}\pi_{\xi_n}^+\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n$$
$$=\frac{21}{8}\big(h'(0)\big)^2\pi\Omega_5\,\mathrm{d}x'.$$

By Lemmas 3.1, 3.3 and some direct computations, we obtain

$$\partial_{\xi_n}^2\partial_{x_n}\sigma_{-2}(D_T^{-2})(x_0)\big|_{|\xi'|=1}=\frac{4-20\xi_n^2}{(1+\xi_n^2)^4}h'(0). \tag{3.30}$$

By the relation of the Clifford action and trace $(AB)=$ trace (BA), we have

$$\operatorname{trace}[\overline{c}(\widetilde{e_n})\overline{c}(\widetilde{e_k})](x_0)=0,\quad k<n.$$

By (3.19) and (3.30), we obtain

$$\operatorname{trace}\Big[\sigma_{-3}(D_T^{-2})\partial_{\xi_n}^2\partial_{x_n}\sigma_{-2}(D_T^{-2})\Big](x_0)=-(h'(0))^2\frac{20i\xi_n-88i\xi_n^3-60i\xi_n^5}{(1+\xi_n^2)^7}.$$

From some direct computations, we obtain

$$\int_{-\infty}^{+\infty}\frac{-20i\xi_n+88i\xi_n^3+60i\xi_n^5}{(1+\xi_n^2)^7}\mathrm{d}\xi_n=\frac{2\pi i}{6!}\left[\frac{-20i\xi_n+88i\xi_n^3+60i\xi_n^5}{(\xi_n+i)^7}\right]^{(6)}\Bigg|_{\xi_n=i}=0.$$

Therefore the first term of (3.29) vanishes. Thus,

$$\text{Case (12)}=\frac{21}{8}\big(h'(0)\big)^2\pi\Omega_5\,\mathrm{d}x'.$$

Case (13): $r=-3$, $\ell=-3$, $k=0$, $j=0$, $|\alpha|=0$. From (2.3) and the Leibniz rule, we have

$$\begin{aligned}\text{Case (13)}&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi_{\xi_n}^+\sigma_{-3}(D_T^{-2})\\&\qquad\times\partial_{\xi_n}\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-3}(D_T^{-2})\\&\qquad\times\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.\end{aligned}$$

By (3.19), (2.1), and some direct calculations, we have

$$\begin{aligned}\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-3}(D_T^{-2})(x_0)\big|_{|\xi'|=1}&=\frac{3i-\xi_n}{4(\xi_n-i)^3}h'(0)\sum_{k<n}\xi_kc(\widetilde{e_k})c(\widetilde{e_n})+h'(0)\frac{7\xi_n-10i}{4(\xi_n-i)^4}\\&\quad-\frac{\xi_n^2-4\xi_n+3i}{8(\xi_n-i)^4}ih'(0)\sum_{k<n}\xi_k\overline{c}(\widetilde{e_k})\overline{c}(\widetilde{e_n}).\end{aligned} \tag{3.31}$$

By the relation of the Clifford action and trace $(AB) = $ trace (BA), we have

$$\sum_{k,l<n} \xi_l\xi_k \mathrm{trace}[\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_k})\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_l})](x_0) = -8\sum_{k<n}\xi_k^2,$$
$$\sum_{k,l<n} \mathrm{trace}[\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})](x_0) = 0.$$

Then, by (3.19), (3.31), we have

$$\begin{aligned}&\mathrm{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-3}(D_T^{-2})\sigma_{-3}(D_T^{-2})\Big](x_0)\\ &\quad= (h'(0))^2\frac{2(10i-7\xi_n)(5i\xi_n+3i\xi_n^3)}{(\xi_n-i)^4(1+\xi_n^2)^3} - (h'(0))^2\frac{(i\xi_n^2+4\xi_n-3i)}{2(\xi_n-i)^4(1+\xi_n^2)^2}\sum_{k<n}\xi_k^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Case (13)} &= i\big(h'(0)\big)^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-3}(D_T^{-2})\\ &\qquad\times\sigma_{-3}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\\ &= i\big(h'(0)\big)^2\frac{2\pi i}{6!}\left[\frac{-3i-96\xi_n-72i\xi_n^2-56\xi_n^3-41i\xi_n^4}{(\xi_n+i)^3}\right]^{(6)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'\\ &\quad+\frac{1}{2}\big(h'(0)\big)^2\frac{2\pi i}{5!}\left[\frac{\xi_n^2-4i\xi_n-3)}{(\xi_n+i)^2}\right]^{(5)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'\\ &= (-\frac{5}{32}-\frac{57}{8})(h'(0))^2\pi\Omega_5\,dx'.\end{aligned}$$

