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On the Construction and Integration of a Hierarchy for the Kaup System with a Self-Consistent Source in the Class of Periodic Functions

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In the paper, we derive a rich hierarchy for the Kaup system with a self-consistent source in the class of periodic functions. We discuss the complete integrability of the constructed systems that is based on the transformation to the spectral data of an associated quadratic pencil of Sturm–Liouville equations with periodic coefficients. In particular, Dubrovin-type equations are derived for the time-evolution of the spectral data corresponding to the solutions of any system in the hierarchy. Moreover, we pick a particular system of the hierarchy and demonstrate the benefits of integrability by proving global existence of solutions for the Cauchy problem and by providing an explicit solution.

Key words: the system of Kaup equations, hierarchy, self-consistent source, quadratic pencil of Sturm–Liouville equations, inverse spectral problem, trace formulas, periodical potential

Mathematical Subject Classification 2010: 39A23, 35Q51, 34K13, 34K29

1. Introduction

Our goal is to construct a hierarchy for the Kaup system with a self-consistent source in the class of periodic functions that can be integrated via the inverse spectral method. The Kaup system [20] is used to simulate the propagation of waves in shallow water. Different techniques were used to construct its solution [26, 31–33]. In [6], the authors presented the Kaup system with a self-consistent source and showed its integrability by using inverse spectral method of the quadratic pencil of Sturm–Liouville equations. The corresponding inverse spectral problems were solved in [1, 2, 11–15]. For a discussion on the relevance and applicability of the nonlinear evolution equations with self-consistent sources we refer the reader to [3, 5, 7–10, 17, 18, 21–23, 27–30, 34–36, 38, 41, 42]. In [19], a method for constructing a hierarchy for the Kaup system without a source was presented in the class of "rapidly decreasing" functions and their complete integrability was shown using the inverse scattering transform for the one-dimensional

Schrödinger equation with an energy-dependent potential. In recent years, hierarchies for soliton equations with self-consistent sources were studied for both physical and mathematical reasons in [4, 16, 24, 25, 37, 39, 40].

In this paper, we make a rather general approach for systems that can in principle be integrated by the inverse spectral transform method for the quadratic pencil of Sturm–Liouville equations with periodic coefficients. This approach leads to a rich hierarchy of systems (2.1)–(2.5) that is indexed by positive integers N and contains additional parameters. In particular, the classical Kaup system is contained for $N = 1$. Our main result, Theorem 5.1, provides for all systems in the hierarchy Dubrovin-type equations for the spectral data that correspond to their solutions. The advantages of having an integrable system are well known. It allows, for example, to prove global existence results for the Cauchy problem, a notoriously hard problem in the theory of nonlinear partial differential equations. Also, it makes the construction of large families of explicit solutions possible that are useful for modeling physical phenomena. And last not least, it is a viable tool for computing solutions numerically as Dubrovin-type equations are well-behaved and the inverse spectral transform is explicit due to trace formulas. We demonstrate part of these advantages by considering a particular system that was already constructed in [19, (5.2)] in the scattering case by an entirely different method and that we can reproduce in our hierarchy. We show global existence for the Cauchy problem for a large class of initial data using in particular trace formulas. Moreover, we present an explicit non-trivial solution that can be expressed in terms of the Jacobi elliptic functions by considering the two-gap case.

The paper is organized as follows. In Sections 2 and 3, we give the formulation of the Cauchy problem considered and provide some basic information about the spectral theory of the quadratic pencil of Sturm–Liouville equations with periodic coefficients. Sections 4 and 5 are devoted to constructing the hierarchy for the Kaup system with self-consistent sources and to deducing the equations that govern the evolution for the corresponding spectral data. In Section 6, some conclusions from our main result given in Section 5 are collected. We then turn to a specific system in the hierarchy with $N = 2$. Section 7 proves global existence of solutions for this system for a rather general class of initial data, where some arguments, in particular, the application of the Picard–Lindelöf theorem are moved to Appendix. Section 8 contains the construction of an explicit non-trivial solution that can be expressed in terms of Jacobi elliptic functions.

2. Formulation of the problem

For any integer $N \geq 1$ we consider the system of equations with a self-consistent source

$$p_t = \frac{1}{4}c'''_{N-1} - qc'_{N-1} - \frac{1}{2}q'c_{N-1} - 2pc'_N - p'c_N + \sum_{k=-\infty}^{\infty} \alpha_k(t)s(\pi, \lambda_k, t)(\psi^2(x, \lambda_k, t))_x, \tag{2.1}$$

$$q_t = \frac{1}{2}c_N''' - 2qc_N' - q'c_N$$
$$+ 2 \sum_{k=-\infty}^{\infty} \alpha_k(t)s(\pi, \lambda_k, t) \left\{ -p_x\psi^2(x, \lambda_k, t) + (\lambda_k - 2p)(\psi^2(x, \lambda_k, t))_x \right\} \quad (2.2)$$

