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Some Properties of the Tsallis Relative Operator φ-Entropy

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In this paper, we introduce the notion of the Tsallis relative operator φ-entropy between two strictly positive operators and verify its properties such as joint convexity, joint subadditivity, and monotonicity. We also give the Shannon type operator inequality and its reverse one satisfied by the Tsallis relative operator φ-entropy.

Key words: perspective function, generalized perspective function, Tsallis relative operator entropy, Tsallis relative operator φ-entropy

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1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional (separable) Hilbert space. By $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})_{\text{sa}}$, and $\mathcal{B}(\mathcal{H})^{++}$ we denote the set of all bounded linear operators, the set of all self-adjoint operators, and the set of all strictly positive operators on $\mathcal{H}$, respectively. A continuous real function f on $[0,\infty)$ is said to be operator monotone (more precisely, operator monotone increasing) if $A \leq B$ implies $f(A) \leq f(B)$ for $A, B \in \mathcal{B}(\mathcal{H})_{\text{sa}}$. For a self-adjoint operator A, the value $f(A)$ is defined via the functional calculus as usual. The function f is called operator convex on an interval I if

$$f(cA_1 + (1-c)A_2) \leq cf(A_1) + (1-c)f(A_2)$$

for all $A_1, A_2 \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ whose spectra are contained in I, and $c \in [0,1]$ and the function f is operator concave if $-f$ is operator convex. A real valued continuous function f on an interval I is said to be operator subadditive on I whenever

$$f(A+B) \leq f(A) + f(B)$$

for all $A, B \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ whose spectra are contained in I and f is operator superadditive if $-f$ is operator subadditive. The function g of two variables is called jointly convex if

$$g(cA_1 + (1-c)A_2, cB_1 + (1-c)B_2) \leq cg(A_1, B_1) + (1-c)g(A_2, B_2)$$

for all $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ and $c \in [0,1]$, and the function g is jointly concave if $-g$ is jointly convex. The function g is called jointly subadditive whenever

$$g(A_1 + A_2, B_1 + B_2) \leq g(A_1, B_1) + g(A_2, B_2)$$

for all $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$, and the function g is jointly supperadditive if $-g$ is jointly subadditive.

Let f and h be two continuous functions defined on $[0, \infty)$ and let h be strictly positive in the sense that $h(A) \in \mathcal{B}(\mathcal{H})^{++}$ for $A \in \mathcal{B}(\mathcal{H})^{++}$. Effros introduced in [3] an operator version of perspectivity for commuting operators and proved that the perspective of an operator convex function is operator convex as a function of two variables. Following Effros' approach, we dropped the commutativity assumption and defined in [2] a fully noncommutative perspective of two variables (associated to f), by choosing an appropriate ordering, as follows:

$$P_f(A, B) := A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

We also defined the operator version of a fully noncommutative generalized perspective of two variables (associated to f and h) as follows:

$$P_{f\Delta h}(A, B) := h(A)^{1/2} f(h(A)^{-1/2} B h(A)^{-1/2}) h(A)^{1/2}$$

for $A \in \mathcal{B}(\mathcal{H})^{++}$ and $B \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$. In fact, $P_{f\Delta h}(A, B) = P_f(h(A), B)$. We remark that the noncommutative case should have a deep influence on quantum mechanics and quantum information theory. We then proved several striking matrix analogues of a classical result for operator convex functions. More precisely, we proved the necessary and sufficient conditions for joint convexity of a fully noncommutative perspective and generalized perspective function. We proved some characterizations of the operator perspective in [19].

The axiomatic theory for connections and means for pairs of positive operators was developed by Nishio and Ando [23]. Kubo and Ando [11] proved the existence of an affine order isomorphism between the class of connections and the class of positive operator monotone functions on $\mathbb{R}^+$. This isomorphism, $\sigma \leftrightarrow f$, is characterized by the relation $A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}$.

Definition 1.1. Let $A, B, C, \ldots$ be strictly positive operators. A binary operation σ defined on the set of strictly positive operators is called a connection if

(i) $A \leq C, B \leq D$ implies $A\sigma B \leq C\sigma D$,

(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$,

(iii) $A_k \downarrow A$ and $B_k \downarrow B$ imply $A_k \sigma B_k \downarrow A\sigma B$.

A connection with the normalization condition $I\sigma I = I$ is called an operator mean.

When f is a positive operator monotone function with $f(1) = 1$, the operator perspective P_f reduces to the operator mean σ with the representing function f. Yanagi et al. [22] defined the notion of the Tsallis relative operator entropy and gave its properties and the generalized Shannon inequalities (see also [16]). In this paper, we give the Shannon type operator inequality and its reverse one satisfied by the Tsallis relative operator φ-entropy and in particular by the Tsallis relative operator entropy. Moreover, we verify some properties of the Tsallis relative operator φ-entropy such as joint convexity, joint subadditivity, and monotonicity.

2. Subadditivity and superadditivity of non-commutative generalized perspectives

We refined Theorem 2.4 and Corollary 2.6 of [2] as follows. For more details and proofs we refer to [3, 15, 18].

