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On a Certain Class of Γ-Contractions

Berrabah Bendoukha

The present paper is aimed to study a certain class of pairs of operators having the symmetrized bidisk as a spectral set. For such pairs, the conditions of Γ-contractivity are given and the functional model is constructed. Some criteria of unitary equivalence are also established.

Key words: functional model, fundamental operator, pure contraction, spectral set, symmetrized bidisc

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1. Introduction and preliminaries

In the following, $\mathcal{H}$ is a separable complex Hilbert space, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators acting in $\mathcal{H}$ with the identity I. If T is a contraction in $\mathcal{H}$, we denote by $D_T = (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ the defect operators of T and by $\mathcal{D}_T = \overline{D_T(\mathcal{H})}$, $\mathcal{D}_{T^*} = \overline{D_{T^*}(\mathcal{H})}$ the corresponding defect subspaces.

Definition 1.1. A contraction T, defined on $\mathcal{H}$, is called completely non unitary (cnu in the following) if there is no non trivial reducing subspace in which T induces a unitary operator. If the sequence T^{*n} strongly converges to 0, then, following [11, Chap. 2, Sect. 4], we say that T is a $C_{\cdot 0}$ contraction.

The following results are well known.

Theorem 1.2 ([11, Chap. 1, Sect. 3]). *For every contraction T in $\mathcal{H}$, there exists a unique orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_T$ such that both $\mathcal{H}_0$ and $\mathcal{H}_T$ are invariant over T, in $\mathcal{H}_0$ the operator T induces a unitary operator and in $\mathcal{H}_T$ it induces a cnu contraction. Moreover,*

$$\mathcal{H}_T = \overline{\operatorname{span}\left\{T^n\left(\mathcal{D}_{T^*}\right),\ T^{*m}\left(\mathcal{D}_T\right),\quad n, m = 0, 1, 2, \ldots\right\}}.$$

Theorem 1.3 ([11, Chap. 2, Sect. 6]). *If the contraction T is cnu and the intersection of its spectrum with the unit circle has a null measure, then*

$$\lim_{n\to+\infty} T^n(x) = \lim_{n\to+\infty} T^{*n}(x) = 0 \quad \textit{for all } x \in \mathcal{H},$$

and thus the operator T is in the class C_{00} of all contractions satisfying the condition

$$\lim_{n\to+\infty} T^n h = \lim_{n\to+\infty} T^{*n} h = 0 \quad \textit{for all } h \in \mathcal{H}.$$

Remark 1.4. The restriction T_1 of T to the reducing subspace $\mathcal{H}_T$ is called the cnu part of T.

If T is a contraction on $\mathcal{H}$, then the analytical operator-valued function Θ_T, defined from the open unit disc $\mathbb{D}$ of $\mathbb{C}$ into the set $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$ of all bounded linear operators from $\mathcal{D}_T$ into $\mathcal{D}_{T^*}$ by

$$\Theta_T(z) = \left[-T + zD_{T^*}\left(I - zT^*\right)^{-1} D_T\right], \quad z \in \mathbb{D},$$

is called the characteristic function of T. It is well known [11, Chap. 6, Sect. 3] that Θ_T is a unitary invariant of T.

Remark 1.5. Following [11, Chap. 5, Sect. 2], we will suppose every function

$$\Theta(z) = V\Theta_T(z)U : \mathcal{E} \to \mathcal{E}'$$

to be equal to $\Theta_T(z)$ for any separable Hilbert spaces $\mathcal{E}$, $\mathcal{E}'$ and any unitary operators U, V acting from $\mathcal{E}$ into D_T and from D_{T^*} into $\mathcal{E}'$ respectively.

If $\mathcal{E}$ is a separable space, design by $\mathcal{O}(\mathbb{D}, \mathcal{E})$ the class of all $\mathcal{E}$-valued analytic functions on $\mathbb{D}$ and consider the following Hilbert space [4]:

$$\mathbb{H}(\mathcal{E}) = \left\{ f \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : f = \sum_{n=0}^{+\infty} a_n z^n \text{ with } a_n \in \mathcal{E} \text{ and } \sum_{n=0}^{+\infty} \|a_n\|^2 < +\infty \right\}.$$

The space $\mathbb{H}(\mathcal{E})$ is given by the reproducing kernel $(1 - \langle z, w\rangle)^{-1} I_{\mathcal{E}}$, and for $\mathcal{E} = \mathbb{C}$, this is the usual Hardy space on the unit disk. Moreover [4], $\mathbb{H}(\mathbb{C}) \otimes \mathcal{E}$ and $\mathbb{H}(\mathcal{E})$ are isometrically isomorphic via the unitary operator $U_{\mathcal{E}}(f \otimes x) = fx$. This allows us to identify the element $f \otimes x$ of $\mathbb{H}(\mathbb{C}) \otimes \mathcal{E}$ with the element fx of $\mathbb{H}(\mathcal{E})$.

Definition 1.6. Let T be a $C_{\cdot 0}$ contraction in $\mathcal{H}$. The space $\mathbb{H}_T = \mathbb{H}(\mathcal{D}_{T^*}) \ominus M_{\Theta_T}(\mathbb{H}(\mathcal{D}_T))$ is called the model space of T. The functional model of T is the restriction of the operator $P_{\mathbb{H}_T}(M_z \otimes I)$ to this space, where $P_{\mathbb{H}_T}$ is the orthogonal projector of $\mathbb{H}(\mathcal{D}_{T^*})$ onto $\mathbb{H}_T$, M_z is the multiplication operator by the independent variable $z \in \mathbb{D}$.

A $C_{\cdot 0}$ contraction T, its model space and functional model are linked by the following fundamental result due to Sz-Nagy and Foias [11, Chap. 6, Sect. 2].