Case (14): $r=-2$, $\ell=-4$, $k=0$, $j=0$, $|\alpha|=0$. From (2.3) and the Leibniz rule, we have that

$$\begin{aligned}\text{Case (14)} &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\partial_{\xi_n}\sigma_{-4}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\\ &= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\sigma_{-4}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.\end{aligned}$$

By Lemma 3.1, (2.1) and some calculations, we get

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} = \frac{i}{2(\xi_n-i)^2}. \tag{3.32}$$

By Lemmas 3.2–3.6 and some direct calculation, we have

$$\begin{aligned}\sigma_{-4}(D_T^{-2})(x_0)\Big|_{|\xi'|=1} &= -12\frac{h'(0)}{(1+\xi_n^2)^3} + \frac{2\big(h'(0)\big)^2}{(1+\xi_n^2)^4}\sum_{k<n,l}\xi_k\xi_l c(\widetilde{e_n})c(\widetilde{e_k})\\ &-\frac{7\big(h'(0)\big)^2}{16(1+\xi_n^2)^2}\sum_{k,l<n}\xi_k\xi_l\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_k})\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_l}) + \frac{21\big(h'(0)\big)^2}{4(1+\xi_n^2)^2}\sum_{k<n}\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_k})\end{aligned}$$

$$+\frac{1}{8(1+\xi_n^2)^2}\sum_{ijkl}R^{\partial_M}_{ijkl}(x_0)\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l})-\frac{T\sum_i c(\widetilde{e_i})\bar{c}(\nabla^{TM}_{\widetilde{e_i}}V)+T^2|V|^2}{(1+\xi_n^2)^2}$$
$$+\frac{2h'(0)\sum_k\xi_k}{(1+\xi_n^2)^3}-\frac{(\sum_{k<n}\xi_n\xi_k)+\xi_n^2}{2(1+\xi_n^2)}\sum_{t<n,s}\left((h'(0))^2-h''(0)\right)\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})$$
$$-4\frac{\sum_{k,l<n}\xi_k\xi_l+\sum_{k<n}\xi_k\xi_n}{(1+\xi_n^2)}\sum_{t<n,s}\left(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\right)\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})$$
$$-\frac{\big(h'(0)\big)^2}{4(1+\xi_n^2)^3}\sum_{k,l<n}\xi_k\xi_l\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_k})\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_l})-\frac{7\big(h'(0)\big)^2}{8(1+\xi_n^2)^2}\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})$$
$$-\frac{\big(h'(0)\big)^2}{8(1+\xi_n^2)^3}\sum_{k,l<n}\xi_k\xi_l c(\widetilde{e_n})c(\widetilde{e_k})\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_l})+\sigma_{-4}(D^{-2})(x_0)\Big|_{|\xi'|=1},\tag{3.33}$$

where

$$\begin{aligned}\sigma_{-4}(D^{-2})(x_0)\Big|_{|\xi'|=1}&=\frac{-\big(h'(0)\big)^2}{4(1+\xi_n^2)^3}c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})c(\widetilde{e_n})-\frac{9\big(h'(0)\big)^2}{(1+\xi_n^2)^3}\xi_n^3\xi_\mu\xi_l\\
&+\frac{\big(h'(0)\big)^2}{4(1+\xi_n^2)^2}\xi_k\xi_l c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})c(\widetilde{e_n})-\frac{1}{4(1+\xi_n^2)^2}s(x_0)\\
&-\frac{5}{3(1+\xi_n^2)^3}\xi_k\xi_l\sum_{i<n}R^{\partial_M}_{ikil}(x_0)-\frac{6}{(1+\xi_n^2)^3}h''(0)\xi_n^2\\
&-\frac{4}{3(1+\xi_n^2)^4}\xi_k\xi_l\xi_\gamma\xi_\delta\sum_{\gamma,\delta<n}\left(R^{\partial_M}_{k\gamma l\delta}(x_0)+R^{\partial_M}_{l\gamma k\delta}(x_0)\right)\\
&-\frac{1}{3(1+\xi_n^2)^3}\xi_\alpha\xi_\beta\sum_{\alpha,\beta<n}\left(R^{\partial_M}_{k\alpha l\beta}(x_0)+R^{\partial_M}_{l\beta k\alpha}(x_0)\right)\\
&+\frac{h''(0)}{(1+\xi_n^2)^3}+\frac{4h''(0)}{(1+\xi_n^2)^4}\xi_n^2\\
&+\frac{2+3\xi_n+10\xi_n^2+12\xi_n^3-4\xi_n^4+9\xi_n^5}{(1+\xi_n^2)^5}\big(h'(0)\big)^2.\end{aligned}\tag{3.34}$$

