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General Decay Result for a Type III Thermoelastic Coupled System with Acoustic Boundary Conditions in the Presence of Distributed Delay

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In the paper, the general decay of energy solutions for a type III thermoelastic coupled system with distributed delay is studied. The coupling is via the acoustic boundary conditions. Our result is obtained under a class of generality of the relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the inequality $g'(t) \leq -\xi(t)\mathcal{H}(t)$ for all $t \geq 0$, where ξ and $\mathcal{H}$ are functions satisfying some specific properties. This work extends previous works with thermoelasticity of type III and improves earlier results in the literature.

Key words: thermoelastic effect, acoustic boundary conditions, viscoelastic damping, general decay

Mathematical Subject Classification 2010: 35B40, 74D05, 74F05, 93D15

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1, \dots, \nu_n)$ represents the unit outward normal to Γ. In this setting, we look into $u, \theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $z : \Gamma_1 \times \mathbb{R}_+ \to \mathbb{R}$ solutions for the type III thermoelastic coupled system

$$u_{tt} - \mathcal{A}u + \int_0^t g(t-s)\mathcal{A}u(s)\,ds + \operatorname{div}(\sigma\theta) = 0 \qquad \text{in } \Omega \times \mathbb{R}_+ \tag{1.1}$$

$$\theta_{tt} - \mathcal{B}\theta - \mathcal{B}\theta_t + \int_0^{+\infty} \mu(s)\theta_t(t-s)\,ds + (\sigma\nabla)u_{tt} = 0 \qquad \text{in } \Omega \times \mathbb{R}_+ \tag{1.2}$$

$$u = 0 \qquad \text{on } \Gamma_0 \times \mathbb{R}_+ \tag{1.3}$$

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_{\mathcal{A}}}(s)\,ds + F(u_t) = hz_t \qquad \text{on } \Gamma_1 \times \mathbb{R}_+ \tag{1.4}$$

$$u_t + fz_t + mz = 0 \qquad \text{on } \Gamma_1 \times \mathbb{R}_+ \tag{1.5}$$

$$\theta = 0 \qquad \text{on } \Gamma \times \mathbb{R}_+ \tag{1.6}$$
$$\theta_t(x,-s) = \phi(x,s) \qquad \text{for } x \in \Omega,\ s \in \mathbb{R}_+ \tag{1.7}$$
$$u(0) = u_0,\ u_t(0) = u_1,\ \theta(0) = \theta_0,\ \theta_t(0) = \theta_1 \qquad \text{in } \Omega \tag{1.8}$$
$$z(0) = z_0 \qquad \text{in } \Gamma_1, \tag{1.9}$$

where

$$\mathcal{A}u = \operatorname{div}(\mathrm{A}\nabla u), \quad \mathrm{A} = \Big(a_{ij}(x)\Big)_{1\le i,j\le n}, \quad \mathcal{B} = \operatorname{div}(\mathrm{B}\nabla u), \quad \mathrm{B} = \Big(b_{ij}(x)\Big)_{1\le i,j\le n}$$

and

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i.$$

Here, the function $u = u(x,t)$ describes the velocity, $\theta = \theta(x,t)$ is the temperature difference and the normal displacement on a part of the boundary is represented by $z = z(x,t)$. The first integral term is the finite memory responsible for the viscoelastic damping, where g is called the relaxation function. The second integral term represents the distributed delay. The function vector σ is assumed to be $\mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$, and $u_0, u_1 : \Omega \to \mathbb{R}$, $z_0 : \Gamma_1 \to \mathbb{R}$, μ, F and ϕ are given functions. The functions $h, f, m : \Gamma_1 \to \mathbb{R}_+$ are essentially bounded such that $h(x) \ge h_0$, $f(x) \ge f_0$ and $m(x) \ge m_0$ for a.e. $x \in \Gamma_1$.

The acoustic boundary layer can be used to describe a vibrating impermeable wall. It is well known that the fluid particles follow the wall motion. Moreover, the acoustic pressure is combined with the displacement at the actual position in order to characterize the impenetrability condition, which is obtained from the continuity of the velocity on the boundary. To the best of our knowledge, the classical acoustic conditions of the wave equation were introduced by Morse and Ingard [29] and developed by Beale [5] and Beale–Rosencrans [6]. In [6], it is shown that each point on the surface Γ_1 reacts to the excess pressure of the wave like a resistive harmonic oscillator. For survey works concerning well-posedness and asymptotic behavior of smooth, as well as weak solutions of wave equations with acoustic boundary conditions, we recall to see [4, 7, 17, 36] and references therein.

Green and Naghdi [18] postulated a new concept in thermoelasticity theories and proposed three models. They rated the heat conduction of type III as that of dissipative nature, where the heat flux is a combination of type I and type II as limiting cases. Many works were devoted to studying the existence, uniqueness and asymptotic stability of solutions of thermoelastic systems, see [26, 31, 39] for examples. Although the majority examples involve only one space dimension, in [15, 38, 41], the authors extended the result to higher space dimensions.

The first work bringing an analysis involving Dirichlet, feedback and mixed boundary conditions system was published by Miranda and Medeiros [28]. More precisely, Braz e Silva et al. [10] generalized the system given in [28]. They studied the existence and uniqueness of solutions and asymptotic stabilization of the energy associated with the nonlinear coupled system of thermoelastic type with acoustic boundary conditions:

$$
\begin{aligned}
& u_{tt} - \alpha\Delta u + \lambda|u|^p u + (a.\nabla)\theta = 0 && \text{in } \Omega \times \mathbb{R}_+ && (1.10)\\
& \theta_t - \beta\left(\int_\Omega \theta\, dx\right)\Delta\theta + (a.\nabla)u_t = 0 && \text{in } \Omega \times \mathbb{R}_+ && (1.11)\\
& u_t + hz_{tt} + fz_t + mz = 0 && \text{on } \Gamma_0 \times \mathbb{R}_+ && (1.12)\\
& \frac{\partial u}{\partial \nu} - z_t + \eta(\cdot, u_t) = 0 && \text{on } \Gamma_1 \times \mathbb{R}_+ && (1.13)\\
& u = 0 && \text{on } \Gamma_0 \times \mathbb{R}_+ && (1.14)\\
& \theta = 0 && \text{on } \Gamma \times \mathbb{R}_+, && (1.15)
\end{aligned}
$$

where α and β are given functions, λ and p are positive real constants, a is a constant known vector of $\mathbb{R}^n$ and the feedback function $\eta(., u_t)$ models a frictional damping on Γ_1. In various works, the results on the existence and decay rate of solutions were obtained by many authors in the presence of internal or boundary memory and/or linear/nonlinear damping term, see [11, 12, 23, 30, 37].

The time delay is still peculiar to many physical, chemical, biological and thermal phenomena. Often this effect destabilizes a dissipative system, which can be disastrous in the long term. Recently, a big interest has been directed to control PDE systems (may be a source of instability and/or ill-posedness due to the time delay). As a result, Datko et al. [13] proved that for a small delay in the boundary of wave equation the system becomes unstable. We refer the reader to [16, 21, 35] for different stability/instability results.

In the case of a viscoelastic wave equation with time delay, many researchers were interested in the connection between the weight of the delay and the damping memory term or the frictional damping term. In [22], Kirane and Said-Houari established the energy decay result under the condition ($\mu_2 < \mu_1$) for the following system:

$$
\begin{aligned}
& u_{tt} - \Delta u + \int_0^t g(s)\Delta u(t-s)\, ds \\
& \qquad\qquad + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0 && \text{in } \Omega \times \mathbb{R}_+ && (1.16)\\
& u = 0 && \text{on } \Gamma \times \mathbb{R}_+ && (1.17)\\
& u_t(x, s) = \phi(x, s) && \text{for } x \in \Omega,\ s \in (-\tau,\ 0), && (1.18)
\end{aligned}
$$

where $\tau > 0$. For the distributed delay term $\left(\int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)\, ds\right)$, Guesmia and Tatar [20] obtained the stability result by combining two kernels for abstract hyperbolic equations with arbitrary decay. Furthermore, Fareh and Messaoudi [14] proved the exponential stabilization of one-dimension type III thermoelastic Timoshenko system in the presence/absence of a frictional damping under the condition of smallness on the weight of delay.

For a larger type of relaxation functions, Mustafa [33] established the optimal and the general decay results of (1.16)–(1.18), when ($\mu_1 = \mu_2 = 0$), under the following general latest assumption:

$$
g'(t) \le -\xi(t)\mathcal{H}(g(t)), \quad t \ge 0, \tag{1.19}
$$

where $\mathcal{H}$ is an increasing and convex function near the origin and ξ is a positive nonincreasing differentiable function. This assumption has wildly attracted considerable attention of a number of researchers in the last few years. For example, Al-Gharabli et al. [1] extended and combined the result with the boundary feedback stabilization.

As highlighted by the following papers, there have been encouraging advances toward obtaining the asymptotic behavior in each case of (1.19).

In the case of $\mathcal{H}(s) = s^p$, $1 \le p < 2$, Messaoudi [27] considered (1.16)–(1.18) for $p = 1$. He established a general decay result for the growth of relaxation function. Otherwise, Mustafa [32] obtained optimal and polynomial decay rates. Moreover, in the presence of a discrete or time-dependent delay, where the relaxation function satisfies (1.19) with nonlinear damping term, see [8, 9].