Theorem 1.7. *Every $C_{\cdot 0}$ contraction T in $\mathcal{H}$ is unitarily equivalent to its functional model. In other words, there exists a unitary operator U from $\mathcal{H}$ onto $\mathbb{H}_T$ such that $T = U^{-1}\mathbb{T}U$.*

In the following, we will suppose that the spectrum $\sigma(T)$ of T is concentrated at the point $a = 1$ and $\dim(\mathcal{D}_T) = 1$. In this case, the operator T is invertible and $\dim(\mathcal{D}_{T^*}) = 1$. Moreover, we have the representation [8]:

$$\langle \Theta_T(z)(u), v\rangle = \exp\left\{\int_0^l \frac{z+1}{z-1} dt\right\} = \exp\left\{l\frac{z+1}{z-1}\right\}, \tag{1.1}$$

where u and v are two vectors such that $\|u\| = \|v\| \prec 1$, which satisfy

$$I - T^*T = \langle \cdot, u\rangle u \text{ and } I - TT^* = \langle \cdot, v\rangle v.$$

Now, in the space $L^2_{[0,\ l]}$ of square integrable functions consider the operator

$$\widetilde{T}f(x) = f(x) - 2e^x \int_x^l e^{-t} f(t)\, dt. \tag{1.2}$$

In the literature (see, e.g., [8]), the operator $\widetilde{T}$ is known as the triangular model of the class of cnu contractions having one-dimensional defect subspaces and the spectrum concentrated at $a = 1$. This finds its justification in the following facts:

(a) Direct calculations give us

$$\widetilde{T}^* f(x) = f(x) - 2e^{-x} \int_0^x e^t f(t) dt, \tag{1.3}$$

$$I - \widetilde{T}^*\widetilde{T} = \langle \cdot, g\rangle g, \quad I - \widetilde{T}\widetilde{T}^* = \langle \cdot, h\rangle h, \tag{1.4}$$

where

$$g\,(x) = \sqrt{2}e^{-x}, \quad h\,(x) = \sqrt{2}e^{x-l}, \quad 0 \le x \le l. \tag{1.5}$$

This proves that $\widetilde{T}$ is a contraction with one-dimensional defect subspaces.

(b) Consider in $L^2_{[0,l]}$ the Volterra integration operator

$$\widetilde{A}f(x) = i \int_x^l f(t) dt.$$

It is known [6, Chap. 1, Sect. 8.2] that $\widetilde{A}$ is a completely non-self-adjoint operator with spectrum concentrated at the point $\mu = 0$ and one-dimensional imaginary part. Moreover, one can easily prove that $\widetilde{T} = -\,\mathcal{K}(\widetilde{A})$, where

$$\mathcal{K}(\widetilde{A}) = \left(\widetilde{A} - iI\right)\left(\widetilde{A} + iI\right)^{-1} = I - 2i\left(\widetilde{A} + iI\right)^{-1} \tag{1.6}$$

is the Cayley transform of $\widetilde{A}$. So, we have the spectral relation

$$\sigma(\widetilde{T}) = \left\{-\,\frac{\mu - i}{\mu + i} :\ \mu \in \sigma\left(\widetilde{A}\right) = \{0\}\right\} = \{1\}$$

which proves that the spectrum of $\widetilde{T}$ is concentrated at the point $\lambda = 1$.

(c) Using (1.6), one obtains

$$\widetilde{A} = iI - 2i\left(\widetilde{T} + I\right)^{-1}, \tag{1.7}$$

$$\widetilde{A^*} = -iI + 2i\left(\widetilde{T}^* + I\right)^{-1}. \tag{1.8}$$

$$\frac{A - A^*}{i} = 2\,(I + T^*)^{-1}\,(I - T^*T)\,(I + T)^{-1}. \tag{1.9}$$

$$\frac{A - A^*}{i} = 2\,(I+T)^{-1}\,(I - TT^*)\,(I+T^*)^{-1}\,. \tag{1.10}$$

Using formulas (1.7), (1.8), one can prove that every subspace H_0 reducing $\widetilde{T}$ reduces also $\widetilde{A}$. Formulas (1.9) and (1.10) show that if $\widetilde{T}$ induces a unitary operator in H_0, then $\widetilde{A}$ induces a self-adjoint operator in H_0. Thus, we have necessarily $H_0 = 0$. In other words, the operator $\widetilde{T}$ is cnu.

(d) According to [8] (see Theorem 2), every cnu contraction with one-dimensional defect subspaces and spectrum concentrated at $a = 1$ is unitarily equivalent to $\widetilde{T}$.

Definition 1.8. A pair (S, T) of commuting bounded linear operators on $\mathcal{H}$ is called a Γ-contraction if it has the symmetrized bidisc

$$\Gamma = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_1| \le 1, |\lambda_2| \le 1\} \subset \mathbb{C}^2$$

as a spectral set. That is (see [1]), the spectrum $\sigma(S, T)$ of the pair (S, T) is contained in Γ and

$$\|f(S,T)\| \le \max_{(z_1,z_2)\in\Gamma} |f(z_1, z_2)|$$

for all functions f that are holomorphic on a neighbourhood of Γ.

It is known [3] that if (S, T) is a Γ-contraction, then the operator T is a contraction ($\|T\| \le 1$). The study of Γ-contractions was introduced and carried out very successfully over several papers by Agler and Young, (see [1] and the references therein). From the paper of Agler and Young, we retain the useful assertion contained in Theorem 1.5.

Theorem 1.9. *Let (S, T) be a pair of commuting operators in $\mathcal{H}$. Then Γ is a spectral set for (S, T) if and only if $\rho(\alpha S, \alpha^2 T) \ge 0$ for all $\alpha \in \mathbb{D}$ and*

$$\rho(S, T) = 2(I - T^*T) - S + S^*T - S^* + T^*S.$$

The key concept in the study of Γ-contractions is the so-called fundamental operator F which is the unique element of $\mathcal{B}(\mathcal{D}_T)$ satisfying the fundamental equation

$$S - S^*T = D_T X D_T.$$

It has a numerical radius $w(F)$ no greater than one and was firstly introduced in [5]. If (S, T) is a Γ-contraction, then so is the pair (S^*, T^*) with fundamental operator G, the unique solution of the operator equation $S^* - ST^* = D_{T^*} Y D_{T^*}$.

Definition 1.10. Two pairs of operators (S, T) and (S', T'), defined on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, are said to be unitarily equivalent if there exists a unitary operator U from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $S' = U^{-1}SU$ and $T' = U^{-1}TU$.

Remark 1.11. It is clear that the pairs (S, T) and (S', T') are unitarily equivalent if and only if the pairs (S^*, T^*) and (S'^*, T'^*) are unitarily equivalent.

Theorem 1.12 ([4]). *Every pure Γ-contraction (S,T) (that is, T is a $C_{\cdot 0}-$contraction) is unitarily equivalent to the pair $(\mathbb{S},\mathbb{T})$, defined in the model space $\mathbb{H}_T$ as follows: the operator $\mathbb{T}$ is the functional model given in definition 1.6, $\mathbb{S}$ is the restriction to $\mathbb{H}_T$ of the operator $P_{\mathbb{H}_T}((I\otimes G^*)+(M_z\otimes G))$. The operators M_z and $P_{\mathbb{H}_T}$ are also taken from definition 1.6, G is the fundamental operator of the Γ-contraction (S^*,T^*).*

The main purposes of the present paper are:

1. to characterize a certain class of linear bounded operators S with difference kernel in the space $L^2_{[0,l]}$ and such that the pairs $(S,\widetilde{T})$ are Γ-contractions;
2. to construct the corresponding functional models;
3. to give some criteria for unitary equivalence between $(S,\widetilde{T})$ and a given Γ-contraction (R,Q) defined on an arbitrary complex separable Hilbert space.