By the relation of the Clifford action and trace $(AB)=\text{trace}\,(BA)$, we have

$$\text{trace}[c(\widetilde{e_n})c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})](x_0)=-8\delta_l^k,\ \text{trace}\left[c(\widetilde{e_i})\bar{c}(\nabla^{TM}_{\widetilde{e_i}}V)\right](x_0)=0.$$

By (3.32)–(3.34), we obtain

$$\begin{aligned}\text{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\sigma_{-4}(D_T^{-2})\Big](x_0)&=\text{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-2})\sigma_{-4}(D^{-2})\Big](x_0)\\
-\frac{2i(h'(0))^2}{(\xi_n-i)^2(1+\xi_n^2)^3}&+\frac{21i\big((h'(0))^2-h''(0)\big)}{(\xi_n-i)^2(1+\xi_n^2)^2}+\frac{is_M}{2(\xi_n-i)^2(1+\xi_n^2)^2}\\
+\frac{21i(h'(0))^2}{2(\xi_n-i)^2(1+\xi_n^2)^2}&+\frac{-i4T^2|V|^2}{(\xi_n-i)^2(1+\xi_n^2)^2}+\frac{8ih'(0)\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^3}\\
-\frac{\big(12i\sum_{k<n}\xi_k\xi_n+12i\xi_n^2\big)\big((h'(0))^2-h''(0)\big)}{(\xi_n-i)^2(1+\xi_n^2)}&-\frac{48ih'(0)}{(\xi_n-i)^2(1+\xi_n^2)^3}\end{aligned}$$

$$-\frac{\left(96i\sum_{k<n}\xi_k\xi_n+96i\sum_{k<n}\xi_k^2\right)\left(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\right)}{(\xi_n-i)^2(1+\xi_n^2)},$$

where

$$\begin{aligned}
\operatorname{trace}&\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-2})\sigma_{-4}(D^{-2})\Big](x_0)\\
&=\frac{-i\big(h'(0)\big)^2}{8(\xi_n-i)^2(1+\xi_n^2)^3}\operatorname{trace}\big[c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})c(\widetilde{e_n})\big]\\
&\quad-\frac{36i\big(h'(0)\big)^2\xi_n^3\sum_{k,l<n}\xi_k\xi_l}{(\xi_n-i)^2(1+\xi_n^2)^3}-\frac{is(x_0)}{(\xi_n-i)^2(1+\xi_n^2)^2}\\
&\quad+\frac{i\big(h'(0)\big)^2\sum_{k,l<n}\xi_k\xi_l}{8(\xi_n-i)^2(1+\xi_n^2)^2}\operatorname{trace}\big[c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})c(\widetilde{e_n})\big]\\
&\quad-\frac{20i\sum_{k,l<n}\xi_k\xi_l\sum_{i<n}R^{\partial_M}_{ikil}(x_0)}{3(\xi_n-i)^2(1+\xi_n^2)^3}-\frac{24ih''(0)\xi_n^2}{(\xi_n-i)^2(1+\xi_n^2)^3}\\
&\quad-\frac{16i\xi_k\xi_l\xi_\gamma\xi_\delta\sum_{\gamma,\delta<n}\left(R^{\partial_M}_{k\gamma l\delta}(x_0)+R^{\partial_M}_{l\gamma k\delta}(x_0)\right)}{3(\xi_n-i)^2(1+\xi_n^2)^4}\\
&\quad-\frac{4i\sum_{k,l<n}\xi_k\xi_l\sum_{\alpha,\beta<n}\left(R^{\partial_M}_{k\alpha l\beta}(x_0)+R^{\partial_M}_{l\beta k\alpha}(x_0)\right)}{3(\xi_n-i)^2(1+\xi_n^2)^3}\\
&\quad+\frac{16ih''(0)\xi_n^2}{(\xi_n-i)^2(1+\xi_n^2)^4}+\frac{4ih''(0)}{(\xi_n-i)^2(1+\xi_n^2)^3}\\
&\quad+\frac{4i\big(h'(0)\big)^2(2+3\xi_n+10\xi_n^2+12\xi_n^3-4\xi_n^4+9\xi_n^5)}{(\xi_n-i)^2(1+\xi_n^2)^5}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\text{Case (14)}=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\sigma_{-4}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'\\
&=s_M(x_0)\frac{2\pi i}{3!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(3)}\Bigg|_{\xi_n=i}\Omega_5\mathrm{d}x'+s_{\partial_M}(x_0)\frac{2\pi i}{4!}\left[\frac{14}{9(\xi_n+i)^3}\right]^{(4)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'\\
&\quad+\big(h'(0)\big)^2\frac{2\pi i}{6!}\Bigg[\frac{-53-72\xi_n-33\xi_n^2-288\xi_n^3+525\xi_n^4-216\xi_n^5}{6(\xi_n+i)^5}\\
&\quad+\frac{217\xi_n^6}{6(\xi_n+i)^5}\Bigg]^{(6)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'+h''(0)\frac{2\pi i}{5!}\left[\frac{-4(1-\xi_n^2-6\xi_n^4)}{(\xi_n+i)^4}\right]^{(5)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'\\
&\quad+\big(48h'(0)+2(h'(0))^2\big)\frac{2\pi i}{4!}\left[\frac{1}{(\xi_n+i)^3}\right]^{(4)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'\\
&\quad+\left(4T^2|V|^2-21\big((h'(0))^2-h''(0)\big)-\frac{1}{2}s_M-\frac{21}{2}(h'(0))^2\right)\\
&\quad\times\frac{2\pi i}{3!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(3)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'-8h'(0)\frac{2\pi i}{4!}\left[\frac{\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'
\end{aligned}$$