Theorem 2.1. *Suppose that f and h are two continuous functions defined on $[0, \infty)$ and $h > 0$.*

(i) *If f is operator convex and h is operator concave with $f(0) \leq 0$, then the generalized perspective function $P_{f\Delta h}$ is jointly convex.*

(ii) *If f and h are operator concave with $f(0) \geq 0$, then the generalized perspective function $P_{f\Delta h}$ is jointly concave.*

The following two results are due to Kubo and Ando.

Theorem 2.2 ([11, 17])**.** *If $A \leq C$, $B \leq D$ and f is a positive operator monotone function on $[0, \infty)$, then*

$$P_f(A, B) \leq P_f(C, D).$$

Theorem 2.3 ([11])**.** *If f is a positive operator monotone function on $[0, \infty)$, then P_f is jointly superadditive.*

Theorem 2.4. *Suppose that f and h are two continuous functions defined on $[0, \infty)$ and $h > 0$. If f is a positive operator monotone function and h is operator superadditive, then $P_{f\Delta h}$ is jointly superadditive.*

Proof. An operator monotone function $f : (0, \infty) \to \mathbb{R}$ is automatically operator concave [9, Corollary 2.2]. Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. Define

$$L = \sum_{i=1}^{n} B_i, \quad R = \sum_{i=1}^{n} h(A_i).$$

Then we have

$$f(R^{-1/2} L R^{-1/2}) = f(\sum_{i=1}^{n} R^{-1/2} B_i R^{-1/2})$$

$$= f(\sum_{i=1}^{n} R^{-1/2}h(A_i)^{1/2}(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})h(A_i)^{1/2}R^{-1/2}).$$

Since

$$\sum_{i=1}^{n} R^{-1/2}h(A_i)^{1/2}h(A_i)^{1/2}R^{-1/2} = I$$

and f is operator concave, Theorem 2.1 of [7] implies

$$f(R^{-1/2}LR^{-1/2}) \geq \sum_{i=1}^{n} R^{-1/2}h(A_i)^{1/2}f(h(A_i)^{-1/2}B_ih(A_i)^{-1/2})h(A_i)^{1/2}R^{-1/2}.$$

Therefore,

$$P_f(R, L) \geq \sum_{i=1}^{n} P_{f\Delta h}(A_i, B_i).$$

Whence,

$$P_f\left(\sum_{i=1}^{n} h(A_i), \sum_{i=1}^{n} B_i\right) \geq \sum_{i=1}^{n} P_{f\Delta h}(A_i, B_i). \tag{2.1}$$

From the superadditivity of h and Theorem 2.2, we detect that

$$P_f\left(h(\sum_{i=1}^{n} A_i), \sum_{i=1}^{n} B_i\right) \geq P_f\left(\sum_{i=1}^{n} h(A_i), \sum_{i=1}^{n} B_i\right). \tag{2.2}$$

We note that the left-hand side of (2.2) is equal to $P_{f\Delta h}\Big(\sum_{i=1}^{n} A_i, \sum_{i=1}^{n} B_i\Big)$. So, in view of (2.1) and (2.2), we observe that $P_{f\Delta h}$ is jointly superadditive. □

In the following theorem, the case of an operator concave function is just the result of Kubo and Ando, see Theorem 2.3.

Theorem 2.5. *Suppose that f is a continuous function defined on $[0, \infty)$. If f is an operator convex (respectively, concave) function, then P_f is jointly subadditive (respectively, superadditive).*

Proof. Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. Define

$$L = \sum_{i=1}^{n} B_i \quad \text{and} \quad R = \sum_{i=1}^{n} A_i.$$

Since f is operator convex (respectively, concave), using the same approach as in the proof of Theorem 2.4, we see that

$$P_f(R, L) \leq (\geq) \sum_{i=1}^{n} P_f(A_i, B_i).$$

□

We now provide some non-trivial examples of operator subadditive (respectively, superadditive) functions. The following functions are non-trivial examples of operator subadditive functions (see [4, Remark 1]):

$$h(t) = \log(t^{-1} + 1),\ t > 0,$$
$$h(t) = t^{\alpha},\ \alpha \in [-1, 0),\ t > 0.$$

It is easy to see that the following functions on $[0, \infty)$ are also operator subadditive (see [21]):

(1) $h(t) = \alpha t + \beta$ $(\alpha \in \mathbb{R},\ \beta > 0)$;

(2) $h(t) = \dfrac{1}{1+t}$.

Moslehian et. al [12] proved that some operator convex functions under certain conditions on the sequence of self adjoint operators are operator superadditive. Indeed, by letting $A = 0$ in [12, Corollary 2.5], one may deduce the following result.

Remark 2.6. Let $\{B_1, \ldots, B_n\}$ be a sequence of strictly positive operators with $B_k \leq M \leq \sum_{i=1}^{n} B_i$ for every $1 \leq k \leq n$ and some $M > 0$. Then every operator convex function f on $[0, \infty)$ with $f(0) \leq 0$ has the operator superadditive property, i.e.,

$$f\left(\sum_{i=1}^{n} B_i\right) \geq \sum_{i=1}^{n} f(B_i). \tag{2.3}$$

Uchiyama et. al [12] proved that some operator convex (respectively, non-negative operator monotone) functions under certain conditions on the sequence of strictly positive operators are operator superadditive (respectively, subadditive).