2. Conditions of Γ-contractivity

The aim of the present section is to characterize a certain class of bounded linear operators S acting in $L^2_{[0,l]}$ and such that the corresponding pairs $(S,\widetilde{T})$ are Γ-contractions.

Proposition 2.1 ([9]). *Every bounded linear operator S on $L^2_{[0,l]}$ admits the representation*

$$Sf(x)=\frac{d}{dx}\left(\int_0^l s(x,t)f(t)\,dt\right), \tag{2.1}$$

where the function $s(x,t)$ is an element of $L^2_{[0,l]}$ for every fixed x in $[0,l]$.

Remark 2.2. As mentioned in [9], the kernel $s(x,t)$ can be chosen such that $s(l,t)=0$ for all $t\in[0,\ l]$ and

$$\int_0^l |s(x+h,t)-s(x,t)|^2\,dt\leq \|S\|^2\,|h|\,.$$

Moreover, the operator S and its adjoint S^* are linked by the relation $S^*=USU$ where, U is the involution $Uf(x)=\overline{f(l-x)}$.

In the following, we will suppose that the operator S has a difference kernel $s(x,t)=s(x-t)$ satisfying the conditions of Remark 2.2.

Proposition 2.3. *A bounded linear operator S in $L^2_{[0,\ l]}$, having a difference kernel $s(x,t)=s(x-t)$, commutes with the operator $\widetilde{T}$ if and only if for every $f\in L^2_{[0,\ l]}$ and $x\in[0,\ l]$,*

$$s(x)\int_O^l e^{-t}f(t)\,dt=\int_x^l e^{x-t}\int_0^l s(t-y)f(y)\,dy\,dt-\int_0^l s(x-t)\int_t^l e^{t-y}f(y)\,dy\,dt.$$

Proof. First calculations give us that for every $f \in L^2_{[0,l]}$,

$$[\widetilde{T}S - S\widetilde{T}]f(x) = 2\int_0^l s(x-t)f(t)\,dt - 2\int_x^l e^{x-t}\int_0^l s(t-y)f(y)\,dy\,dt + 2\frac{d}{dx}\left(\int_0^l e^t s(x-t)\int_t^l e^{-z}f(z)\,dz\right)dt.$$

Setting $x-t=y$, we get

$$\begin{aligned}\frac{d}{dx}\left(\int_0^l e^t s(x-t)\int_t^l e^{-z}f(z)\,dz\right)dt &= \frac{d}{dx}\left(\int_0^l e^{-z}f(z)\int_0^z e^t s(x-t)\,dt\right)dz\\ &= \frac{d}{dx}\left(\int_0^l e^{-z}f(z)\int_{x-z}^x e^{x-y}s(y)\,dy\right)dz\\ &= \int_0^l e^{-z}f(z)\frac{d}{dx}\left(\int_{x-z}^x e^{x-y}s(y)\,dy\right)dz.\end{aligned}$$

On the other hand,

$$\frac{d}{dx}\left(\int_{x-z}^x e^{x-y}s(y)\,dy\right) = \int_{x-z}^x e^{x-y}s(y)\,dy + s(x) - e^z s(x-z).$$

So,

$$\begin{aligned}\frac{d}{dx}\left(\int_0^l e^t s(x-t)\int_t^l e^{-z}f(z)\,dz\right)dt &= \int_0^l e^{-z}f(z)\int_{x-z}^x e^{x-y}s(y)\,dy\,dz\\ &\quad+\int_0^l e^{-z}f(z)s(x)\,dz - \int_0^l f(z)s(x-z)\,dz.\end{aligned}$$

Replacing, we get

$$\begin{aligned}[\widetilde{T}S - S\widetilde{T}]f(x) &= 2\int_0^l e^{-t}f(t)s(x)\,dt - 2\int_x^l e^{x-t}\int_0^l s(t-y)f(y)\,dy\,dt\\ &\quad+2\int_0^l e^{-z}f(z)\int_{x-z}^x e^{x-y}s(y)\,dy\,dz\\ &= 2\int_0^l e^{-t}f(t)s(x)\,dt - 2\int_x^l e^{x-t}\int_0^l s(t-y)f(y)\,dy\,dt\\ &\quad+2\int_0^l e^{-z}f(z)\int_0^t e^t s(x-t)\,dt\,dz\\ &= 2\int_0^l e^{-t}f(t)s(x)\,dt - 2\int_x^l e^{x-t}\int_0^l s(t-y)f(y)\,dy\,dt\\ &\quad+2\int_0^l e^t s(x-t)\int_t^l e^{-y}f(y)\,dy\,dt.\end{aligned}$$

This leads us to the desired result. □

We will now seek the conditions of positivity for the operator $\rho(\alpha S, \alpha^2 \widetilde{T})$, $|\alpha| < 1$. For the reasons of density, it suffices to find these conditions of positivity for derivable functions f such that $f(0) = f(l) = 0$. We have

$$\begin{aligned}\langle \rho(\alpha S, \alpha^2 \widetilde{T}) f, f\rangle &= 2(1 - |\alpha|^4)\, \|f\|^2 + 2\,|\alpha|^4 \,\langle (I - \widetilde{T}^* \widetilde{T}) f, f\rangle \\ &\quad - 2\Re\left(\alpha \langle Sf, f\rangle\right) + 2\,|\alpha|^2\, \Re\left(\alpha \langle \widetilde{T} f, Sf\rangle\right).\end{aligned}$$

Integrating by parts, we get

$$\langle Sf, f\rangle = -\int_0^l \overline{f'(x)} \int_0^l s(x-t) f(t)\, dt\, dx. \tag{2.2}$$