For $\xi(t) = 1$, Lasiecka and Tataru [25] used different approaches to establish the uniform decay of wave equation with frictional damping for some additional constraints imposed on $\mathcal{H}$, where $\mathcal{H}$ is positive, strictly increasing and strictly $\mathcal{C}^2$ convex near the origin with $\mathcal{H}(0) = \mathcal{H}'(0) = 0$. We refer to the previous studies, [2, 19, 24] and [34], where a general decay result was established for $\mathcal{H} \in \mathcal{C}^1(\mathbb{R})$, $\mathcal{H}(0) = 0$.

Motivated by the works mentioned above, we are interested in giving optimal and general decay rates for a type III thermoelastic coupled system with acoustic boundary conditions in the presence of distributed delay. This paper extends the results of [32] to problem (1.1)–(1.9) under general assumption on the nonlinear damping term which was first considered in [25]. Our system (1.1)–(1.9) suggested here applies the system (1.10)–(1.15) studied in [10] in thermoelasticity of type III.

The paper is organized as follows. In Section 2, we give some notations and present some assumptions needed for our work. In Section 3, we state and prove some technical lemmas in order to get our main results. Finally, the decay rate is improved explicitly by using the convexity of the relaxation function g and without imposing any restrictive growth assumption on the damping term. The proof is based on the construction of a suitable Lyapunov functional.

2. Preliminary

In this section, we present some material we use in order to present our results. Let

$$\mathbf{H}(\mathcal{A}, \Omega) = \{u \in \mathrm{H}^1(\Omega) \mid \mathcal{A}u \in \mathrm{L}^2(\Omega)\}$$

be a Hilbert space equipped with the norm

$$\|u\|_{\mathbf{H}(\mathcal{A},\Omega)} = \left(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|\mathcal{A}u\|_2^2\right)^{1/2},$$

where $\mathrm{H}^1(\Omega)$ is the Sobolev space of first order, $\|\cdot\|_2$ is an L^2-norm and $\|\cdot\|_{2,\Gamma_1}$ is an L^2-norm on Γ_1, and $\langle\cdot,\cdot\rangle_{\Gamma_1}$ is the scalar product in $\mathrm{L}^2(\Gamma_1)$.

Denoting by $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Gamma)$ and $\gamma_1 : \mathbf{H}(\mathcal{A}, \Omega) \to \mathrm{L}^2(\Gamma)$ the trace map of order 0 and the Neumann trace map on $\mathbf{H}(\mathcal{A}, \Omega)$ respectively, we have $\gamma_0(u) = u_{|\Gamma}$

and $\gamma_1(u) = \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}}\right)_\Gamma$ for all u in $\mathbf{H}(\mathcal{A}, \Omega)$. Sometimes to simplify the notations we write u and $\frac{\partial u}{\partial \nu_{\mathcal{A}}}$ instead of $\gamma_0(u)$ and $\gamma_1(u)$, respectively.

We denote

$$\mathbf{V} = \left\{u \in \mathrm{H}^1(\Omega) \mid u = 0 \text{ on } \Gamma_0\right\}$$

equipped with the norm equivalent to the usual norm in $\mathrm{H}^1(\Omega)$. The Poincaré inequality holds on $\mathbf{V}$, i.e.,

$$\exists C_* > 0 \ \forall u \in \mathbf{V} \quad \|u(t)\|_2 \leq C_* \|\nabla u(t)\|_2. \tag{2.1}$$

Moreover,

$$\exists \bar{C}_* > 0 \ \forall u \in \mathbf{V} \quad \|u(t)\|_{2,\Gamma_1} \leq \bar{C}_* \|\nabla u(t)\|_2. \tag{2.2}$$

In this study, we will need the following assumptions.

($\mathbf{A}_1$) The coefficients $a_{ij}, b_{ij} \in \mathcal{C}^1(\overline{\Omega})$ are symmetric and there exist two constants $a_0, b_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\eta_i\eta_j \geq a_0|\eta|^2 \qquad \text{for all } x \in \overline{\Omega},\ \eta \in \mathbb{R}^n, \tag{2.3}$$

$$\sum_{i,j=1}^{n} b_{ij}(x)\eta_i\eta_j \geq b_0|\eta|^2, \qquad \text{for all } x \in \overline{\Omega},\ \eta \in \mathbb{R}^n. \tag{2.4}$$

Furthermore, we assume that the weight of delay $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded nonincreasing function satisfying

$$c_1 = b_0 - C_*^2 \int_0^{+\infty} \mu(s)\, ds > 0. \tag{2.5}$$

($\mathbf{A}_2$) The relaxation function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded $\mathcal{C}^1$ nonincreasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)\, ds = \ell > 0, \tag{2.6}$$

and there exists a function $\mathcal{H} : \mathbb{R}_+ \to \mathbb{R}_+$ which is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0,\ r]$, for a positive constant $r \leq g(0)$, with $\mathcal{H}(0) = \mathcal{H}'(0) = 0$ such that

$$g'(t) \leq -\xi(t)\mathcal{H}(g(t)) \quad \text{for all } t \geq 0, \tag{2.7}$$

where ξ is a positive nonincreasing differentiable function.

($\mathbf{A}_3$) $F : \mathbb{R} \to \mathbb{R}$ is an increasing $\mathcal{C}^0$ function such that there exists a strictly increasing function $F_0 \in \mathcal{C}^1(\mathbb{R}_+)$ with $F_0(0) = 0$. Furthermore, there exist two constants $c_1', c_2' > 0$ such that

$$c_1'|s| \leq |F(s)| \leq c_2'|s| \qquad \text{for all } |s| \geq \varepsilon, \tag{2.8}$$

$$F_0(|s|) \leq |F(s)| \leq F_0^{-1}(|s|) \qquad \text{for all } |s| \leq \varepsilon. \tag{2.9}$$

In addition, we assume that the function G, defined by $\mathrm{G}(s) = \sqrt{s}F_0(\sqrt{s})$, is a strictly convex $\mathcal{C}^2$ function on $(0,\ r_1]$ $(r_1 > 0)$. This hypothesis was first considered in [25].

Let us introduce the following notation:

$$(g \diamond u)(t) = \int_0^t g(t-s)a(u(t)-u(s), u(t)-u(s))\,ds,$$

where

$$a(u(t), v(t)) = \int_\Omega \mathrm{A}\nabla u(t)\nabla v(t)\,dx = \sum_{i,j=1}^n \int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial v(t)}{\partial x_i}\,dx.$$

Then we have

$$\begin{aligned}\frac{d}{dt}(g \diamond u)(t) &= (g' \diamond u)(t) - 2\int_0^t g(t-s)a(u(s), u_t(t))\,ds \\ &\quad + \frac{d}{dt}\left(a(u(t), u(t))\int_0^t g(s)\,ds\right) - g(t)a(u(t), u(t)). \qquad (2.10)\end{aligned}$$

Note that

$$b(u(t), v(t)) = \int_\Omega \mathrm{B}\nabla u(t)\nabla v(t)\,dx.$$

It is easy to verify that the bilinear forms $a(\cdot,\cdot) : \mathbf{V}\times\mathbf{V} \to \mathbb{R}$ and $b(\cdot,\cdot) : \mathrm{H}_0^1(\Omega)\times \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ are symmetric, continuous and coercive by using (2.3) and (2.4).

We mention some additional remarks that will be used in arguments and proofs.

Remark 2.1 ([33]). We should note the following.

1) We can easily deduce from $(\mathbf{A}_2)$ that there is t_0 large enough while $g(t_0) = r$. Hence, for all $t \le t_0$,

$$0 < g(t_0) \le g(t) \le g(0), \quad 0 < \xi(t_0) \le \xi(t) \le \xi(0),$$

which implies that there are two positive constants ϵ_1 and ϵ_2 such that

$$\epsilon_1 \le \xi(t)\mathcal{H}(g(t)) \le \epsilon_2.$$

Therefore,

$$\exists \zeta_1 = \frac{\epsilon_1}{g(0)} > 0 \quad \forall t \le t_0 \quad g'(t) \le -\zeta_1 g(t). \qquad (2.11)$$

2) If (2.7) holds, then $\mathcal{H}$ has an extension $\overline{\mathcal{H}}$, which is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $\mathbb{R}_+$. For example, if we set $\mathcal{H}(r) = \alpha$, $\mathcal{H}'(r) = \beta$ and $\mathcal{H}''(r) = \gamma$, we can define $\overline{\mathcal{H}}$ for all $t > r$ by

$$\overline{\mathcal{H}}(t) = \frac{\gamma}{2}t^2 + (\beta - \gamma r)t + \left(\alpha + \frac{\gamma}{2}r^2 - \beta r\right).$$

3) Suppose φ is a non-negative measurable function satisfying $\int_\Omega \varphi(x)dx = 1$. If f is any real-valued measurable function and Ψ is convex over the range of f, then Jensen's inequality states that

$$\Psi\left[\int_\Omega f(x)\varphi(x)dx\right] \le \int_\Omega \Psi[f(x)]\varphi(x)\,dx.$$