Remark 2.7. Let $\{B_1, \ldots, B_n\}$ be a sequence of positive operators with $\sum_{i\neq j}^{n} B_i B_j \geq 0$. With this assumption Uchiyama et. al [24, Theorem 2.1] proved that every operator convex function f on $[0, \infty)$ with $f(0) \leq 0$ has the operator superadditive property (2.3) and every non-negative operator monotone function f on $[0, \infty)$ has the operator subadditive property, i.e.,

$$f\left(\sum_{i=1}^{n} B_i\right) \leq \sum_{i=1}^{n} f(B_i).$$

Substituting the operator convex function $t \log t$ on $[0, \infty)$ to Remark 2.7 deduces that if $\sum_{i\neq j}^{n} B_i B_j \geq 0$ for strictly positive operators B_i $(i = 1, 2, \ldots, n)$, then

$$\left(\sum_{i=1}^{n} B_i\right)\left(\log \sum_{i=1}^{n} B_i\right) \geq \sum_{i=1}^{n} B_i \log B_i.$$

The following corollary is a well-known result, see [6, Theorem 5.7], and Theorem 2.5 confirms it by considering $f(t) = t^p$, $p \in [0, 1]$.

Corollary 2.8. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ and $p \in [0, 1]$. Then*

$$\sum_{j=1}^{n} A_j \#_p B_j \leq \left(\sum_{j=1}^{n} A_j \right) \#_p \left(\sum_{j=1}^{n} B_j \right).$$

The notion of the operator (α, β)-geometric mean as a generalization of the notion of the operator α-geometric mean was introduced by Nikoufar et al. [13] for $\alpha, \beta \geq 0$ as follows:

$$A \#_{(\alpha,\beta)} B := A^{\beta/2} \left(A^{-\beta/2} B A^{-\beta/2} \right)^{\alpha} A^{\beta/2}.$$

The following interesting result indicates that the operator (α, β)-geometric mean is jointly superadditive under a certain condition.

Corollary 2.9. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ with $A_k \leq M \leq \sum_{i=1}^{n} A_i$ for every $1 \leq k \leq n$ and some $M > 0$. If $\alpha \in [0, 1]$, $\beta \in [1, 2]$, then*

$$\sum_{j=1}^{n} A_j \#_{(\alpha,\beta)} B_j \leq \left(\sum_{j=1}^{n} A_j \right) \#_{(\alpha,\beta)} \left(\sum_{j=1}^{n} B_j \right).$$

Proof. The function $h(t) = t^{\beta}$, $\beta \in [1, 2]$ is operator convex with $f(0) = 0$. So, by Remark 2.6, h satisfies the operator superadditive property for the sequence of strictly positive operators $\{A_1, A_2, \ldots, A_n\}$. Apply Theorem 2.4 for the operator concave function $f(t) = t^{\alpha}$, $\alpha \in [0, 1]$ and the operator superadditive function h to obtain the result. $\square$

Note that, by Remark 2.7, if the sequence of strictly positive operators $\{A_1, A_2, \ldots, A_n\}$ satisfies the condition $\sum_{i \neq j} A_i A_j > 0$, then Corollary 2.9 holds again.

3. Tsallis relative operator φ-entropy

The concepts of the relative operator entropy and the operator entropy play an important role in different subjects, such as statistical mechanics, information theory, dynamical systems, etc. The Tsallis relative operator entropy was introduced by Yanagi et al. [22] and defined by

$$T_{\lambda}(A|B) := \frac{A^{1/2}(A^{-1/2} B A^{-1/2})^{\lambda} A^{1/2} - A}{\lambda}$$

for $A, B \in \mathcal{B}(\mathcal{H})^{++}$ and $0 < \lambda \leq 1$. Some operator inequalities related to the Tsallis relative operator entropy were proved in [25].

The operator perspective corresponding to the operator monotone functions t^{λ} and $\frac{t^{\lambda}-1}{\lambda}$ for $0 < \lambda \leq 1$ are called the weighted geometric mean and the Tsallis relative operator entropy between A and B, respectively. Note that the weighted geometric mean is denoted by $A\#_\lambda B$ and it is the perspective of the function t^λ. The Tsallis relative operator entropy is denoted by $T_\lambda(A,B)$ and it is the perspective of the function $\frac{t^\lambda-1}{\lambda}$. Indeed,

$$A\#_\lambda B = P_{t^\lambda}(A,B) = A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2},$$
$$T_\lambda(A,B) = P_{\frac{t^\lambda-1}{\lambda}}(A,B) = \frac{A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2} - A}{\lambda}.$$

So, we consider the Tsallis relative operator entropy for $\lambda \in \mathbb{R}\backslash\{0\}$ and denote it again by $T_\lambda(A,B)$. It is a well-known monotonicity for $A\#_\lambda B$ and $T_\lambda(A,B)$.

Corollary 3.1 ([5, Proposition 2.1])**.** *Let $\lambda \in [0,1]$ and $A,B,C,D \in \mathcal{B}(\mathcal{H})^{++}$.*

(i) *If $A \le C$ and $B \le D$, then $A\#_\lambda B \le C\#_\lambda D$.*

(ii) *If $B \le D$, then $T_\lambda(A,B) \le T_\lambda(A,D)$.*

Monotonicity of the Tsallis relative operator entropy can be generalized as follows.