On the other hand,

$$\begin{aligned}\langle \widetilde{T} f, Sf\rangle &= \int_0^l \left[f(x) - 2e^x \int_x^l e^{-t} f(t)\, dt\right] \frac{d}{dx}\left(\int_0^l \overline{s(x-y) f(y)}\, dy\right) dx \\ &= \int_0^l f(x) \frac{d}{dx}\left(\int_0^l \overline{s(x-y) f(y)}\, dy\right) dx \\ &\quad - 2\int_0^l e^x \int_x^l e^{-t} f(t)\, dt\, \frac{d}{dx}\left(\int_0^l \overline{s(x-y) f(y)}\, dy\right) dx \\ &= -\int_0^l f'(x) \int_0^l \overline{s(x-y) f(y)}\, dy\, dx + 2\int_0^l e^{-t} f(t)\, dt \int_0^l \overline{s(-y) f(y)}\, dy \\ &\quad + 2\int_0^l \left[e^x \int_x^l e^{-t} f(t) dt - f(x)\right] \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \\ &= -\int_0^l \left(f'(x) + 2f(x)\right) \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \\ &\quad + 2\int_0^l e^{-t} f(t)\, dt \int_0^l \overline{s(-y) f(y)}\, dy \\ &\quad + 2\int_0^l e^x \int_x^l e^{-t} f(t) dt \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \\ &= -\int_0^l \left(f'(x) + 2f(x)\right) \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \\ &\quad + 2\int_0^l e^{-t} f(t)\, dt \int_0^l \overline{s(-y) f(y)}\, dy \\ &\quad + 2\int_0^l e^x \left[e^{-x} f(x) + \int_x^l e^{-t} f'(t)\, dt\right] \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \\ &= \int_0^l \left[-f'(x) + 2e^x \int_x^l e^{-t} f'(t)\, dt\right] \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \\ &\quad + 2\int_0^l e^{-t} f(t)\, dt \int_0^l \overline{s(-y) f(y)}\, dy.\end{aligned}$$

Replacing $\langle Sf, f\rangle$ and $\langle \widetilde{T} f, Sf\rangle$ by their found expressions, we obtain the final result.

Proposition 2.4. *An operator $\rho(\alpha S, \alpha^2 \widetilde{T})$ is positive if and only if for every derivable function f such that $f(0) = f(l) = 0$ the quantity*

$$\begin{aligned} A(\alpha, \widetilde{T}, S, f) = (1 - |\alpha|^4)\, \|f\|^2 + 2\,|\alpha|^4 \left| \int_0^l e^{-t} f(t)\, dt \right|^2 \\ + \Re \left(\alpha \int_0^l f'(x) \int_0^l s(x-t) f(t)\, dt \right) \\ + |\alpha|^2\, \Re \left(\alpha \int_0^l \left[-f'(x) + 2e^x \int_x^l e^{-t} f(t)\, dt \right] \right. \\ \left. \times \int_0^l \overline{s(x-y) f(y)}\, dy\, dx \right) \\ + |\alpha|^2\, \Re \left(\alpha \int_0^l e^{-t} f(t)\, dt \int_0^l \overline{s(-y) f(y)}\, dy \right) \end{aligned}$$

is also positive. Here the symbol $\Re$ designs the real part.

Summarizing, we get

Theorem 2.5. *If S is an operator of the form (2.1) with a difference kernel $s(x,t)$ satisfying the conditions of Remark 2.2, the conclusions of Propositions 2.3 and 2.4, then $(S, \widetilde{T})$ is a Γ-contraction in the space $L^2_{[0,l]}$.*

We end this section by giving the method for obtaining a certain class of operators commuting with $\widetilde{T}$. Since the interval $[0, l]$ is finite, the space $L^2_{[0,l]}$ is contained in $L_{[0,l]}$. Equipped with the Duhamel convolution product

$$(f, g) \mapsto f * g(x) = \int_0^x f(x-t) g(t)\, dt = \int_0^x f(t) g(x-t)\, dt$$

as a multiplication, $L_{[0,l]}$ becomes a Duhamel convolution algebra [7, Chap. 1, Sect. 1.1]. If $\widehat{A}$ is the Volterra integration operator in $L_{[0,l]}$, then $L^2_{[0,l]}$ is invariant for $\widehat{A}$ and the restriction to $L^2_{[0,l]}$ of $\widehat{A}$ coincides with $\widetilde{A}$ (the Volterra integration operator in $L^2_{[0,l]}$). Let now $\widehat{S}$ be any bounded linear operator acting in $L_{[0,l]}$ and commuting with $\widehat{A}$. According to [7, Chap. 1, Sect. 1.3, Theorem 1.1.2], $\widehat{S}$ is a multiplier of the Duhamel convolution algebra $L_{[0,l]}$. That is,

$$S(f * g) = S(f) * g, \quad f, g \in L_{[0,l]}. \tag{2.3}$$

Clearly, formula (2.3) remains true if $f, g \in L^2_{[0,l]}$. Suppose now that $L^2_{[0,l]}$ is invariant for $\widehat{S}$ and let S be the restriction of $\widehat{S}$ to $L^2_{[0,l]}$. Since the operator $\widetilde{T}$ is the Cayley transform (up to a sign) of $\widetilde{A}$, it is not difficult to see that acting in $L^2_{[0,l]}$ the operators $\widetilde{T}$ and S commute.

Summarizing, we get

Proposition 2.6. *Every multiplier of the Duhamel convolution algebra $L_{[0,l]}$ having $L^2_{[0,l]}$ as an invariant subspace generates by restriction to $L^2_{[0,l]}$ an operator commuting with $\widetilde{T}$.*

Proposition 2.6 admits the following converse.

Proposition 2.7. *Let $\widehat{S}$ be a bounded linear operator on $L_{[0,l]}$ having $L^2_{[0,l]}$ as invariant for $\widehat{S}$. Assume that*

1. *The operators $\widehat{S}$ and $\widehat{A}$ commute on the orthogonal of $L^2_{[0,l]}$.*
2. *The restriction to $L^2_{[0,l]}$ of $\widehat{S}$ has the form (2.1) with a difference kernel $s(x,t)$ satisfying the conditions of Remark 2.2 and the conclusion of Proposition 2.3.*

Then the operator $\widehat{S}$ is a multiplier of the Duhamel convolution algebra $L_{[0,\ l]}$.