$$+12\big((h'(0))^2-h''(0)\big)\frac{2\pi i}{2!}\left[\frac{\xi_n^2}{(\xi_n+i)}\right]^{(2)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'$$
$$+96\Big(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\Big)\frac{2\pi i}{2!}\left[\frac{1}{(\xi_n+i)}\right]^{(2)}\Bigg|_{\xi_n=i}\Omega_5\,\mathrm{d}x'$$
$$=\left(\frac{-1}{4}s_M(x_0)-\left(\frac{45}{4}+\frac{5i}{8}\right)h'(0)-\left(\frac{23}{12}+\frac{3i}{2}\right)\big(h'(0)\big)^2\right.$$
$$\left.+\frac{235}{64}h''(0)+\frac{47}{96}s_{\partial_M}(x_0)-T^2|V|^2\right)\pi\Omega_5\,\mathrm{d}x'.$$

Case (15): $r=-4$, $\ell=-2$, $k=0$, $j=0$, $|\alpha|=0$. From (2.3), we have that

$$\text{Case (15)}=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi^+_{\xi_n}\sigma_{-4}(D_T^{-2})$$
$$\times\,\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'.$$

By the Leibniz rule, the trace property and "$++$" and "$--$" vanishing after the integration over ξ_n in [7], we have

$$\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\pi^+_{\xi_n}\sigma_{-4}(D_T^{-2})\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n$$
$$=\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\sigma_{-4}(D_T^{-2})\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n$$
$$-\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\sigma_{-4}(D_T^{-2})\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n.$$

By Case (14), we obtain

$$i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\Big[\sigma_{-4}(D_T^{-2})\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)\,\mathrm{d}\xi_n\sigma(\xi')\,\mathrm{d}x'$$
$$=\left(\frac{-1}{4}s_M(x_0)-\left(\frac{45}{4}+\frac{5i}{8}\right)h'(0)-\left(\frac{23}{12}+\frac{3i}{2}\right)\big(h'(0)\big)^2\right.$$
$$\left.+\frac{235}{64}h''(0)+\frac{47}{96}s_{\partial_M}(x_0)-T^2|V|^2\right)\pi\Omega_5\,\mathrm{d}x'.$$

By Lemma 3.1 and some calculations, we obtain

$$\partial_{\xi_n}\sigma_{-2}(D_T^{-2})(x_0)\Big|_{|\xi'|=1}=\frac{-2\xi_n}{(1+\xi_n^2)^2}.\tag{3.35}$$