For completeness, let us take a new variable φ first introduced in [40]:

$$\varphi(x,t) = \int_0^t \theta(x,s)\,ds + \Phi(x), \quad \text{forall } x \in \Omega,\ t \ge 0, \tag{2.12}$$

where $\Phi \in \mathrm{H}_0^1(\Omega)$ solves

$$\begin{aligned} &\mathcal{B}\Phi = \theta_1 - \mathcal{B}\theta_0 + \int_0^{+\infty} \mu(s)\theta(-s)\,ds + (\sigma\nabla)u_1 && \text{in } \Omega \\ &\Phi = 0 && \text{on } \Gamma. \end{aligned}$$

As in [35], let us set

$$\omega(x,p,s,t) = \varphi_t(x, t - sp), \quad (x,p,s) \in \mathrm{Q} = \Omega \times (0,\ 1) \times \mathbb{R}_+, \quad t \ge 0.$$

Then our system leads to

$$\begin{aligned}
&u_{tt} - \mathcal{A}u + \int_0^t g(t-s)\mathcal{A}u(s)\,ds + \operatorname{div}(\sigma\varphi_t) = 0 && \text{in } \Omega \times \mathbb{R}_+ && (2.13)\\
&\varphi_{tt} - \mathcal{B}\varphi - \mathcal{B}\varphi_t + \int_0^{+\infty} \mu(s)\omega(x,1,s,t)\,ds \\
&\qquad\qquad\qquad\qquad + (\sigma\nabla)u_t = 0 && \text{in } \Omega \times \mathbb{R}_+ && (2.14)\\
&s\omega_t + \omega_p = 0 && \text{in } \mathrm{Q} \times \mathbb{R}_+ && (2.15)\\
&f z_t + m z + u_t = 0 && \text{on } \Gamma_1 \times \mathbb{R}_+ && (2.16)\\
&u = 0 && \text{on } \Gamma_0 \times \mathbb{R}_+ && (2.17)\\
&\frac{\partial u}{\partial \nu_{\mathcal{A}}} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_{\mathcal{A}}}(s)\,ds + F(u_t) = h z_t && \text{on } \Gamma_1 \times \mathbb{R}_+ && (2.18)\\
&\varphi = 0 && \text{on } \Gamma \times \mathbb{R}_+ && (2.19)\\
&\omega(x,p,s,0) = \omega_0(x,p,s) && \text{for } (x,p,s) \in \mathrm{Q} && (2.20)\\
&u(0) = u_0,\ u_t(0) = u_1,\ \varphi(0) = \Phi,\ \varphi_t(0) = \theta_0 && \text{in } \Omega && (2.21)\\
&z(0) = z_0 && \text{in } \Gamma_1. && (2.22)
\end{aligned}$$

In the following proposition, we state the global existence of solution for system (2.13)–(2.22) without proof which can be established by means of the Faedo–Galerkin method. We refer the reader to [7, 11].

Proposition 2.2. *Let $u_0 \in \mathrm{H}^2(\Omega) \cap \mathbf{V}$, $u_1 \in \mathbf{V}$, $\Phi \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$, $\theta_0 \in \mathrm{H}_0^1(\Omega)$, $\omega_0 \in \mathrm{L}^2(\mathrm{Q})$ and $z_0 \in \mathrm{L}^2(\Gamma_1)$ be given. Assume that* (**A**$_1$), (**A**$_3$), *and* (2.6) *hold. Then problem* (2.13)–(2.22) *has a unique global solution such that*

$$u, u_t \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbf{V}), \qquad u_{tt} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\Omega)), \qquad u \in \mathbf{H}(\mathcal{A}, \Omega);$$

$$\varphi, \varphi_t \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{H}_0^1(\Omega)), \qquad \varphi_{tt} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\Omega)), \qquad \varphi \in \mathbf{H}(\mathcal{B}, \Omega);$$
$$\omega \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\mathrm{Q})); \qquad z, z_t \in \mathrm{L}^\infty(\mathbb{R}_+; \mathrm{L}^2(\Gamma_1)).$$

3. Technical lemmas

In this section, we establish several lemmas needed for our main result. We define the energy E of system (2.13)–(2.22) for all $t \geq 0$ by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right) a(u(t), u(t)) + \frac{1}{2}(g \diamond u)(t) \\ & + \frac{1}{2} b(\varphi(t), \varphi(t)) + \frac{1}{2}\|\varphi_t(t)\|_2^2 + \frac{1}{2}\|h^{1/2} m^{1/2} z(t)\|_{2,\Gamma_1}^2 \\ & + \frac{1}{2}\int_{\mathrm{Q}} s\mu(s)\omega^2(x, p, s, t)\,ds\,dp\,dx. \end{aligned} \tag{3.1}$$

Lemma 3.1. *The energy functional, in view of* (2.13)–(2.22)*, is a nonincreasing function and satisfies the estimate*

$$\begin{aligned} E'(t) \leq {} & \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2} g(t) a(u(t), u(t)) - \|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2 \\ & - \Big\langle F(u_t(t)), u_t(t) \Big\rangle_{\Gamma_1} - c_1 \|\nabla \varphi_t(t)\|_2^2. \end{aligned} \tag{3.2}$$

Proof. Multiplying (2.13) by u_t, (2.14) by φ_t and integrating over Ω, then adding it to the inner product of (2.15) with $s\mu(s)$ in $\mathrm{L}^2(\mathrm{Q})$, by using Green's formula, we arrive at

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\Big(\|u_t(t)\|_2^2 + a(u(t), u(t)) + b(\varphi(t), \varphi(t)) + \int_{\mathrm{Q}} s\mu(s)\omega^2(x, p, s, t)\,ds\,dp\,dx \\ & \quad + \|\varphi_t(t)\|_2^2 \Big) - \int_0^t g(t-s) a(u(s), u_t(t))\,ds - \langle h(x) z_t(t), u_t(t) \rangle_{\Gamma_1} \\ & \quad + b(\varphi_t(t), \varphi_t(t)) + \frac{1}{2}\int_\Omega \int_0^{+\infty} \mu(s) \left[\omega^2(x, 1, s, t) - \omega^2(x, 0, s, t)\right] ds\,dx \\ & \quad + \langle F(u_t(t)), u_t(t) \rangle_{\Gamma_1} + \int_\Omega \varphi_t(t) \int_0^{+\infty} \mu(s)\omega(x, 1, s, t)\,ds\,dx = 0. \end{aligned} \tag{3.3}$$

We notice from (2.16) that

$$\begin{aligned} -\langle h(x) u_t(t), z_t(t) \rangle_{\Gamma_1} & = \|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2 + \langle h(x) m(x) z(t), z_t(t) \rangle_{\Gamma_1} \\ & = \|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2 + \frac{1}{2}\frac{d}{dt}\|h^{1/2} m^{1/2} z(t)\|_{2,\Gamma_1}^2. \end{aligned} \tag{3.4}$$

The last term of the right-hand side of (3.3) can be estimated as follows:

$$\begin{aligned} & -\int_\Omega \varphi_t(t) \int_0^{+\infty} \mu(s)\omega(x, 1, s, t)\,ds\,dx \\ & \quad \leq \frac{C_*^2}{2}\left(\int_0^{+\infty} \mu(s)\,ds\right) \|\nabla \varphi_t(t)\|_2^2 + \frac{1}{2}\int_\Omega \int_0^{+\infty} \mu(s)\omega^2(x, 1, s, t)\,ds\,dx. \end{aligned} \tag{3.5}$$

Substituting (2.10), (3.4), and (3.5) into (3.3), we get (3.2) from (2.5). $\square$

Now we are going to construct a Lyapunov functional equivalent to the energy functional by resorting to the following functions (K_i; $i = 1, 2, 3, 4$), with which we can show the desired result.

Lemma 3.2. *Let (u, φ, ω, z) be a solution of (2.13)–(2.22). Then the functional*

$$K_1(t) = \int_\Omega u_t(t)u(t)\,dx + \langle hz(t), u(t)\rangle_{\Gamma_1} + \frac{1}{2}\|h^{1/2}f^{1/2}z(t)\|^2_{2,\Gamma_1},$$

satisfies the estimate, for any $0 < \alpha < 1$, for all $t \geq 0$,

$$\begin{aligned}K_1'(t) \leq \|u_t(t)\|_2^2 - \frac{\ell}{2}a(u(t), u(t)) + \frac{C_1 C_\alpha}{4a_0}(k \diamond u)(t) + C_2\int_{\Gamma_1} F^2(u_t(t))\,d\Gamma \\ + C_3\|\nabla\varphi_t(t)\|_2^2 + C_4\|h^{1/2}f^{1/2}z_t(t)\|^2_{2,\Gamma_1} - \|h^{1/2}m^{1/2}z(t)\|^2_{2,\Gamma_1},\end{aligned} \tag{3.6}$$

where

$$\begin{gathered}C_\alpha = \int_0^\infty \frac{g^2(s)}{\alpha g(s) - g'(s)}ds, \quad C_1 = \frac{6a_1 n}{a_0\ell}, \quad C_2 = \frac{2\overline{C}_*^2}{a_0\ell}, \\ C_3 = 2\frac{C_*^4\|\mathrm{div}(\sigma)\|_\infty^2 + C_*^2\|\sigma\|^2}{a_0\ell}, \quad C_4 = \frac{6\overline{C}_*^2\|h\|_\infty\|f\|_\infty}{a_0\ell f_0^2}, \\ a_1 = \max_{j=1}^{n}\left(\sum_{i=1}^n \|a_{ij}\|_\infty^2\right), \; k(t) = \alpha g(t) - g'(t), \; \|\sigma\| = \sup_{x\in\Omega}\sup_{i=1,\ldots,n}|\sigma_i(x)|,\end{gathered} \tag{3.7}$$