Proof. Conditions 1 and 2 mean that the operators $\widehat{S}$ and $\widehat{A}$ commute in the whole space $L_{[0,\ l]}$. To conclude, it suffices to apply [7, Chap. 1, Sect. 1.3, Theorem 1.1.2]. □

3. Functional model

We begin this section by giving the explicit form of the elements of the model space $\mathbb{H}_{\widetilde{T}}$. For this, consider once again the functions

$$g(x) = \sqrt{2}e^{-x}, \quad h(x) = \sqrt{2}e^{x-l}, \quad x \in [0,l]$$

which are linked with the operator $\widetilde{T}$ by formula (1.4). We have

$$D^2_{\widetilde{T}} = I - \widetilde{T}^*\widetilde{T} = \langle\cdot, g\rangle g \to D_{\widetilde{T}} = \frac{\langle\cdot, g\rangle}{\|g\|}g. \tag{3.1}$$

Similarly,

$$D^2_{\widetilde{T}^*} = I - \widetilde{T}\widetilde{T}^* = \langle\cdot, h\rangle h \to D_{\widetilde{T}^*} = \frac{\langle\cdot, h\rangle}{\|h\|}h. \tag{3.2}$$

Note also that according to (1.1) and taking in account the equality $\|g\| = \|h\| = \sqrt{1-e^{-2l}}$, we get

$$[\Theta_{\widetilde{T}}(z)](g) = e^{l\frac{z+1}{z-1}}h, \quad z \in \mathbb{D}. \tag{3.3}$$

Consider now the linear operator $J : L^2_{[0,l]} \to \mathbb{H}(\mathcal{D}_{\widetilde{T}^*})$ defined by

$$\zeta \in \mathcal{H}_{\widetilde{T}} \mapsto J(\zeta)(z) = \sum_{n=0}^{+\infty} D_{\widetilde{T}^*}\widetilde{T}^{*n}(\zeta)z^n = D_{\widetilde{T}^*}(I - z\widetilde{T}^*)^{-1}(\zeta). \tag{3.4}$$

It is known (see proofs of Theorem 3.7. in [2] and Theorem 3.1. in [4]) that since $\widetilde{T} \in C_{\cdot 0}$, then J is an isometry and the model space $\mathbb{H}_{\widetilde{T}}$ coincides with the range of J. This leads to the following result.

Proposition 3.1. *A function $\widetilde{f}$ belongs to the model space $\mathbb{H}_{\widetilde{T}}$ if and only if there exists a function $f \in L^2_{[0,l]}$ such that $\widetilde{f}(z) = \langle f,\ H(z)\rangle_{L^2}.\frac{h}{\|h\|}$, where $z \in \mathbb{D}$ and*

$$[H(z)](x) = [(I - \overline{z}\widetilde{T})^{-1}(h)](x) = \frac{\sqrt{2}}{1-\overline{z}}e^{\frac{1+\overline{z}}{1-\overline{z}}(x-l)}.$$

Proof. Since the functional model space $\mathbb{H}_{\widetilde{T}}$ coincides with the range of J, then

$$\mathbb{H}_{\widetilde{T}} = \operatorname{ran}(J) = \left\{ \widetilde{f} = J(f) : f \in L^2_{[0,l]} \right\}.$$

Consequently, if $\widetilde{f} = J(f)$, then, using (3.2), we get

$$D_{\widetilde{T}^*}(I - z\widetilde{T}^*)^{-1}(f) = \langle (I - z\widetilde{T}^*)^{-1}(f), h\rangle_{L^2} \frac{h}{\|h\|} = \langle f, (I - \overline{z}\widetilde{T})^{-1}(h)\rangle_{L^2} \frac{h}{\|h\|}.$$

Let us now find $(I - \overline{z}\widetilde{T})^{-1}(h)$. We have

$$(I - \overline{z}\widetilde{T})^{-1}(h) = h_1 \Leftrightarrow h_1(x) = \frac{h(x)}{1-\overline{z}} - \frac{2\overline{z}e^x}{1-\overline{z}} \int_x^l e^{-t} h_1(t)\, dt, \quad x \in [0, l]\,. \tag{3.5}$$

So, we need to find the expression of the function

$$H_1(x) = \int_x^l e^{-t} h_1(t) dt, \quad x \in [0,\ l]\,.$$

Using the relation $H_1'(x) = -e^{-x}h_1(x)$, it is not difficult to see that H_1 satisfies the Cauchy problem

$$H_1'(x) = \frac{2\overline{z}}{1-\overline{z}} H_1(x) - \frac{\sqrt{2}e^{-l}}{1-\overline{z}}, \quad H_1(l) = 0,$$

which admits a unique solution

$$H_1(x) = \frac{\sqrt{2}e^{-l}}{1-\overline{z}} \left\{ e^{\frac{2\overline{z}}{1-\overline{z}}(x-l)} - 1 \right\}.$$

Substituting H_1 in (3.5), we get

$$h_1(x) = [(I - \overline{z}\widetilde{T})^{-1}(h)](x) = \frac{\sqrt{2}}{1-\overline{z}} e^{\frac{1+\overline{z}}{1-\overline{z}}(x-l)}.$$

This completes the proof of the proposition. □

Let $(S, \widetilde{T})$ be a Γ-contraction in the space $L^2_{[0,l]}$ as defined in Theorem 2.5. The fundamental operator F of $(S, \widetilde{T})$ satisfies the equality $S - S^*\widetilde{T} = D_{\widetilde{T}} F D_{\widetilde{T}}$. Since $\dim(D_{\widetilde{T}}) = 1$, there exists a complex constant λ such that $F(f) = \lambda f$ for all $f \in D_{\widetilde{T}}$. So,

$$S - S^*\widetilde{T} = D_{\widetilde{T}} F D_{\widetilde{T}} = \lambda D^2_{\widetilde{T}} = \ \lambda\langle \cdot, g\rangle g. \tag{3.6}$$

Similarly, for the fundamental operator G of $(S^*, \widetilde{T}^*)$ there exists a complex constant λ_* such that $G(f) = \lambda_* f$ for all $f \in D_{\widetilde{T}^*}$. Hence,

$$S^* - S\widetilde{T}^* = D_{\widetilde{T}^*} G D_{\widetilde{T}^*} = \lambda_* D^2_{\widetilde{T}^*} = \ \lambda_*\langle \cdot, h\rangle h. \tag{3.7}$$

Taking in account the equalities

$$\widetilde{T}\,(g) = e^{-l}h, \quad \widetilde{T}^*\,(h) = e^{-l}g, \quad \|g\|^2 = \|h\|^2 = 1 - e^{-2l},$$

we obtain that the constants λ and λ_* satisfy the relations

$$S(g) - e^{-l} S^*(h) = \lambda \left(1 - e^{-2l}\right) g \quad \text{and} \quad S^*(h) - e^{-l} S(g) = \lambda_* \left(1 - e^{-2l}\right) h.$$

Finally,

$$\lambda = \frac{\langle S(g), g\rangle - e^{-l} \langle S^*(h), g\rangle}{\left(1 - e^{-2l}\right)^2} \quad \text{and} \quad \lambda_* = \frac{\langle S^*(h), h\rangle - e^{-l} \langle S(g), h\rangle}{\left(1 - e^{-2l}\right)^2}. \tag{3.8}$$