By (3.33), (3.34), and (3.35), we get

$$\operatorname{trace}\Big[\sigma_{-4}(D_T^{-2})\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\Big](x_0)$$
$$=\operatorname{trace}\Big[\sigma_{-4}(D^{-2})\partial_{\xi_n}\sigma_{-2}(D^{-2})\Big](x_0)-\frac{2s_M\xi_n}{(1+\xi_n^2)^4}$$

$$
\begin{aligned}
&+\frac{192h'(0)\xi_n}{(1+\xi_n^2)^5}-\frac{16(h'(0))^2\xi_n}{(1+\xi_n^2)^5}+\frac{12\big((h'(0))^2-h''(0)\big)\xi_n}{(1+\xi_n^2)^4}\\
&+\frac{12(h'(0))^2\xi_n}{(1+\xi_n^2)^2}-\frac{16T^2|V|^2}{(1+\xi_n^2)^4}-\frac{32h'(0)\sum_k\xi_k\xi_n}{(1+\xi_n^2)^3}\\
&-\frac{48\sum_{k<n}\xi_k\xi_n^2\big((h'(0))^2-h''(0)\big)}{(1+\xi_n^2)^5}-\frac{48\xi_n^2\big((h'(0))^2-h''(0)\big)\xi_n^3}{(1+\xi_n^2)^4}\\
&-\frac{-384\sum_{k<n}\xi_k\xi_n^2\Big(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\Big)}{(1+\xi_n^2)^3}\\
&-\frac{384\sum_{k<n}\xi_k^2\Big(\frac{3}{8}\big(h'(0)\big)^2-\frac{1}{4}h''(0)\Big)\xi_n}{(1+\xi_n^2)^3},
\end{aligned}
$$

where

$$
\begin{aligned}
&\operatorname{trace}\Big[\sigma_{-4}(D^{-2})\partial_{\xi_n}\sigma_{-2}(D^{-2})\Big](x_0)\\
&\quad=\frac{\xi_n\big(h'(0)\big)^2}{2(1+\xi_n^2)^5}\operatorname{trace}\big[c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})c(\widetilde{e_n})\big]+\frac{144\big(h'(0)\big)^2\xi_n^4\sum_{k,l<n}\xi_k\xi_l}{(1+\xi_n^2)^5}\\
&\qquad-\frac{\big(h'(0)\big)^2\sum_{k,l<n}\xi_k\xi_l}{2(1+\xi_n^2)^4}\operatorname{trace}\big[c(\widetilde{e_k})c(\widetilde{e_n})c(\widetilde{e_l})c(\widetilde{e_n})\big]-\frac{4s(x_0)\xi_n}{(1+\xi_n^2)^4}\\
&\qquad+\frac{80\xi_n\sum_{k,l<n}\xi_k\xi_l\sum_{i<n}R^{\partial_M}_{ikil}(x_0)}{3(1+\xi_n^2)^5}+\frac{96h''(0)\xi_n^3}{(1+\xi_n^2)^5}-\frac{18\xi_nh''(0)}{(1+\xi_n^2)^5}\\
&\qquad-\frac{64\xi_n\sum_{k,l,\gamma,\delta<n}\xi_k\xi_l\xi_\gamma\xi_\delta}{3(1+\xi_n^2)^6}\sum_{\gamma,\delta<n}\Big(R^{\partial_M}_{k\gamma l\delta}(x_0)+R^{\partial_M}_{l\gamma k\delta}(x_0)\Big)\\
&\qquad+\frac{16\xi_n\sum_{k,l<n}\xi_k\xi_l}{3(1+\xi_n^2)^5}\sum_{\alpha,\beta<n}\Big(R^{\partial_M}_{k\alpha l\beta}(x_0)+R^{\partial_M}_{l\beta k\alpha}(x_0)\Big)-\frac{64\xi_n^3h''(0)}{(1+\xi_n^2)^6}\\
&\qquad-\frac{16\xi_n\big(h'(0)\big)^2(2+3\xi_n+10\xi_n^2+12\xi_n^3-4\xi_n^4+9\xi_n^5)}{(1+\xi_n^2)^7}.
\end{aligned}
$$

By direct calculations, we get

$$
-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\big[\sigma_{-4}(D_T^{-2})\partial_{\xi_n}\sigma_{-2}(D_T^{-2})\big](x_0)\,\mathrm{d}\xi_n=3i(h'(0))^2\pi\Omega_5\,\mathrm{d}x'.
$$

Therefore,

$$
\begin{aligned}
\text{Case (15)}=\Big(&\frac{-1}{4}s_M(x_0)-(\frac{45}{4}+\frac{5i}{8})h'(0)-(\frac{23}{12}-\frac{3i}{2})\big(h'(0)\big)^2\\
&+\frac{235}{64}h''(0)+\frac{47}{96}s_{\partial_M}(x_0)-T^2|V|^2\Big)\pi\Omega_5\,\mathrm{d}x'.
\end{aligned}
$$