Corollary 3.2. *If $A \le C$, $B \le D$ and $\lambda \in (0,1]$, then*

$$T_\lambda(A,B) \le T_\lambda(C,D) + \frac{C-A}{\lambda}.$$

Proof. By using Corollary 3.1, we obtain

$$T_\lambda(A,B) = \frac{A\#_\lambda B - A}{\lambda} \le \frac{C\#_\lambda D - A}{\lambda} = T_\lambda(C,D) + \frac{C-A}{\lambda}. \qquad \square$$

Moreover, by virtue of Theorem 2.2, we infer the following result.

Corollary 3.3. *If $A \le C$ and $B \le D$, then*

$$T_\lambda(A, A+B) \le T_\lambda(C, C+D)$$

for $A,B \in \mathcal{B}(\mathcal{H})^{++}$ and $\lambda \in [-1,0)\cup(0,1]$.

Proof. Consider

$$f(t) = \frac{(t+1)^\lambda - 1}{\lambda}, \quad t \ge 0.$$

Then the function f is positive and operator monotone for $\lambda \in [-1,0)\cup(0,1]$. Since $A \le C$ and $B \le D$, it follows from Theorem 2.2 that $P_f(A,B) \le P_f(C,D)$. On the other hand,

$$\begin{aligned} P_f(A,B) &= A^{1/2}\frac{(A^{-1/2}BA^{-1/2}+1)^\lambda - 1}{\lambda}A^{1/2} \\ &= \frac{A^{1/2}(A^{-1/2}(B+A)A^{-1/2})^\lambda A^{1/2} - A}{\lambda} = T_\lambda(A, A+B). \end{aligned} \tag{3.1}$$

In the same way, $P_f(C,D) = T_\lambda(C, C+D)$ and so the result follows. $\square$

We know that the Tsallis relative operator entropy can be interpreted as the perspective of the function $f(t) = \frac{t^\lambda - 1}{\lambda}$, namely, $T_\lambda(A,B) = P_f(A,B)$. Thus, due to Theorem 2.5, we conclude the following theorem.

Theorem 3.4. *$T_\lambda(A,B)$ is jointly superadditive for $\lambda \in [-1,0) \cup (0,1]$ and jointly subadditive for $\lambda \in [1,2]$.*

We introduce the notion of the Tsallis relative operator φ-entropy between the operators $A, B \in \mathcal{B}(\mathcal{H})^{++}$ and establish its properties. Our results are generalizations of the superadditivity and the subadditivity of the Tsallis relative operator entropy [5].

Definition 3.5. For $A, B \in \mathcal{B}(\mathcal{H})^{++}$ and $\lambda \in \mathbb{R}\backslash\{0\}$,

$$T_\lambda^\varphi(A,B) = \frac{\varphi(A)^{1/2}(\varphi(A)^{-1/2}B\varphi(A)^{-1/2})^\lambda \varphi(A)^{1/2} - \varphi(A)}{\lambda}$$

is called the Tsallis relative operator φ-entropy between A and B, where $\varphi > 0$ is a function defined on $[0, \infty)$.

We would remark that $T_\lambda^\varphi(A,B) = T_\lambda(\varphi(A), B)$ for $\lambda \in \mathbb{R}\backslash\{0\}$. In particular, when φ is the identity operator and $0 < \lambda \leq 1$, the Tsallis relative operator φ-entropy is indeed the Tsallis relative operator entropy.

A crucial property of the Tsallis relative operator φ-entropy can be stated as follows in the framework of quantum information theory. The quantum relative entropies, no matter in which form (Umegaki, quasi, α-z, ...), can be considered as a measurement to distinguish two quantum states. If the relative entropy of ρ with respect to σ is small, then ρ and σ should be 'close'. In particular, if the relative entropy is 0, then ρ and σ must be the same, so that we can not tell one from the other. Corollary 3.3 indicates that the Tsallis relative operator φ-entropy evaluated on two quantum states ρ and σ are subjected to $\varphi(\rho) \leq \rho$ can not increase. Indeed,

$$T_\lambda^\varphi(\rho, \varphi(\rho) + \sigma) = T_\lambda(\varphi(\rho), \varphi(\rho) + \sigma) \leq T_\lambda(\rho, \rho + \sigma)$$

subject to $\varphi(\rho) \leq \rho$. One may note that $\varphi(t) = t^\beta$ for $\beta \in [-1,0) \cup (0,1]$ and $t > 1$ and $\varphi(t) = \log(t)$ for $t > 1$ are two concrete examples satisfying the above conditions.

We verify when the Tsallis relative operator φ-entropy is jointly convex (respectively, jointly concave) and when it is jointly subadditive (respectively, jointly superadditive).

Corollary 3.6. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. If the function φ is operator superadditive and $\lambda \in (0,1]$, then*

$$T_\lambda^\varphi\left(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i\right) \geq \sum_{i=1}^n T_\lambda^\varphi(A_i, B_i) - \frac{1}{\lambda}\left(\varphi\left(\sum_{i=1}^n A_i\right) - \sum_{i=1}^n \varphi(A_i)\right). \quad (3.2)$$

Moreover, the function φ is additive if and only if T_λ^φ is jointly superadditive.