Theorem 3.2. *Let $(S, \widetilde{T})$ be a Γ-contraction in the space $L^2_{[0,l]}$ as defined in Theorem 2.5. Then the corresponding functional model $(\mathbb{S}, \widetilde{\mathbb{T}})$ is given in the model space $\mathbb{H}_{\widetilde{T}}$ by*

$$\widetilde{\mathbb{T}}\left(\langle f, H(\cdot)\rangle_{L^2} \frac{h}{\|h\|}\right) = \widetilde{\mathbb{P}}\left(M_z\left(\langle f, H(\cdot)\rangle_{L^2}\right) \frac{h}{\|h\|}\right),$$
$$\widetilde{\mathbb{S}}\left(\langle f, H(\cdot)\rangle_{L^2} \frac{h}{\|h\|}\right) = \widetilde{\mathbb{P}}\left(\left(\overline{\lambda_*} + \lambda_* M_z\right)\left(\langle f, H(\cdot)\rangle_{L^2}\right) \frac{h}{\|h\|}\right),$$

where $\widetilde{\mathbb{P}}$ is the orthogonal projection of $\mathbb{H}\left(\mathcal{D}_{\widetilde{T}^*}\right)$ onto $\mathbb{H}_{\widetilde{T}}$, $f \in L^2_{[0,l]}$, $H(\cdot)$ is the function of complex argument, defined in Proposition 3.1, and λ_* is the complex constant given by (3.8).

Proof. Using the identification of the element $\langle f, H(z)\rangle_{L^2} \frac{h}{\|h\|}$ of $\mathbb{H}\left(\mathcal{D}_{\widetilde{T}^*}\right)$ with the element $\langle f, H(z)\rangle_{L^2} \otimes \frac{h}{\|h\|}$ of $\mathbb{H}(\mathbb{C}) \otimes \mathcal{D}_{\widetilde{T}^*}$ and according to the theory [3, 10], the functional model of the Γ-contraction $(S, \widetilde{T})$ is given in $\mathbb{H}_{\widetilde{T}}$ by

$$\begin{aligned}
\widetilde{\mathbb{T}}\left(\langle f, H(\cdot)\rangle_{L^2} \frac{h}{\|h\|}\right) &= \widetilde{\mathbb{T}}\left(\langle f, H(\cdot)\rangle_{L^2} \otimes \frac{h}{\|h\|}\right) \\
&= \widetilde{\mathbb{P}}\left((M_z \otimes I)\left(\langle f, H(\cdot)\rangle_{L^2} \otimes \frac{h}{\|h\|}\right)\right) \\
&= \widetilde{\mathbb{P}}\left(M_z\left(\langle f, H(\cdot)\rangle_{L^2}\right) \otimes \frac{h}{\|h\|}\right) \\
&= \widetilde{\mathbb{P}}\left(M_z(\langle f, H(\cdot)\rangle_{L^2}) \frac{h}{\|h\|}\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathbb{S}}\left(\langle f, H(\cdot)\rangle_{L^2} . \frac{h}{\|h\|}\right) &= \widetilde{\mathbb{S}}\left(\langle f, H(\cdot)\rangle_{L^2} \otimes \frac{h}{\|h\|}\right) \\
&= \widetilde{\mathbb{P}}\left((I \otimes G^* + M_z \otimes G)\left(\langle f, H(\cdot)\rangle_{L^2} \otimes \frac{h}{\|h\|}\right)\right) \\
&= \widetilde{\mathbb{P}}\left(\langle f, H(\cdot)\rangle_{L^2} \otimes \frac{\overline{\lambda_*} h}{\|h\|} + M_z(\langle f, H(\cdot)\rangle_{L^2}) \otimes \frac{\lambda_* h}{\|h\|}\right) \\
&= \widetilde{\mathbb{P}}\left(\overline{\lambda_*} \langle f, H(\cdot)\rangle_{L^2} \otimes \frac{h}{\|h\|} + \lambda_* M_z(\langle f, H(\cdot)\rangle_{L^2}) \otimes \frac{h}{\|h\|}\right) \\
&= \widetilde{\mathbb{P}}\left(\left(\overline{\lambda_*} + \lambda_* M_z\right)\left(\langle f, H(\cdot)\rangle_{L^2}\right) \otimes \frac{h}{\|h\|}\right).
\end{aligned}$$

□

Theorem 3.3. *Let T be a cnu contraction in $\mathcal{H}$ with a spectrum concentrated at $a = 1$ and one-dimensional defect spaces. Then there exists a unitary operator U from $\mathcal{H}$ onto $L^2_{[0,l]}$ such that for every operator S satisfying the conditions of Theorem 2.5, the pair (U^*SU, T) is a Γ-contraction which is unitarily equivalent to the pair $(\widetilde{\mathbb{S}}, \widetilde{\mathbb{T}})$ of Theorem 3.2.*

Proof. Under the assumptions of the theorem, the operators T and $\widetilde{T}$ have the same characteristic function and thus are unitarily equivalent. Therefore, there exists a unitary operator U from $\mathcal{H}$ onto $L^2_{[0,l]}$ such that $T = U^*\widetilde{T}U$. If S is any operator in $L^2_{[0,l]}$ satisfying the conditions of Theorem 2.5, then $(S, \widetilde{T})$ is a Γ-contraction. Consequently, we have the commuting relations

$$\begin{aligned}(U^*SU)T &= (U^*SU)(U^*\widetilde{T}U) = U^*S\widetilde{T}U = U^*\widetilde{T}SU \\ &= (U^*\widetilde{T}U)(U^*SU) = T(U^*SU),\end{aligned}$$

and for every $\alpha \in \mathbb{D}$,

$$\begin{aligned}\rho(\alpha U^*SU, \alpha^2 T) &= \rho(\alpha U^*SU, \alpha^2 U^*\widetilde{T}U) = 2(I - |\alpha|^4 U^*\widetilde{T}^*UU^*\widetilde{T}U) - \alpha U^*SU \\ &\quad + \overline{\alpha}\alpha^2 U^*S^*UU^*\widetilde{T}U - \overline{\alpha}U^*S^*U + \alpha\overline{\alpha}^2 U^*\widetilde{T}^*UU^*SU \\ &= U^*\left\{2(I - |\alpha|^4\widetilde{T}^*\widetilde{T}) - \alpha S + \overline{\alpha}\alpha^2 S^*\widetilde{T} - \overline{\alpha}S^* + \alpha\overline{\alpha}^2\widetilde{T}^*S\right\}U \\ &= U^*\rho(\alpha S, \alpha^2\widetilde{T})U \geq 0.\end{aligned}$$

Thus, the pair (U^*SU, T) is a Γ-contraction. By the construction, (U^*SU, T) is unitarily equivalent to the pair $(S, \widetilde{T})$ which itself is unitarily equivalent to the pair $(\mathbb{S}, \widetilde{\mathbb{T}})$. We can hence conclude that (U^*SU, T) is unitarily equivalent to $(\widetilde{\mathbb{S}}, \widetilde{\mathbb{T}})$. □

4. Some unitary equivalence results

Let now S be a fixed on the space $L^2_{[0,l]}$ bounded linear operator of the form (2.1) with a difference kernel $s(x,t) = s(x-t)$ satisfying the properties of Remark 2.2. We will suppose that the pair $(S, \widetilde{T})$ is a pure Γ-contraction on $L^2_{[0,l]}$.