Proof. Direct computations, by using (2.13) and (2.17), gives

$$\begin{aligned}K_1'(t) = \|u_t(t)\|_2^2 - a(u(t), u(t)) + \int_0^t g(t-s)a(u(s), u(t))\,ds \\ - \langle F(u_t(t)), u(t)\rangle_{\Gamma_1} - \|h^{1/2}m^{1/2}z(t)\|^2_{2,\Gamma_1} \\ + 2\langle h(x)z_t(t), u(t)\rangle_{\Gamma_1} - \int_\Omega \mathrm{div}(\sigma(x)\varphi_t(t))u(t)\,dx.\end{aligned} \tag{3.8}$$

Now, by using Cauchy–Schwarz's inequality, we get

$$\begin{aligned}&\int_\Omega\left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\right)^2 dx \\ &\quad = \int_\Omega\left(\int_0^t \frac{g(t-s)}{\sqrt{k(t-s)}}\sqrt{k(t-s)}(\nabla u(s) - \nabla u(t))\,ds\right)^2 dx \leq \frac{C_\alpha}{a_0}(k \diamond u)(t).\end{aligned}$$

Using Young's inequality, (2.3) and (2.6), we obtain, for some constant $\delta > 0$,

$$\begin{aligned}&\int_0^t g(t-s)a(u(s), u(t))\,ds \\ &\quad = \int_0^t g(t-s)\int_\Omega \mathrm{A}\,(\nabla u(s) - \nabla u(t))\,\nabla u(t)\,dx\,ds + \left(\int_0^t g(s)\,ds\right)a(u(t), u(t)),\end{aligned}$$

$$\leq \left[(1-\ell)+\delta\frac{a_1}{a_0}\right]a(u(t),u(t))+\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t), \tag{3.9}$$

By using Cauchy–Schwarz's, Young's inequalities and (2.2), we get, for $\delta_1>0$,

$$\begin{aligned}\left|\left\langle h(x)z_t(t),u(t)\right\rangle_{\Gamma_1}\right| &\leq \frac{\|h\|_\infty^{1/2}\|f\|_\infty^{1/2}}{f_0}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}\|u(t)\|_{2,\Gamma_1}\\ &\leq \delta_1\frac{\overline{C}_*^2}{a_0}a(u(t),u(t))+\frac{\|h\|_\infty\|f\|_\infty}{4\delta_1 f_0^2}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2, \end{aligned}\tag{3.10}$$

for $\delta_2>0$,

$$\begin{aligned}\left|\int_\Omega \operatorname{div}(\sigma(x)\varphi_t(t))u(t)\,dx\right| &\leq \frac{C_*^2\|\operatorname{div}(\sigma)\|_\infty^2+\|\sigma\|^2}{4\delta_2}\|\nabla\varphi_t(t)\|_2^2\\ &\quad+\frac{\delta_2 C_*^2}{a_0}a(u(t),u(t)),\end{aligned}\tag{3.11}$$

and for $\delta_3>0$,

$$-\left\langle F(u_t(t)),u(t)\right\rangle_{\Gamma_1}\leq\frac{1}{4\delta_3}\int_{\Gamma_1}F^2(u_t(t))\,d\Gamma+\frac{\delta_3\overline{C}_*^2}{a_0}a(u(t),u(t)). \tag{3.12}$$

Substituting (3.9)–(3.12) into (3.8), we find that

$$\begin{aligned}K_1'(t)\leq{}&\|u_t(t)\|_2^2+\left[-\ell+\delta\frac{a_1}{a_0}+2\delta_1\frac{\overline{C}_*^2}{a_0}+\frac{\delta_2C_*^2}{a_0}+\frac{\delta_3\overline{C}_*^2}{a_0}\right]a(u(t),u(t))\\ &+\frac{nC_\alpha}{4a_0\delta}(k\diamond u)(t)+\frac{C_*^2\|\operatorname{div}(\sigma)\|^2+\|\sigma\|^2}{4\delta_2}\|\nabla\varphi_t(t)\|_2^2-\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2\\ &+\frac{\|h\|_\infty\|f\|_\infty}{2\delta_1f_0^2}\|h^{1/2}f^{1/2}z_t(t)\|_{2,\Gamma_1}^2+\frac{1}{4\delta_3}\int_{\Gamma_1}F^2(u_t(t))\,d\Gamma.\end{aligned}$$

Let us choose

$$\delta=\frac{a_0\ell}{8a_1},\quad \delta_1=\frac{a_0\ell}{16\overline{C}_*^2},\quad \delta_2=\frac{a_0\ell}{8C_*^2}\quad\text{and}\quad \delta_3=\frac{a_0\ell}{8\overline{C}_*^2}.$$

Then we obtain (3.6). □

Lemma 3.3. *Let (u,φ,ω,z) be a solution of (2.13)–(2.22). Then the functional*

$$K_2(t)=\int_\Omega\varphi_t(t)\varphi(t)\,dx+\frac{1}{2}b(\varphi,\varphi)+\int_\Omega(\sigma(x).\nabla)u(t)\varphi(t)\,dx$$

satisfies the estimate, for all $t\geq 0$,

$$\begin{aligned}K_2'(t)\leq{}&\frac{C_*^2}{2a_0}a(u(t),u(t))+C_5\|\nabla\varphi_t\|_2^2-\frac{1}{2}b(\varphi(t),\varphi(t))\\ &+C_6\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\,ds\,dx,\end{aligned}\tag{3.13}$$

where $2C_5=2C_*^2+\|\sigma\|^2$ and $C_6=\frac{C_*^2}{2b_0}\left(\int_0^{+\infty}\mu(s)\,ds\right)$.

Proof. By exploiting (2.14) and using Green's formula, we have

$$K_2'(t) = \|\varphi_t(t)\|_2^2 - b(\varphi(t),\varphi(t)) - \int_\Omega (\sigma(x).\nabla)\varphi_t(t)u(t)\,dx \\ - \int_\Omega \varphi(t)\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\,ds\,dx. \tag{3.14}$$

We can estimate the last two terms above for similar calculations in (3.9) in the following:

$$-\int_\Omega (\sigma(x)\nabla)\varphi_t(t)u(t)\,dx \le \frac{\|\sigma\|^2}{2}\|\nabla\varphi_t\|_2^2 + \frac{C_*^2}{2a_0}a(u(t),u(t)), \tag{3.15}$$

and for $\delta > 0$,

$$-\int_\Omega \varphi(t)\int_0^{+\infty}\mu(s)\omega(x,1,s,t)\,ds\,dx \\ \le \frac{C_*^2}{4\delta b_0}b(\varphi(t),\varphi(t)) + \delta\left(\int_0^{+\infty}\mu(s)\,ds\right)\int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\,ds\,dx. \tag{3.16}$$

Let us choose $\delta = \frac{C_*^2}{2b_0}$. Inserting (3.15) and (3.16) into (3.14), we get (3.13). □

Lemma 3.4. *Let (u,φ,ω,z) be a solution of (2.13)–(2.22). Then the functional*

$$K_3(t) = \int_Q se^{-sp}\mu(s)\omega^2(x,p,s,t)\,dp\,ds\,dx$$

satisfies the estimate, for all $t \ge 0$,

$$K_3'(t) \le C_7\|\nabla\varphi_t(t)\|_2^2 - \int_\Omega\int_0^{+\infty}\mu(s)\omega^2(x,1,s,t)\,ds\,dx \\ - \int_Q s\mu(s)\omega^2(x,p,s,t)\,dp\,ds\,dx, \tag{3.17}$$

where $C_7 = C_*^2\int_0^{+\infty}\mu(s)\,ds$.