Proof. In view of Corollary 3.2, we have

$$T_\lambda^\varphi\left(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i\right) = T_\lambda\left(\varphi\left(\sum_{i=1}^n A_i\right), \sum_{i=1}^n B_i\right)$$
$$\geq T_\lambda\left(\sum_{i=1}^n \varphi(A_i), \sum_{i=1}^n B_i\right) - \frac{1}{\lambda}\left(\varphi\left(\sum_{i=1}^n A_i\right) - \sum_{i=1}^n \varphi(A_i)\right).$$

Taking into account Theorem 3.4, we obtain

$$T_\lambda^\varphi\left(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i\right) \geq \sum_{i=1}^n T_\lambda(\varphi(A_i), B_i) - \frac{1}{\lambda}\left(\varphi\left(\sum_{i=1}^n A_i\right) - \sum_{i=1}^n \varphi(A_i)\right)$$
$$= \sum_{i=1}^n T_\lambda^\varphi(A_i, B_i) - \frac{1}{\lambda}\left(\varphi\left(\sum_{i=1}^n A_i\right) - \sum_{i=1}^n \varphi(A_i)\right).$$

Moreover, if the function φ is additive, then

$$\varphi\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \varphi(A_i),$$

and it follows from (3.2) that T_λ^φ is jointly superadditive. Conversely, let T_λ^φ be jointly superadditive. Then

$$T_\lambda^\varphi\left(A + B, \frac{1}{2}I + \frac{1}{2}I\right) \geq T_\lambda^\varphi\left(A, \frac{1}{2}I\right) + T_\lambda^\varphi\left(B, \frac{1}{2}I\right).$$

So, by simplifying the above inequality, one can deduce

$$\varphi(A+B)^{1-\lambda} - \varphi(A+B) \geq \frac{1}{2^\lambda}\varphi(A)^{1-\lambda} - \varphi(A) + \frac{1}{2^\lambda}\varphi(B)^{1-\lambda} - \varphi(B).$$

Let $\lambda \to 1^-$. This entails that

$$\varphi(A+B) \leq \varphi(A) + \varphi(B) - I,$$

and since $\varphi(A), \varphi(B) > 0$, we reach

$$\varphi(A+B) \leq \varphi(A) + \varphi(B).$$

Hence, φ is subadditive. By assumption, φ is superadditive, and thus φ is additive. □

Note that in Corollary 3.6 the additivity of φ implies $\varphi(0) = 0$ and the only additive function is the function of the form $\varphi(t) = ct$, $t \geq 0$ and $c > 0$, see [21, Theorem 1].

Corollary 3.7. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$. If the function φ is operator superadditive and $\lambda \in [-1, 0) \cup (0, 1]$, then*

$$T_\lambda^\varphi\left(\sum_{i=1}^n A_i, \varphi\left(\sum_{i=1}^n A_i\right) + \sum_{i=1}^n B_i\right) \geq \sum_{i=1}^n T_\lambda^\varphi(A_i, \varphi(A_i) + B_i).$$

Proof. In view of Corollary 3.3 and Theorem 3.4, one gets

$$\begin{aligned} T_\lambda^\varphi\left(\sum_{i=1}^n A_i, \varphi\left(\sum_{i=1}^n A_i\right) + \sum_{i=1}^n B_i\right) &= T_\lambda\left(\varphi\left(\sum_{i=1}^n A_i\right), \varphi\left(\sum_{i=1}^n A_i\right) + \sum_{i=1}^n B_i\right) \\ &\geq T_\lambda\left(\sum_{i=1}^n \varphi(A_i), \sum_{i=1}^n \varphi(A_i) + \sum_{i=1}^n B_i\right) \\ &\geq \sum_{i=1}^n T_\lambda\big(\varphi(A_i), \varphi(A_i) + B_i\big) \\ &= \sum_{i=1}^n T_\lambda^\varphi\big(A_i, \varphi(A_i) + B_i\big). \end{aligned}$$

□

The function $\varphi(t) = t^\beta$, $t \geq 0$ and $\beta \in [1,2]$, is an interesting example of an operator superadditive function (Remark 2.6) under a certain condition on the sequence of strictly positive operators. For the Tsallis relative operator (α, β)-entropy introduced in [14, 18] and in the sense of Corollary 3.6, 3.7, respectively, we have the following results. Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ with $A_k \leq M \leq \sum_{i=1}^n A_i$ for every $1 \leq k \leq n$ and some $M > 0$.

(i) Let $\alpha \in (0,1]$, $\beta \in [1,2]$. Then

$$T_{\alpha,\beta}\Big(\sum_{i=1}^n A_i, \sum_{i=1}^n B_i\Big) \geq \sum_{i=1}^n T_{\alpha,\beta}(A_i, B_i) - \frac{(\sum_{i=1}^n A_i)^\beta - \sum_{i=1}^n A_i^\beta}{\lambda}.$$

(ii) Let $\alpha \in [-1,0) \cup (0,1]$, $\beta \in [1,2]$. Then

$$T_{\alpha,\beta}\Big(\sum_{i=1}^n A_i, (\sum_{i=1}^n A_i)^\beta + \sum_{i=1}^n B_i\Big) \geq \sum_{i=1}^n T_{\alpha,\beta}(A_i, A_i^\beta + B_i).$$

We may note, by Remark 2.7, that if the sequence of strictly positive operators $\{A_1, A_2, \ldots, A_n\}$ satisfies the condition $\sum_{i \neq j} A_i A_j > 0$, then (i) and (ii) hold again.