Theorem 4.1. *If a Γ-contraction (R, Q) defined on $\mathcal{H}$ is unitarily equivalent to $(S, \widetilde{T})$, then (R, Q) is pure. Moreover, there exist in $\mathcal{H}$ two non null vectors q_1 and q_2 such that:*

$$I - Q^*Q = \langle \cdot, q_1\rangle q_1 \quad \textit{and} \quad I - QQ^* = \langle \cdot, q_2\rangle q_2, \tag{4.1}$$

$$\langle (R - R^*Q)(q_1), q_1\rangle = \langle S(g), g\rangle - e^{-l}\langle S^*(h), g\rangle \tag{4.2}$$

and

$$\langle (R^* - RQ^*)(q_2), q_2\rangle = \langle S^*(h), h\rangle - e^{-l}\langle S(g), h\rangle, \tag{4.3}$$

where the functions g and h are given by the representations (1.4) and (1.5).

Proof. Let (R, Q) be a Γ-contraction on $\mathcal{H}$ and F_* be its fundamental operator. As we know, the pair (R^*, Q^*) is also a Γ-contraction on $\mathcal{H}$ with fundamental operator G_*. Suppose now that (R, Q) is unitarily equivalent to $(S, \widetilde{T})$. There exists then a unitary operator $W : \mathcal{H} \to L^2_{[0,l]}$ such that $R = W^*SW$ and $Q = W^*\widetilde{T}W$. Since the contractions $\widetilde{T}$ and $\widetilde{T}^*$ are in the class $C_{\cdot 0}$, it follows immediately that both operators $Q = W^*\widetilde{T}W$, $Q^* = W^*\widetilde{T}^*W$ are also in $C_{\cdot 0}$ and Γ-contractions (R, Q), (R^*, Q^*) are pure. We have also that (R^*, Q^*) is unitarily equivalent to $(S^*, \widetilde{T}^*)$ by the same unitary operator W. Setting $q_1 = W^*(g)$ and $q_2 = W^*(h)$, we get

$$\begin{aligned} I - Q^*Q &= W^*W - W^*\widetilde{T}^*WW^*\widetilde{T}W = W^*\left(I - \widetilde{T}^*\widetilde{T}\right)W \\ &= \langle W(\cdot), g\rangle W^*(g) = \langle \cdot, W^*(g)\rangle W^*(g) = \langle \cdot, q_1\rangle q_1 \end{aligned}$$

and similarly,

$$\begin{aligned} I - QQ^* &= W^*W - W^*\widetilde{T}WW^*\widetilde{T}^*W = W^*\left(I - \widetilde{T}\widetilde{T}^*\right)W \\ &= \langle W(\cdot), h\rangle W^*(h) = \langle \cdot, W^*(h)\rangle W^*(h) = \langle \cdot, q_2\rangle q_2. \end{aligned}$$

Thus, relations (4.1) are satisfied. On the other hand, according to [4] (see the proof of Proposition 4.2.), $V = W|_{\mathcal{D}_T}$ defines a unitary operator from $\mathcal{D}_T$ onto $\mathcal{D}_{\widetilde{T}}$ such that $F_* = V^*FV$, where F is the fundamental operator of the pair $(S, \widetilde{T})$. Since F is a homothety with ratio

$$\lambda = \frac{\langle S(g), g\rangle - e^{-l}\langle S^*(h), g\rangle}{(1 - e^{-2l})^2},$$

then the operator F_* is also a homothety with the same ratio λ. Consequently, from the relations

$$R - R^*Q = W^*D_TF_*D_TW = \lambda W^*D_T^2W = \lambda\langle \cdot, q_1\rangle q_1,$$

and

$$\|q_1\|^4 = \|W^*(g)\|^4 = \left(\sqrt{1 - e^{-2l}}\right)^4 = \left(1 - e^{-2l}\right)^2,$$

it follows that

$$\begin{aligned} \langle R(q_1), q_1\rangle - \langle Q(q_1), R(q_1)\rangle &= \lambda\|q_1\|^4 = \frac{\langle S(g), g\rangle - e^{-l}\langle S^*(h), g\rangle}{(1 - e^{-2l})^2}\|q_1\|^4 \\ &= \langle S(g), g\rangle - e^{-l}\langle S^*(h), g\rangle. \end{aligned}$$

Thus, relation (4.2) is also satisfied. Reasoning similarly with unitarily equivalent Γ-contractions (R^*, Q^*) and $(S^*, \widetilde{T}^*)$, one can establish relation (4.3). □

Theorem 4.1 admits the following partial converse.

Theorem 4.2. *Let (R, Q) be a pure Γ-contraction on $\mathcal{H}$ and $\Theta_Q(\cdot)$ be the characteristic function of Q. Suppose that there exists a unitary operator U from $\mathcal{D}_Q$ onto $\mathcal{D}_{\widetilde{T}}$ and there exists a unitary operator V from $\mathcal{D}_{\widetilde{T}^*}$ onto $\mathcal{D}_{Q^*}$ such that $\Theta_Q(z) = V\Theta_{\widetilde{T}}(z)U$ for all $z \in \mathbb{D}$. Suppose also that*

$$I - QQ^* = \langle \cdot, V(h)\rangle V(h)$$

and

$$\langle (R^* - RQ^*) V(h), V(h)\rangle = \left\{\langle S^*(h), h\rangle - e^{-l}\langle S(g), h\rangle\right\}.$$

Then (R, Q) and $(S, \widetilde{T})$ are unitarily equivalent.