Now Φ is the sum of the Case (1), ..., Case (15), so

$$
\Phi=\sum_{I=1}^{15}\text{Case (I)}=\Bigg(-\frac{1}{2}s_M(x_0)+\frac{35}{24}s_{\partial_M}(x_0)-2T^2|V|^2-\Bigg(\frac{45}{2}+\frac{5i}{4}\Bigg)h'(0)
$$

$$+\left(\frac{45i}{32}-\frac{3947}{384}\right)\left(h'(0)\right)^2+\left(\frac{247}{32}-\frac{45i}{32}\right)h''(0)\right)\pi\Omega_5\,\mathrm{d}x'.$$

Hence we have the conclusion as follows.

Theorem 3.7. *Let M be a 7-dimensional spin compact manifold with the boundary ∂M. Then we get the volumes associated to the Witten deformation D_T on $\widehat{M}$:*

$$\widetilde{\mathrm{Wres}}[\pi^+D_T^{-2}\circ\pi^+D_T^{-2}]=\int_{\partial M}\left(-\frac{1}{2}s_M(x_0)+\frac{35}{24}s_{\partial_M}(x_0)-\left(\frac{45}{2}+\frac{5i}{4}\right)h'(0)\right.$$
$$\left.-2T^2|V|^2+\left(\frac{45i}{32}-\frac{3947}{384}\right)\left(h'(0)\right)^2+\left(\frac{247}{32}-\frac{45i}{32}\right)h''(0)\right)\pi\Omega_5\,\mathrm{d}x'.$$

4. The gravitational action for 7-dimensional manifolds with boundary

Firstly, we recall the Einstein–Hilbert action for manifolds with boundary (see [17] or [18]),

$$I_{\mathrm{Gr}}=\frac{1}{16\pi}\int_M s\,\mathrm{dvol}_M+2\int_{\partial M}K\,\mathrm{dvol}_{\partial_M}:=I_{\mathrm{Gr,i}}+I_{\mathrm{Gr,b}},\tag{4.1}$$

where

$$K=\sum_{1\leq i,j\leq n-1}K_{i,j}g_{\partial M}^{i,j},\quad K_{i,j}=-\Gamma_{i,j}^n,\tag{4.2}$$

and $K_{i,j}$ is the second fundamental form or the extrinsic curvature. Taking the metric in Section 2, for $n=7$, we have

$$K(x_0)=-\frac{5}{2}h'(0),\quad I_{\mathrm{Gr,b}}=-5h'(0)\mathrm{Vol}_{\partial M}.\tag{4.3}$$

Then we obtain

$$\widetilde{\mathrm{Wres}}[(\pi^+D_T^{-2})^2]_i=0,\tag{4.4}$$

$$\widetilde{\mathrm{Wres}}[(\pi^+D_T^{-2})^2]_b=\int_{\partial M}\Phi=Q_0\pi\Omega_5\mathrm{Vol}_{\partial M},\tag{4.5}$$

where

$$Q_0=-\frac{1}{2}s_M(x_0)+\frac{35}{24}s_{\partial_M}(x_0)-2T^2|V|^2-\left(\frac{45}{2}+\frac{5i}{4}\right)h'(0)$$
$$+\left(\frac{45i}{32}-\frac{3947}{384}\right)\left(h'(0)\right)^2+\left(\frac{247}{32}-\frac{45i}{32}\right)h''(0).\tag{4.6}$$

By (4.1)–(4.6), we obtain

Corollary 4.1. *Let M be a 7-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above and let D_T be the Witten deformation on $\widehat{M}$. Then*

$$I_{\mathrm{Gr,b}}=-\frac{5h'(0)}{Q_0\pi\Omega_5}\widetilde{\mathrm{Wres}}[(\pi^+D_T^{-1})^2]_b.$$

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Теорема типу Кастлера–Калау–Вальце про деформацію Віттена для 7-вимірних многовидів з межею

Kai Hua Bao, Ai Hui Sun, and Kun Ming Hu

У цій роботі ми методом повного перебору доводимо теорему типу Кастлера–Калау–Вальце про деформацію Віттена для 7-вимірних многовидів з межею і даємо теоретичне пояснення гравітаційної дії для 7-вимірних многовидів з межею.

Ключові слова: деформація Віттена, некомутативний залишок для многовидів з межею, низьковимірний об'єм