Corollary 3.8. *Suppose that the function φ is operator concave.*

(i) *If $\lambda \in [1,2]$, then T_λ^φ is jointly convex.*

(ii) *If $\lambda \in [-1,0)$, then T_λ^φ is jointly concave.*

Proof. (i) Define $f(t) = \frac{t^\lambda - 1}{\lambda}$. Then we have $T_\lambda^\varphi(A,B) = P_{f\Delta\varphi}(A,B)$. Due to the operator convexity of f and $f(0) = -\frac{1}{\lambda} < 0$, the operator concavity of φ and Theorem 2.1 (i), we get the result.

(ii) According to the operator concavity of f and $f(0) = -\frac{1}{\lambda} > 0$, the operator concavity of φ, and Theorem 2.1 (ii), we reach the result. □

The special case $\varphi(t) = ct$ has an essential meaning for our discussion when $0 < c < 1$. Since one of the subjects was $\varphi(\rho) \le \rho$ in the framework of quantum information theory, clearly, the condition $c\rho \le \rho$ holds and one can use the Tsallis relative operator φ-entropy in this situation.

Corollary 3.9. *Suppose that $\varphi(t) = ct$, $c > 0$.*

(i) *If $\lambda \in [1, 2]$, then T_λ^φ is jointly subadditive.*

(ii) *If $\lambda \in [-1, 0) \cup (0, 1]$, then T_λ^φ is jointly superadditive.*

Corollary 3.10. *Suppose that $\varphi(t) = ct$, $c > 0$. If $\lambda \in (0, 1]$, then T_λ^φ is jointly concave.*

We give the Shannon type operator inequality and its reverse one satisfied by the Tsallis relative operator φ-entropy. In particular, when φ is the identity operator in Theorems 3.11 (ii) and 3.12, we obtain the Shannon type operator inequality and its reverse one satisfied by the Tsallis relative operator entropy [22, Theorem 1].

Theorem 3.11. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$.*

(i) *If $\lambda \in (0, 1] \cup [-1, 0)$ and φ is strictly operator superadditive, then*

$$\sum_{i=1}^n T_\lambda^\varphi(A_i, \varphi(A_i) + B_i) \le T_\lambda^\varphi(I, \varphi(I) + I).$$

The reverse inequality holds when φ is strictly operator subadditive.

(ii) *If $\lambda \in (0, 1] \cup [-1, 0)$ and $\varphi(t) = ct$, $c > 0$, then*

$$\sum_{i=1}^n T_\lambda^\varphi(A_i, B_i) \le T_\lambda^\varphi(I, I).$$

The reverse inequality holds when $\lambda \in [1, 2]$.

Proof. (i) From Corollary 3.3, Theorem 3.4, and strict superadditivity of φ, it follows that

$$\begin{aligned}
T_\lambda^\varphi(I, \varphi(I) + I) &= T_\lambda(\varphi(I), \varphi(I) + I) \\
&= T_\lambda\left(\varphi\left(\sum_{i=1}^n A_i\right), \varphi\left(\sum_{i=1}^n A_i\right) + \sum_{i=1}^n B_i\right) \\
&\ge T_\lambda\left(\sum_{i=1}^n \varphi(A_i), \sum_{i=1}^n \varphi(A_i) + \sum_{i=1}^n B_i\right) \\
&\ge \sum_{i=1}^n T_\lambda(\varphi(A_i), \varphi(A_i) + B_i) = \sum_{i=1}^n T_\lambda^\varphi(A_i, \varphi(A_i) + B_i).
\end{aligned}$$

(ii) By applying Theorem 3.4, we get

$$T_\lambda^\varphi(I,I) = T_\lambda(\varphi(I),I) = T_\lambda\left(\varphi\left(\sum_{i=1}^n A_i\right), \sum_{i=1}^n B_i\right) = T_\lambda\left(c\sum_{i=1}^n A_i, \sum_{i=1}^n B_i\right)$$
$$\geq \sum_{i=1}^n T_\lambda(cA_i,B_i) = \sum_{i=1}^n T_\lambda(\varphi(A_i),B_i) = \sum_{i=1}^n T_\lambda^\varphi(A_i,B_i). \qquad \square$$

Theorem 3.12. *Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathcal{H}$ and $\sum_{i=1}^n \varphi(A_i) = I$.*

(i) *If $\lambda \in [-1,0) \cup (0,1]$, then*

$$-\frac{1}{\lambda}\left(I - \left(\sum_{i=1}^n \varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{-\lambda}\right) \leq \sum_{i=1}^n T_\lambda^\varphi(A_i,B_i).$$

(ii) *If $\lambda \in [1,2]$, then*

$$\sum_{i=1}^n T_{-\lambda}^\varphi(A_i,B_i) \leq \frac{1}{\lambda}\left(I - \left(\sum_{i=1}^n \varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{\lambda}\right).$$

Proof. (i) Assume that $\lambda \in (0,1]$. Applying the operator convexity of the function $f(t) = t^{-\lambda}$ and [8, Theorem 2.1], we deduce

$$\begin{aligned}\left(\sum_{i=1}^n \varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{-\lambda} &= \left(\sum_{i=1}^n \varphi(A_i)^{1/2}(\varphi(A_i)^{1/2}B_i^{-1}\varphi(A_i)^{1/2})\varphi(A_i)^{1/2}\right)^{-\lambda}\\ &\leq \sum_{i=1}^n \varphi(A_i)^{1/2}\left(\varphi(A_i)^{1/2}B_i^{-1}\varphi(A_i)^{1/2}\right)^{-\lambda}\varphi(A_i)^{1/2}\\ &= \sum_{i=1}^n \varphi(A_i)^{1/2}\left(\varphi(A_i)^{-1/2}B_i\varphi(A_i)^{-1/2}\right)^{\lambda}\varphi(A_i)^{1/2}.\end{aligned}$$

Subtracting the identity operator I from both sides, letting $I = \sum_{i=1}^n \varphi(A_i)$ in the right-hand side of the above inequality and dividing by λ, we conclude the desired result.