Proof. Note first that the condition

$$\Theta_Q(z) = V\Theta_{\widetilde{T}}(z)U, \quad z \in \mathbb{D},$$

implies that the characteristic functions $\Theta_Q(\cdot)$ and $\Theta_{\widetilde{T}}(\cdot)$ coincide in the sense of Remark 1.5 and thus the operators Q and $\widetilde{T}$ are unitarily equivalent. We already know (see formulas (3.7) and (3.8)) that the fundamental operator F_* of the pair $(S^*, \widetilde{T}^*)$ is the homothety with ratio

$$\lambda_* = \frac{\langle S^*(h), h\rangle - e^{-l}\langle S(g), h\rangle}{\|h\|^4}.$$

It follows from the relation

$$I - QQ^* = \langle \cdot, V(h)\rangle V(h)$$

that $\dim\left(\mathcal{D}^2_{Q^*}\right) = 1$ and thus the fundamental operator G_* of the pair (R^*, Q^*) is a homothety with ratio η_*. According to [4] (see Proposition 4.3.), to prove the theorem, it suffices to prove that $\eta_* = \lambda_*$. The relations

$$I - QQ^* = \langle \cdot, V(h)\rangle V(h)$$

and

$$R^* - RQ^* = \eta_* D^2_{Q^*} = \eta_* \langle \cdot, V(h)\rangle V(h)$$

imply that

$$\langle (R^* - RQ^*) V(h), V(h)\rangle = \eta_* \|V(h)\|^4 = \eta_* \|h\|^4.$$

Finally,

$$\eta_* = \frac{\langle (R^* - RQ^*) V(h), V(h)\rangle}{\|h\|^4} = \frac{\langle S^*(h), h\rangle - e^{-l}\langle S(g), h\rangle}{\|h\|^4} = \lambda_*. \qquad \square$$

The following result also can be regarded as a partial converse of Theorem 4.1.

Theorem 4.3. *Let (R, Q) be a pure Γ-contraction on $\mathcal{H}$ and $\Theta_Q(\cdot)$ be the characteristic function of Q. Suppose that there exists a unitary operator U from $\mathcal{D}_Q$ into $\mathcal{D}_{\widetilde{T}}$ and there exists a unitary operator V from $\mathcal{D}_{\widetilde{T}^*}$ into $\mathcal{D}_{Q^*}$ such that $\Theta_Q(z) = V\Theta_{\widetilde{T}}(z)U$ for all $z \in \mathbb{D}$. Suppose also that*

$$I - Q^*Q = \langle \cdot, U^*(g)\rangle U^*(g)$$

and

$$\langle (R - R^*Q)\, U^*(g), U^*(g)\rangle = \left\{\langle S(g), g\rangle - e^{-l}\langle S^*(h), g\langle\right\}.$$

Then (R, Q) and $(S, \widetilde{T})$ are unitarily equivalent.

Proof. This is a direct application of the previous theorem to the pure Γ-contractions (R^*, Q^*) and $(S^*, \widetilde{T}^*)$ with taking in account Remark 1.11 and the identity

$$\Theta_Q(z) = V\Theta_{\widetilde{T}}(z)U \Leftrightarrow \Theta_{Q^*}(z) = U^*\Theta_{\widetilde{T}^*}(z)V^*, \quad z \in \mathbb{D}. \qquad \square$$

Notice that the proof of Theorem 4.2 reduces to show that in addition to the given condition $\Theta_Q(z) = V\Theta_{\widetilde{T}}(z)U$, the fundamental operator G_* of the pair (R^*, Q^*) is the homothety with ratio

$$\lambda_*(S^*) = \frac{\langle S^*(h), h\rangle - e^{-l}\langle S(g), h\rangle}{\|h\|^4}. \tag{4.4}$$

By the same way, in Theorem 4.3, it consists to show that the fundamental operator G of the pair (R, Q) is the homothety with ratio

$$\lambda(S) = \frac{\langle S(g), g\rangle - e^{-l}\langle S^*(h), g\rangle}{\|g\|^4}. \tag{4.5}$$

This leads to the following result.

Theorem 4.4. *Let (R, Q) be a pure Γ-contraction on $\mathcal{H}$. Then the following assertions are equivalent:*

1. *(R, Q) is unitarily equivalent to $(S, \widetilde{T})$.*
2. *The characteristic functions of operators Q and $\widetilde{T}$ coincide in the sense of Remark 1.5, moreover, the fundamental operators of (R^*, Q^*) and $(S^*, \widetilde{T}^*)$ are homothecies with the same ratio $\lambda(S^*)$ given by formula (4.4).*
3. *The characteristic functions of operators Q and $\widetilde{T}$ coincide in the sense of Remark 1.5, moreover, the fundamental operators of (R, Q) and $(S, \widetilde{T})$ are homothecies with the same ratio $\lambda(S)$ given by formula (4.5).*

Finally, the main result of the present section is

Theorem 4.5. *Let (R_1, Q_1) and (R_2, Q_2) be pure Γ-contractions on $\mathcal{H}$ and $\mathcal{H}'$. Assume that*

1. *The operators Q_1 and Q_2 have the same spectrum concentrated at the point $a = 1$ and one-dimensional defect spaces.*

2. *The characteristic functions of operators Q_1 and Q_2 coincide in the sense of Remark 1.5.*

3. *There exists in $L^2_{[0,l]}$ a linear operator S satisfying the conditions of Theorem 2.5 such that the fundamental operators of (R_1^*, Q_1^*) and (R_2^*, Q_2^*) are homothecies with the same ratio $\lambda(S^*)$ given by formula (4.4).*

Then (R_1, Q_1) and (R_2, Q_2) are pure and unitarily equivalent.

Proof. Conditions 1 and 2 imply that Q_1 and Q_2 are C_{00} and unitarily equivalent to the same $\widetilde{T}$. According to Theorem 2.5, the pair $(S, \widetilde{T})$ is a pure Γ-contraction. By the second point of Theorem 4.4 and condition 3, each of (R_1, Q_1) and (R_2, Q_2) is unitarily equivalent to $(S, \widetilde{T})$. This completes the proof. □

Remark 4.6. According to the third point of Theorem 4.4, condition 3. can be replaced by the following:

3′. There exists in $L^2_{[0,l]}$ a linear operator S satisfying the conditions of Theorem 2.5 such that the fundamental operators of (R_1, Q_1) and (R_2, Q_2) are homothecies with the same ratio $\lambda(S)$ given by formula (4.5).

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Berrabah Bendoukha,
University of Mostaganem, Route Nationale 11, Mostaganem, 27000, Algeria,
E-mail: bbendoukha@gmail.com

Щодо певного класу Γ-стискань

Berrabah Bendoukha

Метою роботи є вивчення певного класу пар операторів, для яких спектральним набором є симетризований бідиск. Для таких пар надано умови Γ-стискання, а функціональна модель побудована в просторі квадратично інтегровних функцій. Також встановлено деякі критерії унітарної еквівалентності.

Ключові слова: функціональна модель, фундаментальний оператор, чисте стискання, спектральний набір, симетризований бідиск