Proof. By differentiating K_3 with respect to t, we obtain

$$\begin{aligned} K_3'(t) &= -2\int_\Omega\int_0^{+\infty}\mu(s)\int_0^1 e^{-sp}\omega(p,s,t)\omega_p(p,s,t)\,dp\,ds\,dx,\\ &= -\int_\Omega\int_0^{+\infty}\mu(s)\left(e^{-s}\omega^2(1,s,t) - \omega^2(0,t) + s\int_0^1 e^{-sp}\omega(p,s,t)dp\right)ds\,dx,\\ &\le C_7\|\nabla\varphi_t(t)\|_2^2 - \int_\Omega\int_0^{+\infty}\mu(s)\omega^2(1,s,t)\,ds\,dx\\ &\quad - \int_Q s\mu(s)\omega^2(p,s,t)\,dp\,ds\,dx. \end{aligned}$$

□

Lemma 3.5. *Let (u, φ, ω, z) be a solution of (2.13)–(2.22). Then the functional*

$$K_4(t) = -\int_\Omega u_t(t) \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx$$

satisfies the estimate, for $\delta > 0$, for all $t \geq 0$,

$$\begin{aligned} K_4'(t) \leq &- \left(\int_0^t g(s)\,ds - \delta \right) \|u_t(t)\|_2^2 + C_8(\delta) a(u(t), u(t)) \\ &+ \left(\frac{C_\alpha C_9(\delta) + C_{10}(\delta)}{4a_0} \right) (k \diamond u)(t) + C_{11} \|\nabla \varphi_t(t)\|_2^2 \\ &+ C_{12} \|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2 + C_{13} \int_{\Gamma_1} F^2(u_t(t))\,d\Gamma, \end{aligned} \tag{3.18}$$

where

$$C_8(\delta) = \delta \frac{a_1}{a_0}, \quad C_9(\delta) = 1 + 4\sqrt{n a_1} + \frac{2\alpha^2 C_*^2 + n}{\delta}, \quad C_{10}(\delta) = \frac{2k_1 C_*^2}{\delta},$$
$$C_{11} = 3C_*^2 \|\sigma\|^2, \ C_{12} = \frac{3\bar{C}_*^2 \|h\|_\infty \|f\|_\infty}{f_0^2}, \ C_{13} = 3\bar{C}_*^2, \ k_1 = \int_0^\infty k(s)\,ds.$$

Proof. By exploiting (1.1) and using Green's formula, we have

$$\begin{aligned} K_4'(t) = &\left(1 - \int_0^t g(s)\,ds \right) \int_\Omega \mathrm{A}\nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &+ \int_\Omega \int_0^t g(t-s)\mathrm{A}(\nabla u(t) - \nabla u(s))\,ds \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &- \left\langle h z_t(t), \int_0^t g(t-s)(u(t)-u(s))\,ds \right\rangle_{\Gamma_1} - \left(\int_0^t g(s)\,ds \right) \|u_t(t)\|_2^2 \\ &- \int_\Omega u_t(t) \int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\ &- \int_\Omega \sigma(x)\varphi_t(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &+ \langle F(u_t(t)), \int_0^t g(t-s)(u(t)-u(s))\,ds \rangle_{\Gamma_1}. \end{aligned} \tag{3.19}$$

Using the similar calculations in (3.9), we obtain

$$\begin{aligned} &\int_\Omega \mathrm{A}\nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &\qquad \leq \delta \frac{a_1}{a_0} a(u(t), u(t)) + \frac{nC_\alpha}{4a_0\delta}(k \diamond u)(t) \end{aligned} \tag{3.20}$$

for $\delta > 0$ and

$$-\int_\Omega \sigma(x)\varphi_t(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx$$

$$\leq \delta_1 C_*^2 \|\sigma\|^2 \|\nabla \varphi_t(t)\|_2^2 + \frac{C_\alpha}{4a_0\delta_1}(k \diamond u)(t) \tag{3.21}$$

for $\delta_1 > 0$. By repeating the same arguments of (3.9)–(3.10), we get

$$\begin{aligned} &-\big\langle h z_t(t), \int_0^t g(t-s)(u(t)-u(s))\,ds\big\rangle_{\Gamma_1} \\ &\qquad \leq \frac{\bar{C}_*^2 C_\alpha}{4\delta_2 a_0}(k \diamond u)(t) + \frac{\delta_2 \|h\|_\infty \|f\|_\infty}{f_0^2}\|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2 \end{aligned} \tag{3.22}$$

for $\delta_2 > 0$. We arrive to the estimate

$$\begin{aligned} &\int_\Omega \int_0^t g(t-s)\mathrm{A}(\nabla u(t) - \nabla u(s))\,ds \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &\qquad \leq \frac{\sqrt{a_1 n} C_\alpha}{a_0}(k \diamond u)(t). \end{aligned} \tag{3.23}$$

Using (3.2), Cauchy–Schwarz's and Young's inequalities, we obtain

$$\begin{aligned} &-\int_\Omega u_t(t) \int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\ &\qquad \leq \delta_3 \|u_t(t)\|_2^2 + \frac{\alpha^2 [C_\alpha + k_1] C_*^2}{2\delta_3 a_0}(k \diamond u)(t) \end{aligned} \tag{3.24}$$

for $\delta_3 > 0$ and

$$\begin{aligned} &\big\langle F(u_t(t)), \int_0^t g(t-s)(u(t)-u(s))\,ds\big\rangle_{\Gamma_1} \\ &\qquad \leq \delta_4 \int_{\Gamma_1} F^2(u_t(t))d\Gamma + \frac{\bar{C}_*^2 C_\alpha}{4\delta_4 a_0}(k \diamond u)(t) \end{aligned} \tag{3.25}$$

for $\delta_4 > 0$. Inserting (3.20)–(3.25) into (3.19), we obtain

$$\begin{aligned} K_4'(t) \leq &-\left(\int_0^t g(s)\,ds - \delta_4\right)\|u_t(t)\|_2^2 + \delta \frac{a_1}{a_0} a(u(t),u(t)) + \frac{2k_1 C_*^2}{4\delta_3 a_0}(k \diamond u)(t) \\ &+ \frac{C_\alpha}{4a_0}\left[\frac{n}{\delta} + \frac{1}{\delta_1} + \frac{\bar{C}_*^2}{\delta_2} + 4\sqrt{a_1 n} + \frac{2\alpha^2 C_*^2}{\delta_3} + \frac{\bar{C}_*^2}{\delta_4}\right](k \diamond u)(t) \\ &+ \delta_1 C_*^2 \|\sigma\|^2 \|\nabla \varphi_t(t)\|_2^2 + \delta_4 \int_{\Gamma_1} F^2(u_t(t))\,d\Gamma \\ &+ \delta_2 \frac{\|h\|_\infty \|f\|_\infty}{f_0^2}\|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2. \end{aligned}$$

If we choose $\delta_1 = 3$ and $\delta_2 = \delta_4 = 3\bar{C}_*^2$, then we get (3.18). □

Next, we use the functional

$$K_5(t) = \int_0^t p(t-s) a(u(s),u(s))\,ds,$$

where $p(t) = \int_t^\infty g(s)\,ds$.

Lemma 3.6. *Assume that (2.3) and (2.6) hold. In view of (2.13)–(2.22), the functional K_5 satisfies the estimate*

$$K_5'(t) \leq -\frac{1}{2}(g \diamond u)(t) + \rho(1-\ell)a(u(t),u(t)), \tag{3.26}$$

where $\rho = \frac{2a_1 n}{a_0^2} + 1$.

Proof. By Young's inequality and the fact that $p'(t) = -g(t)$, we see that

$$\begin{aligned} K_5'(t) &= p(0)a(u(t),u(t)) - \int_0^t g(t-s)a(u(s),u(s))\,ds \\ &= -(g \diamond u)(t) + 2\int_0^t g(t-s)a(u(t)-u(s),u(t))\,ds + p(t)a(u(t),u(t)). \end{aligned}$$

But

$$\begin{aligned} \int_0^t g(t-s)a(u(t)-u(s),u(t))\,ds \leq{}& \frac{a_1 n}{a_0^2}(1-\ell)a(u(t),u(t)) \\ &+ \frac{1}{4(1-\ell)}\int_0^t g(s)\,ds\,(g \diamond u)(t). \end{aligned}$$

Then, as $p(t) \leq p(0) = (1-\ell)$ and $\int_0^t g(s)\,ds \leq (1-\ell)$, we get (3.26). $\square$

We define a Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t) = NE(t) + N_1K_1(t) + N_2K_2(t) + N_3K_3(t) + N_4K_4(t), \tag{3.27}$$

where N_i, $i = 1,2,3,4$, are positive constants to be fixed later. We choose N so large that $\mathcal{L}$ is equivalent to E.

Lemma 3.7. *For each $i = 1,2,3,4$, N_i is large enough while N is so large that the functional $\mathcal{L}$ defined by (3.27) satisfies*

$$\begin{aligned} \mathcal{L}'(t) \leq{}& -\|u_t(t)\|_2^2 - (\rho+1)(1-\ell)a(u(t),u(t)) + \frac{1}{4}(g \diamond u)(t) - \frac{N_2}{2}b(\varphi(t),\varphi(t)) \\ &- \frac{Nc_1}{2C_*^2}\|\varphi_t(t)\|_2^2 + (C_2N_1 + C_{13}N_4)\int_{\Gamma_1} F^2(u_t(t))\,d\Gamma \\ &- N_1\|h^{1/2}m^{1/2}z(t)\|_{2,\Gamma_1}^2 - N_3\int_Q s\mu(s)\omega^2(x,p,s,t)\,dp\,ds\,dx \end{aligned} \tag{3.28}$$

for all $t \geq 0$.