Assume that $\lambda \in [-1,0)$. Using the operator concavity of the function $f(t) = t^{-\lambda}$ and [8, Theorem 2.1], we find that

$$\left(\sum_{i=1}^n \varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{-\lambda} \geq \sum_{i=1}^n \varphi(A_i)^{1/2}\left(\varphi(A_i)^{-1/2}B_i\varphi(A_i)^{-1/2}\right)^{\lambda}\varphi(A_i)^{1/2}.$$

Consequently, by subtracting the operator I from both sides and dividing by negative λ, we deduce the result.

(ii) Using the operator convexity of the function $f(t)=t^{\lambda}$ and [8, Theorem 2.1], we find that

$$\left(\sum_{i=1}^{n}\varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{\lambda}\leq\sum_{i=1}^{n}\varphi(A_i)^{1/2}\left(\varphi(A_i)^{-1/2}B_i\varphi(A_i)^{-1/2}\right)^{-\lambda}\varphi(A_i)^{1/2}.$$

Subtracting the operator I from both sides and dividing by negative $-\lambda$, we conclude the result. $\square$

Theorem 3.13. *Suppose the hypothesis of Theorem 3.12 is satisfied and $m\leq A_i,B_i\leq M$.*

(i) *If $\lambda\in[-1,0)\cup(0,1]$, then*

$$\sum_{i=1}^{n}T_{\lambda}^{\varphi}(A_i,B_i)\leq-\frac{1}{\lambda}\left(I-\beta(\lambda,\gamma)\left(\sum_{i=1}^{n}\varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{-\lambda}\right).$$

(ii) *If $\lambda\in[1,2]$, then*

$$\sum_{i=1}^{n}T_{-\lambda}^{\varphi}(A_i,B_i)\leq\frac{1}{\lambda}\left(I-\beta(\lambda,\gamma)\left(\sum_{i=1}^{n}\varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{\lambda}\right),$$

where

$$\beta(\lambda,\gamma)=\frac{\gamma-\gamma^{-\lambda}}{(1+\lambda)(\gamma-1)}\left\{\frac{\lambda(\gamma-\gamma^{-\lambda})}{(1+\lambda)(1-\gamma^{-\lambda})}\right\}^{\lambda}$$

is the generalized Kantorovich constant and $\gamma=\frac{M}{m}$.

Proof. (i) Since the function $f(t)=t^{-\lambda}$ is strictly convex for $\lambda\in(0,1]$, by [20, Theorem 8.3], one can deduce

$$\begin{aligned}
&\beta(\lambda,\gamma)\left(\sum_{i=1}^{n}\varphi(A_i)B_i^{-1}\varphi(A_i)\right)^{-\lambda}\\
&=\beta(\lambda,\gamma)\left(\sum_{i=1}^{n}\varphi(A_i)^{1/2}(\varphi(A_i)^{1/2}B_i^{-1}\varphi(A_i)^{1/2})\varphi(A_i)^{1/2}\right)^{-\lambda}\\
&\geq\sum_{i=1}^{n}\varphi(A_i)^{1/2}\left(\varphi(A_i)^{1/2}B_i^{-1}\varphi(A_i)^{1/2}\right)^{-\lambda}\varphi(A_i)^{1/2}\\
&=\sum_{i=1}^{n}\varphi(A_i)^{1/2}\left(\varphi(A_i)^{-1/2}B_i\varphi(A_i)^{-1/2}\right)^{\lambda}\varphi(A_i)^{1/2}. \qquad (3.3)
\end{aligned}$$

Subtracting I from both sides, letting $I=\sum_{i=1}^{n}\varphi(A_i)$ in the right-hand side of the above inequality and dividing by λ, we conclude the desired result.

For the case where $\lambda\in[-1,0)$, the function $f(t)=t^{-\lambda}$ is strictly concave, and thus the reverse inequality in (3.3) holds. By subtracting the operator I from both sides and dividing by negative λ, the result follows.

(ii) Using the convexity of the function $f(t) = t^\lambda$ and [20, Theorem 8.3], we find that

$$\begin{aligned}\beta(\lambda,\gamma)&\left(\sum_{i=1}^n \varphi(A_i)B_i^{-1}\varphi(A_i)\right)^\lambda \\ &\leq \sum_{i=1}^n \varphi(A_i)^{1/2}\left(\varphi(A_i)^{-1/2}B_i\varphi(A_i)^{-1/2}\right)^{-\lambda}\varphi(A_i)^{1/2}.\end{aligned}$$

Subtracting the identity operator I from both sides, letting $I = \sum_{i=1}^n \varphi(A_i)$ in the right-hand side of the above inequality and dividing by negative $-\lambda$, we reach the result. □

Finally, we show the informational monotonicity of the Tsallis relative operator φ-entropy.