Proof. Let

$$g_1 = \int_0^{t_0} g(s)\,ds > 0.$$

By combining (3.28), (3.6), (3.13), (3.17)-(3.18), taking $\delta = \ell a_0/(4a_1N_4)$ and $N_2 = \ell a_0/(2C_*^2)$, we obtain

$$\mathcal{L}'(t) \leq -\left(g_1N_4 - \frac{\ell a_0}{4a_1} - N_1\right)\|u_t(t)\|_2^2 - \left(\frac{\ell}{2}N_1 - \frac{\ell}{2}\right)a(u(t),u(t))$$

$$
\begin{aligned}
&- \frac{\ell a_0}{4C_*^2} b(\varphi(t), \varphi(t)) - \left(\frac{N}{2} - \frac{C_\alpha}{4a_0} (C_1 N_1 + C_9 N_4) - \frac{C_{10} N_4}{4a_0} \right) (k \diamond u)(t) \\
&- N_3 \int_Q s\mu(s) \omega^2(x, p, s, t) \, dp \, ds \, dx + (C_2 N_1 + C_{13} N_4) \int_{\Gamma_1} F^2(u_t(t)) \, d\Gamma \\
&- (N - C_4 N_1 - C_{12} N_4) \|h^{1/2} f^{1/2} z_t(t)\|_{2,\Gamma_1}^2 - \frac{N c_1}{2C_*^2} \|\varphi_t(t)\|_2^2 \\
&- \left(N_3 - \frac{C_6 \ell a_0}{2C_*^2} \right) \int_\Omega \int_0^{+\infty} \mu(s) \omega^2(x, 1, s, t) \, ds \, dx - N_1 \|h^{1/2} m^{1/2} z(t)\|_{2,\Gamma_1}^2 \\
&- \left(\frac{c_1 N}{2} - C_3 N_1 - \frac{C_5 l a_0}{2C_*^2} - C_7 N_3 - C_{11} N_4 \right) \|\nabla \varphi_t(t)\|_2^2 + \frac{\alpha N}{2} (g \diamond u)(t).
\end{aligned}
$$

At this point, we choose N_1 large enough such that

$$
\frac{\ell}{2} N_1 - \frac{\ell}{2} > (\rho + 1)(1 - \ell),
$$

and therefore N_4 large enough such that

$$
g_1 N_4 - \frac{\ell a_0}{4a_1} - N_1 > 1.
$$

By choosing $N_3 > \frac{C_6 \ell a_0}{2C_*^2}$, we are to choose N so large that $N > N_0$, where

$$
N_0 = \max \left\{ \frac{C_{10} N_4}{2a_0}, C_4 N_1 + C_{12} N_4, \frac{2}{c_1} \left(C_3 N_1 + \frac{C_5 \ell a_0}{2C_*^2} + C_7 N_3 + C_{11} N_4 \right) \right\}.
$$

As

$$
\frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s),
$$

it is easy to show, using the Lebesgue dominated convergence theorem, that

$$
\alpha C_\alpha = \int_0^\infty \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} \, ds \to 0 \quad \text{as } \alpha \to 0.
$$

Hence, there is $0 < \alpha_0 < 1$ such that if $\alpha < \alpha_0$, then

$$
\alpha C_\alpha < \frac{1}{4(C_1 N_1 + C_9 N_4)} \quad \text{and} \quad \alpha = \frac{1}{2N} < \alpha_0,
$$

which means

$$
\frac{N}{2} - \frac{C_\alpha}{4a_0} [C_1 N_1 + C_9 N_4] - \frac{C_{10} N_4}{4a_0} > 0.
$$

Thus (3.28) is proven. □

4. General decay

In this section, we give an optimal and general decay rates of the energy. These theorems are consequently divided according to the nature of the function F_0 defined on (**A**$_3$). In Subsection 4.1, we give the general decay theorem when F_0 is linear. In the case of F_0 being nonlinear, the general stability result is given in Subsection 4.2.

4.1. First general theorem

Theorem 4.1. *Assume that* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold and* F_0 *is linear. Then there exist two constants* $k_1, k_2 > 0$ *such that for all* $t \geq t_0$,

$$E(t) \leq k_1 \mathcal{H}_1^{-1}\left(k_2 \int_{t_0}^{t} \xi(s)\, ds\right), \tag{4.1}$$

where $\mathcal{H}_1(t) = \int_t^r \frac{ds}{s\mathcal{H}'(s)}$, *and* $\mathcal{H}_1$ *is strictly decreasing and convex on* $(0,\ r]$ *with* $\lim_{t\to 0} \mathcal{H}_1(t) = +\infty$.

Proof. We start using (2.11) and (3.2) to obtain

$$\int_0^{t_0} g(s)a(u(t) - u(t-s), u(t) - u(t-s))\, ds \leq -\frac{1}{\zeta_1}(g' \diamond u)(t) \leq -c_2 E'(t), \tag{4.2}$$

where c_2 is a positive constant. Inserting (4.2) into (3.28), for all $t \geq t_0$, we get

$$\begin{aligned} \mathcal{F}'(t) \leq{}& -\beta_1 E(t) + \beta_2 \int_{t_0}^{t} g(s)a(u(t) - u(t-s), u(t) - u(t-s))\, ds \\ &+ \int_{\Gamma_1} \beta_3 F^2(u_t(t))\, d\Gamma, \end{aligned} \tag{4.3}$$

where β_i, $i = 1, 2, 3$, are positive constants and $\mathcal{F} = (\mathcal{L} + c_2 E) \sim E$.

Now we study the stability under a suitable assumption of $\mathcal{H}$.

Case 1: $\mathcal{H}$ *is linear.* Multiplying (4.3) by $\xi(t)$, using (2.7), and (3.2) for some t_0 small enough, we have

$$\begin{aligned} \xi(t)\mathcal{F}'(t) &\leq -\beta_1 \xi(t) E(t) + \beta_2 \int_{t_0}^{t} \xi(s) g(s) a(u(t) - u(t-s), u(t) - u(t-s))\, ds \\ &\quad + \beta_3 \xi(t) \int_{\Gamma_1} F^2(u_t(t))\, d\Gamma \\ &\leq -\beta_1 \xi(t) E(t) - \beta_2 (g' \diamond u)(t) + \beta_3 c_2' \xi(t) \big\langle F(u_t(t)), u_t(t) \big\rangle_{\Gamma_1} \\ &\leq -\beta_1 \xi(t) E(t) - \beta_4 E'(t), \end{aligned}$$

where $\beta_4 = (\beta_2 - \beta_3 c_2' \xi(0))$. Since ξ is a positive nonincreasing function, then

$$(\xi\mathcal{F} + \beta_4 E)'(t) \leq -\beta_1 \xi(t) E(t). \tag{4.4}$$

Using the fact that $(\xi\mathcal{F} + \beta_4 E) \sim E$, we conclude, for all $t \geq t_0$, that

$$E(t) \leq k_1 e^{-k_2 \int_{t_0}^{t} \xi(s)\, ds}.$$

Case 2: $\mathcal{H}$ *is nonlinear.* First, we consider

$$\mathcal{L}_1(t) = \mathcal{L}(t) + K_5(t)$$

to be nonnegative, and it follows from (3.2), (3.26), and (3.28) that

$$\mathcal{L}_1'(t) \leq -c_2 E(t) + \gamma_2 c_2' \big\langle F(u_t(t)), u_t(t) \big\rangle_{\Gamma_1} \leq -c_2 E(t) - \gamma_2 c_2' E'(t),$$

where c_2 is a positive constant. Therefore, for all $t \geq t_0$,

$$c_2 \int_{t_0}^{t} E(s)\,ds \leq \mathcal{L}_2(t_0) - \mathcal{L}_2(t) \leq \mathcal{L}_2(t_0) < \infty,$$

where $\mathcal{L}_2 = (\mathcal{L}_1 + \gamma_2 c_2' E) \sim E$.

Next, we define the function

$$I(t) = c_3 \int_{t_0}^{t} a(u(t) - u(t-s), u(t) - u(t-s))\,ds.$$

Note that for $c_3 > 0$ and for all $t \geq t_0$, $I(t) > 0$. Otherwise, an exponential decay is concluded. After that, for a constant $C > 0$, for all $t \geq t_0$, we have

$$\begin{aligned}\int_{t_0}^{t} a(u(t) - u(t-s), u(t) - u(t-s))\,ds &\leq 2C \int_{t_0}^{t} \left(\|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2\right) ds \\ &\leq \frac{8C}{a_0(1-\ell)} \int_{t_0}^{t} E(t)\,ds < \infty,\end{aligned}$$

and therefore for a constant $0 < c_3 < 1$ chosen so that

$$0 < I(t) < 1. \tag{4.5}$$

We also define the functional

$$\lambda(t) = -\int_{t_0}^{t} g'(s) a(u(t) - u(t-s), u(t) - u(t-s))\,ds,$$

for each t_0 small enough. By using (3.2), we observe that for a constant $c_4 > 0$,

$$\lambda(t) \leq -c_4 E'(t). \tag{4.6}$$

Since $\mathcal{H}$ is strictly convex on $(0,\ r]$ and $\mathcal{H}(0) = 0$, then

$$H(\vartheta x) \leq \vartheta H(x), \quad \vartheta \in [0,\ 1],\ x \in (0,\ r]. \tag{4.7}$$

From the hypothesis ($\mathbf{A}_2$), the using of (4.5), (4.7) and Jensen's inequality leads to

$$\begin{aligned}\lambda(t) &= \frac{1}{c_3 I(t)} \int_{t_0}^{t} I(t)[-g'(s)] c_3 a(u(t) - u(t-s), u(t) - u(t-s))\,ds \\ &\geq \frac{1}{c_3 I(t)} \int_{t_0}^{t} I(t)\xi(s)\mathcal{H}(g(s)) c_3 a(u(t) - u(t-s), u(t) - u(t-s))\,ds \\ &\geq \frac{\xi(t)}{c_3 I(t)} \int_{t_0}^{t} \mathcal{H}(I(t)g(s)) c_3 a(u(t) - u(t-s), u(t) - u(t-s))\,ds \\ &\geq \frac{\xi(t)}{c_3} \overline{\mathcal{H}} \left(c_3 \int_{t_0}^{t} g(s) a(u(t) - u(t-s), u(t) - u(t-s))\,ds \right),\end{aligned} \tag{4.8}$$

where $\overline{\mathcal{H}}$ is an extension of $\mathcal{H}$ such that $\overline{\mathcal{H}}$ is a strictly increasing and strictly convex $\mathcal{C}^2$ function on $\mathbb{R}_+$. This implies that

$$\int_{t_0}^{t} g(s)a(u(t)-u(t-s),u(t)-u(t-s))\,ds \leq \frac{1}{c_3}\overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right).$$

Thus, (4.3) becomes

$$\mathcal{F}_1'(t) \leq -\beta_1 E(t) + \frac{\beta_2}{c_3}\overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right), \tag{4.9}$$

where $\mathcal{F}_1 = (\mathcal{F} - \beta_3 c_2'\xi(0)E) \sim E$.

Now, for $\varepsilon_0 > 0$ and $\beta_5 > 0$, we define the functional

$$\mathcal{F}_2(t) = \overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) + \beta_5 E(t).$$

Using (4.9) and the fact that $E' < 0$, $\overline{\mathcal{H}}' > 0$, $\overline{\mathcal{H}}'' > 0$, we conclude that $\mathcal{F}_2 \sim E$, and

$$\begin{aligned}\mathcal{F}_2'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}\overline{\mathcal{H}}''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1(t) + \overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\mathcal{F}_1'(t) + \beta_5 E'(t)\\ &\leq -\beta_1 E(t)\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\beta_2}{c_3}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right) + \beta_5 E'(t).\end{aligned} \tag{4.10}$$

On the other hand, dew to the argument given in [3, pages 61-64], we have

$$\overline{\mathcal{H}}^*(s) = s(\overline{\mathcal{H}}')^{-1}(s) - \overline{\mathcal{H}}[(\overline{\mathcal{H}}')^{-1}(s)] \leq s(\overline{\mathcal{H}}')^{-1}(s) \quad \text{for all } s > 0, \tag{4.11}$$

where $\overline{\mathcal{H}}^*$ is the convex conjugate of H such that

$$\overline{\mathcal{H}}^*(s) = \sup_{t\in\mathbb{R}_+}\{st - \overline{\mathcal{H}}(t)\},$$

and $\overline{\mathcal{H}}^*$ satisfies the following Young's inequality:

$$AB \leq \overline{\mathcal{H}}^*(A) + \overline{\mathcal{H}}(B). \tag{4.12}$$

In view of (4.11) and (4.12) with $A = \overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$ and $B = \overline{\mathcal{H}}^{-1}\left(\frac{c_3\lambda(t)}{\xi(t)}\right)$, (4.10) gives

$$\mathcal{F}_2'(t) \leq -(\beta_1 E(0) - \varepsilon_0)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \beta_2\frac{\lambda(t)}{\xi(t)} + \beta_5 E'(t).$$

So, multiplying by $\xi(t)$, using (4.6) and the fact that, as

$$\varepsilon_0\frac{E(t)}{E(0)} < r, \quad \overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = \mathcal{H}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right),$$

we find that

$$\mathcal{F}_3'(t) \leq \xi(t)\mathcal{F}_2'(t) \leq -(\beta_1 E(0)-\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)-(\beta_2 c_4-\beta_5\xi(0))E'(t),$$

where $\mathcal{F}_3 = (\xi\mathcal{F}_2)$. Let us choose ε_0 and β_5 small enough such that

$$\beta_6 = \beta_1 E(0) - \varepsilon_0 > 0 \quad \text{and} \quad \beta_2 c_4 - \beta_5\xi(0) > 0.$$

Then, for some constants $\alpha_1, \alpha_2 > 0$,

$$\alpha_1\mathcal{F}_3(t) \leq E(t) \leq \alpha_2\mathcal{F}_3(t)$$

and

$$\mathcal{F}_3'(t) \leq -\beta_6\xi(t)\frac{E(t)}{E(0)}\overline{\mathcal{H}}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -\beta_6\xi(t)H_2\left(\frac{E(t)}{E(0)}\right), \tag{4.13}$$

where $\mathcal{H}_2(t) = t\mathcal{H}'(\varepsilon_0 t)$. Since $\mathcal{H}_2'(t) = \mathcal{H}'(\varepsilon_0 t) + \varepsilon_0 t\mathcal{H}''(\varepsilon_0 t)$, then, by using the strict convexity of $\mathcal{H}$ on $(0,\ r]$, we find that $\mathcal{H}_2, \mathcal{H}_2' > 0$ on $(0, 1]$. Let

$$R(t) = \frac{\alpha_1\mathcal{F}_3(t)}{E(0)}.$$

From the fact that $R \sim E$, equation (4.13) yields

$$R'(t) \leq -\beta_6\xi(t)\mathcal{H}_2(R(t)).$$

A simple integration over (t_0, t), taking into consideration that $\varepsilon_0 R(t_0) < r$, gives

$$\mathcal{H}_1(\varepsilon_0 R(t)) \geq \int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_0)} \frac{ds}{s\mathcal{H}'(s)} \geq \beta_6 \int_{t_0}^{t} \xi(s)\, ds.$$

Using the fact that $\mathcal{H}_1$ is a strictly decreasing function on $(0, r]$ and

$$\lim_{t\to 0}\mathcal{H}_1(t) = +\infty,$$

we get (4.1). □

4.2. Second general theorem

Theorem 4.2. *Assume that* **(A$_1$)**–**(A$_3$)** *hold and F_0 is nonlinear. There exist two constants $k_1, k_2 > 0$ such that for all $t \geq t_0$,*

$$E(t) \leq k_1 \mathrm{G}_1^{-1}\left(k_2\int_{t_0}^{t}\xi(s)\, ds\right) \tag{4.14}$$

if $\mathcal{H}$ is linear, where $\mathrm{G}_1(t) = \int_t^{r_1}\frac{ds}{s\mathrm{G}'(s)}$. Moreover, if $\mathcal{H}$ is nonlinear, then there exist other two constants $k_3, k_4 > 0$ such that for all $t > t_0$,

$$E(t) \leq k_3[t-t_0]\mathrm{W}_1^{-1}\left(\frac{k_4}{[t-t_0]\int_{t_0}^{t}\xi(s)\, ds}\right), \tag{4.15}$$

where $\mathrm{W}_1(s) = s\mathrm{W}'(\varepsilon_1 s)$ and $\mathrm{W} = \left(\overline{\mathrm{G}}^{-1} + \overline{\mathcal{H}}^{-1}\right)^{-1}$.

Proof. First, we assume that $\max\{r_1, F_0(r_1)\} < \varepsilon$; otherwise r_1 is small enough. Let $\varepsilon_1 = \min\{r_1, F_0(r_1)\}$. From $(\mathbf{A}_3)$, we have

$$\begin{aligned} F_0(|s|) \le |F(s)| \le F_0^{-1}(|s|) & \qquad \text{for all } |s| < \varepsilon_1 \\ c_1'|s| \le |F(s)| \le c_2'|s| & \qquad \text{for all } |s| \ge \varepsilon_1. \end{aligned}$$

Then, for all $|s| \le \varepsilon_1$,

$$\mathrm{G}(F^2(s)) = |F(s)|F_0(|F(s)|) \le sF(s),$$

which gives

$$F^2(s) \le \mathrm{G}^{-1}(sF(s)) \quad \text{for all } |s| \le \varepsilon_1. \tag{4.16}$$

The following partition was first introduced by Komornik [23]:

$$\Gamma_{11} = \{x \in \Gamma_1 \mid |u_t(t)| \ge \varepsilon_1\}, \quad \Gamma_{12} = \{x \in \Gamma_1 \mid |u_t(t)| \le \varepsilon_1\}. \tag{4.17}$$

Note that for a constant $c_5 > 0$,

$$J(t) = \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t(t) F(u_t(t))\, d\Gamma \le -c_5 E'(t). \tag{4.18}$$

Then, by using (3.2), (4.16), (4.17) and Jensen's inequality, we get

$$\int_{\Gamma_1} F^2(u_t(t))\, d\Gamma \le \mathrm{G}^{-1}(J(t)) - c_2' E'(t). \tag{4.19}$$

Case 1: *$\mathcal{H}$ is linear.* Multiplying (4.3) by $\xi(t)$ and using (4.19), we have

$$\mathcal{F}_4'(t) \le -\beta_1 \xi(t) E(t) + \beta_3 \xi(t) \mathrm{G}^{-1}(J(t)), \tag{4.20}$$

where $\mathcal{F}_4 = (\xi \mathcal{F} + (c_2' + \beta_4) E) \sim E$.