Corollary 3.14. *Let Φ be a unital positive linear map from the set of the bounded linear operators on a Hilbert space to itself. If φ is operator concave, then*

$$\Phi(T_\lambda^\varphi(A,B)) \leq T_\lambda^\varphi(\Phi(A),\Phi(B)) + \frac{\varphi(\Phi(A)) - \Phi(\varphi(A))}{\lambda}$$

for every $A, B \in \mathcal{B}(\mathcal{H})^{++}$ and $\lambda \in (0,1]$.

Proof. The Choi–Davis–Jensen inequality [1] shows that $\Phi(\varphi(A)) \leq \varphi(\Phi(A))$. By the informational monotonicity of the Tsallis relative operator entropy [5, Proposition 2.3] and Corollary 3.2, we get

$$\begin{aligned}\Phi(T_\lambda^\varphi(A,B)) &= \Phi(T_\lambda(\varphi(A),B)) \leq T_\lambda(\Phi(\varphi(A)),\Phi(B)) \\ &\leq T_\lambda(\varphi(\Phi(A)),\Phi(B)) + \frac{\varphi(\Phi(A)) - \Phi(\varphi(A))}{\lambda} \\ &= T_\lambda^\varphi(\Phi(A),\Phi(B)) + \frac{\varphi(\Phi(A)) - \Phi(\varphi(A))}{\lambda}.\end{aligned}$$ □

Corollary 3.15. *Let Φ be a unital positive linear map from the set of the bounded linear operators on a Hilbert space to itself. If φ is operator concave, then*

$$\Phi(T_\lambda^\varphi(A,\varphi(A)+B)) \leq T_\lambda^\varphi(\Phi(A),\varphi(\Phi(A))+\Phi(B)) \tag{3.4}$$

for every $A, B \in \mathcal{B}(\mathcal{H})^{++}$ and $\lambda \in [-1,0)\cup(0,1]$.

Proof. Similar to that of Corollary 3.14 and by using Corollary 3.3, one can deduce

$$\begin{aligned}\Phi(T_\lambda^\varphi(A,\varphi(A)+B)) &= \Phi(T_\lambda(\varphi(A),\varphi(A)+B)) \leq T_\lambda(\Phi(\varphi(A)),\Phi(\varphi(A))+\Phi(B)) \\ &\leq T_\lambda(\varphi(\Phi(A)),\varphi(\Phi(A))+\Phi(B)) = T_\lambda^\varphi(\Phi(A),\varphi(\Phi(A))+\Phi(B)).\end{aligned}$$ □

Remark 3.16. In the case where $\lambda \in (1,2]$, Corollaries 3.14 and 3.15 do not work properly because of the crucial role of Corollaries 3.2 and 3.3 in their proofs, respectively. By an example, we show that Corollary 3.2 does not hold for $\lambda = 2$. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 4 & 0 \\ 0 & 18 \end{bmatrix} = T_2(A,B) \geq T_2(C,D) + \frac{C-A}{2} = \begin{bmatrix} -15/16 & 0 \\ 0 & -25/54 \end{bmatrix}. \tag{3.5}$$

While for

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad B = C = I_2, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

the inequality is reversed in (3.5). One can find a similar example to show that Corollary 3.3 does not work for $\lambda \in (1,2]$.

The multiplicative domain M_Φ of a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined as

$$M_\Phi = \{X \in \mathcal{B}(\mathcal{H}) : \Phi(XY) = \Phi(X)\Phi(Y), \Phi(YX) = \Phi(Y)\Phi(X), Y \in \mathcal{B}(\mathcal{H})\}.$$

Note that when Φ is positive, M_Φ is closed under the adjoint operation, and the restriction of Φ onto M_Φ is a $*$-homomorphism. By [10, Lemma 2.5], for any unital positive linear map Φ and any normal element A in M_Φ, $\Phi(A)$ is also normal, and for any function φ on $\sigma(A) \cup \sigma(\Phi(A))$, $\varphi(\Phi(A)) = \Phi(\varphi(A))$. So, we have the following result.

Corollary 3.17. *Let Φ be a unital positive linear map from the set of the bounded linear operators on a Hilbert space to itself. Then*

$$\Phi(T_\lambda^\varphi(A,B)) \leq T_\lambda^\varphi(\Phi(A), \Phi(B))$$

for every $A \in M_\Phi \cap \mathcal{B}(\mathcal{H})^{++}$, $B \in \mathcal{B}(\mathcal{H})^{++}$ and $\lambda \in (0,1]$.

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Деякі властивості відносної операторної φ-ентропії Цалліса

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У цій роботі ми вводимо поняття відносної операторної φ-ентропії Цалліса між двома суворо позитивними операторами і перевіряємо її властивості, такі як спільна опуклість, спільна субадитивність та монотонність. Ми також наводимо операторну нерівність типу Шеннона та обернену нерівність, які задовольняє відносна операторна φ-ентропія Цалліса.

Ключові слова: перспективна функція, узагальнена перспективна функція, відносна операторна ентропія Цалліса, відносна операторна φ-ентропія Цалліса