Now, for $0 < \varepsilon_1 < r_1$ and $\beta_7 > 0$, by using (4.20) and the fact that $E' \le 0$, $\mathrm{G}' > 0$, $\mathrm{G}'' > 0$ on $(0, r_1]$, we find that the functional $\mathcal{F}_5$, defined by

$$\mathcal{F}_5(t) = \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathcal{F}_4(t) + \beta_7 E(t),$$

satisfies

$$\alpha_3 \mathcal{F}_5(t) \le E(t) \le \alpha_4 \mathcal{F}_5(t)$$

for some $\alpha_3, \alpha_4 > 0$.

Using the convex conjugate of G in the sense of Young, as in (4.12), with $A = \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right)$ and $B = \mathrm{G}^{-1}(J(t))$, by using (4.18) and again the fact that $E' \le 0$, $\mathrm{G}' > 0$, $\mathrm{G}'' > 0$, we obtain

$$\begin{aligned} \mathcal{F}_5'(t) &= \varepsilon_1 \frac{E'(t)}{E(0)} \mathrm{G}''\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathcal{F}_4(t) + \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathcal{F}_4'(t) + \beta_7 E'(t) \\ &\le -\beta_1 \xi(t) E(t) \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) + \beta_3 \xi(t) \mathrm{G}'\left(\varepsilon_1 \frac{E(t)}{E(0)}\right) \mathrm{G}^{-1}(J(t)) + \beta_7 E'(t) \end{aligned}$$

$$\leq -(\beta_1 E(0) - \varepsilon_1)\xi(t)\frac{E(t)}{E(0)}\mathrm{G}'\left(\varepsilon_1\frac{E(t)}{E(0)}\right) + (\beta_7 - c_5)E'(t).$$

Consequently, with a suitable choice of ε_1 and β_7 we obtain

$$\mathcal{F}_5'(t) \leq -\beta_8\xi(t)\frac{E(t)}{E(0)}\mathrm{G}'\left(\varepsilon_1\frac{E(t)}{E(0)}\right) = -\beta_8\xi(t)\mathrm{G}_2\left(\frac{E(t)}{E(0)}\right)$$

for all $t \geq t_0$, where $\beta_8 = \beta_1 E(0) - \varepsilon_1 > 0$ and $\mathrm{G}_2(t) = t\mathrm{G}'(\varepsilon_1 t)$. Since $\mathrm{G}_2'(t) = \mathrm{G}'(\varepsilon_1 t) + \varepsilon_1 t\mathrm{G}''(\varepsilon_1 t)$, then, using the strict convexity of G on $(0,\ r_1]$, we find that $\mathrm{G}_2, \mathrm{G}_2' > 0$ on $(0, 1]$. Let

$$R_1(t) = \alpha_3\frac{\mathcal{F}_5(t)}{E(0)}.$$

Using the fact that $R_1 \sim E$, we have

$$R_1'(t) \leq -\beta_8\xi(t)\mathrm{G}_2(R_1(t)), \quad \text{for all} t \geq t_0.$$

This concludes (4.14).

Case 2: $\mathcal{H}$ *is nonlinear:* First, we define the functional

$$I_1(t) = \frac{c_6}{(t-t_0)}\int_{t_0}^{t} a(u(t) - u(t-s), u(t) - u(t-s))\, ds,$$

and choose $0 < c_6 < 1$ small enough such that

$$0 < I_1(t) < 1$$

for all $t > t_0$. In the same way as in (4.8), we obtain

$$\int_{t_0}^{t} g(s)a(u(t) - u(t-s), u(t) - u(t-s))\, ds \leq \frac{(t-t_0)}{c_6}\overline{\mathcal{H}}^{-1}\left(\frac{c_6\lambda(t)}{(t-t_0)\xi(t)}\right).$$

Hence, we can write (4.3) as follows:

$$\mathcal{F}'(t) \leq -\beta_1 E(t) + \frac{(t-t_0)}{c_6}\overline{\mathcal{H}}^{-1}\left(\frac{c_6\lambda(t)}{(t-t_0)\xi(t)}\right) + \mathrm{G}^{-1}\left(J(t)\right) - \beta_4 E'(t).$$

Since

$$\lim_{t\to\infty}\frac{c_6}{t-t_0} = 0,$$

there exists $t_1 \geq t_0$ such that $\frac{c_6}{t-t_0} < 1$ for all $t > t_1$. Combining this with the strictly increasing and strictly convex properties of $\overline{\mathrm{G}}$, using (4.7), for all $t \geq t_1$, we obtain

$$\mathcal{F}_6'(t) \leq -\beta_1 E(t) + \frac{(t-t_0)}{c_6}\overline{\mathcal{H}}^{-1}\left(\frac{c_6\lambda(t)}{(t-t_0)\xi(t)}\right) + \frac{(t-t_0)}{c_6}\overline{\mathrm{G}}^{-1}\left(\frac{c_6 J(t)}{(t-t_0)}\right),$$

where $\mathcal{F}_6 = (\mathcal{F} + \beta_4 E) \sim E$.

Let $r_0 = \min\{r, r_1\}$ and $\chi(t) = c_6 \max\left\{\frac{\lambda(t)}{(t-t_0)\xi(t)}, \frac{J(t)}{(t-t_0)}\right\}$. Thus,

$$\mathcal{F}_6'(t) \leq -\beta_1 E(t) + \frac{(t-t_0)}{c_6} \mathrm{W}^{-1}(\chi(t)). \tag{4.21}$$

Now, for $0 < \varepsilon_2 < r_0$, using (4.21) and the fact that $E' \leq 0$, $\mathrm{W}' > 0$, $\mathrm{W}'' > 0$ on $(0, r_0]$, we find that the functional $\mathcal{F}_7$, defined by

$$\mathcal{F}_7(t) = \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) \mathcal{F}_6(t),$$

satisfies

$$\alpha_5 \mathcal{F}_7(t) \leq E(t) \leq \alpha_6 \mathcal{F}_7(t)$$

for some $\alpha_5, \alpha_6 > 0$.

As above, using the convex conjugate of W, we get

$$\begin{aligned}
\mathcal{F}_7'(t) &= \left(\frac{-\varepsilon_2}{(t-t_0)^2} + \frac{\varepsilon_2}{(t-t_0)} \frac{E'(t)}{E(0)}\right) \mathrm{W}''\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) \mathcal{F}_6(t) \\
&\quad + \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) \mathcal{F}_6'(t) \\
&\leq -\beta_1 E(t) \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) + \frac{(t-t_0)}{c_6} \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) \mathrm{W}^{-1}(\chi(t)) \\
&\leq -(\beta_1 E(0) - \varepsilon_2) \frac{E(t)}{E(0)} \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) + \frac{(t-t_0)}{c_6} \chi(t).
\end{aligned} \tag{4.22}$$

From (4.6) and (4.18), we observe that for $c_7 > 0$,

$$\frac{(t-t_0)}{c_6} \xi(t) \chi(t) \leq -c_7 E'(t).$$

After multiplying (4.22) by $\xi(t)$, from the fact that $\varepsilon_2 \frac{E(t)}{E(0)} \leq r_0$, it follows that

$$\mathcal{F}_8'(t) \leq -(\beta_1 E(0) - \varepsilon_2) \xi(t) \frac{E(t)}{E(0)} \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right),$$

where $\mathcal{F}_8 = (\xi \mathcal{F}_7 + c_7 E) \sim E$. Let us choose ε_2 small enough such that

$$\beta_9 = \beta_1 E(0) - \varepsilon_2 > 0.$$

Therefore, for all $t \geq t_1$,

$$\beta_9 \xi(t) \frac{E(t)}{E(0)} \mathrm{W}'\left(\frac{\varepsilon_2}{(t-t_0)} \frac{E(t)}{E(0)}\right) \leq -\mathcal{F}_8'(t). \tag{4.23}$$

Integrating (4.23) and multiplying the result by $\frac{1}{(t-t_0)}$, by using the fact that $\mathrm{W}', \mathrm{W}'' > 0$ and taking into consideration the nonincreasing property of E, we deduce that for all $t \geq t_1$,

$$\beta_9 \mathrm{W}_1\left(\frac{1}{(t-t_0)} \frac{E(t)}{E(0)}\right) \int_{t_1}^{t} \xi(s)\, ds \leq \int_{t_1}^{t} \beta_9 \mathrm{W}_1\left(\frac{1}{(s-t_0)} \frac{E(s)}{E(0)}\right) \xi(s)\, ds \leq \frac{\mathcal{F}_8(t_1)}{(t-t_0)}.$$

Consequently, (4.15) is derived as follows:

$$E(t) \leq E(0)(t-t_0)\mathrm{W}_1^{-1}\left(\frac{\mathcal{F}_8(t_1)}{\beta_9(t-t_0)\int_{t_1}^{t}\xi(s)\,ds}\right). \qquad \square$$

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Результат загального згасання для зв'язаної системи термопружності третього типу з акустичними граничними умовами за наявності розподіленої затримки

Abdelaziz Limam, Yamna Boukhatem, and Benyattou Benabderrahmane

У статті вивчаються розв'язки загального згасання енергії для термопружної зв'язаної системи третього типу з розподіленою затримкою часу. Зв'язування відбувається завдяки акустичним граничним умовам. Наш результат одержано в класі загальності функції релаксації і тому ця робота суттєво покращує попередні результати в термопружності.

Ключові слова: ефект термопружності, акустичні граничні умови, в'язкопружне демпфірування, загальне згасання