in the class of real-valued π-periodic (with respect to the spatial variable x) functions $p = p(x, t)$ and $q = q(x, t)$ which satisfy the regularity assumptions

$$p(x, t),\ q(x, t) \in C_x^{2N+1}(t > 0) \cap C_t^1(t > 0) \cap C(t \geq 0)$$

and the initial conditions

$$p(x, t)|_{t=0} = p_0(x), \qquad q(x, t)|_{t=0} = q_0(x). \quad (2.3)$$

Here $p_0(x), q_0(x) \in C^{2N+1}(R)$ are given real-valued π-periodic functions which satisfy the condition given in the authors' previous work [37]. The functions $c_{N-1} = c_{N-1}(x, t)$ and $c_N = c_N(x, t)$ satisfy the recursion relations

$$c_0 = d_0(t),$$
$$c_1 = pc_0 + d_1(t),$$
$$c_k = -\frac{1}{4}\left[c_{k-2}'' - 2qc_{k-2} - 4pc_{k-1}\right]$$
$$+ \frac{1}{2}\int_0^x \left[2pc_{k-1}' + qc_{k-2}'\right] dx + d_k(t), \quad k = 2, 3, \ldots, N, \quad (2.4)$$

where the functions $d_k(t)$, $k = 0, 1, \ldots, N$, are parameters of the construction and we only require their continuity. Varying N, we obtain the hierarchy for the Kaup system with a self-consistent source (2.1), (2.2) that is advertised in the title of this paper. In system (2.1), (2.2), the functions $\alpha_k(t)$, $k \in Z$, can be chosen freely within the class of real-valued continuous functions having the uniform asymptotic decay $\alpha_k = O\left(1/k^2\right)$, $k \to \pm\infty$, thus providing uniform convergence of the series in equations (2.1), (2.2). We denote by $\psi_+(x, \lambda, t)$ and $\psi_-(x, \lambda, t)$ the Floquet solutions (normalized by the condition $\psi_\pm(0, \lambda, t) = 1$) of the quadratic pencil of Sturm–Liouville equations

$$T(\lambda, t)y \equiv -y'' + qy + 2\lambda py - \lambda^2 y = 0, \quad x \in R. \quad (2.5)$$

One can show that $\psi_+(x, \lambda_k, t) = \psi_-(x, \lambda_k, t) =: \psi(x, \lambda_k, t)$ with λ_k denoting the zeros of the function $\Delta^2(\lambda) - 4$, where $\Delta(\lambda) = c(\pi, \lambda, t) + s'(\pi, \lambda, t)$. As usual, we write $c(x, \lambda, t)$ and $s(x, \lambda, t)$ for the solutions of equation (2.5) satisfying the initial conditions $c(0, \lambda, t) = 1$, $c'(0, \lambda, t) = 0$ and $s(0, \lambda, t) = 0$, $s'(0, \lambda, t) = 1$, respectively. Using the expression for the Floquet solutions, one may derive the identity

$$s(\pi, \lambda_k, t)\psi^2(\tau, \lambda_k, t) = s(\pi, \lambda_k, t, \tau), \quad (2.6)$$

where $s(\pi, \lambda, t, \tau)$ is the solution of the quadratic pencil of Sturm–Liouville equations with coefficients $p(x + \tau, t)$ and $q(x + \tau, t)$ satisfying the initial conditions $s(0, \lambda, t, \tau) = 0$, $s'(0, \lambda, t, \tau) = 1$.

The aim of this work is to provide a procedure for constructing the solution $(p(x, t),\ q(x, t),\ \psi(x, \lambda_k, t))$ of problem (2.1)–(2.5) using the inverse spectral theory for the quadratic pencil of Sturm–Liouville equations (2.5).

3. Preliminaries

For the sake of completeness, in this section, we summarize some facts from the inverse spectral theory of the quadratic pencil of Sturm–Liouville equations (see [1, 2, 11–15]).

The boundaries λ_n of the spectrum of (2.5) and the spectral parameters ξ_n, σ_n of the quadratic pencil (2.5) are defined as in [37]. For the solution $s(x, \lambda, t)$, the equality

$$s(\pi,\, \lambda,\, t) = \pi \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\xi_k(t) - \lambda}{k} \tag{3.1}$$

is fulfilled.

If we consider $p(x+\tau, t)$ and $q(x+\tau, t)$ in (2.5) instead of $p(x,t)$ and $q(x,t)$, then the spectrum of the new problem is independent of the parameters τ and t, but the spectral parameters are. From the x-periodicity of p and q it follows that the spectral parameters are π-periodic in τ. In addition, the spectral parameters satisfy the system of Dubrovin differential equations

$$\frac{\partial \xi_n}{\partial \tau} = 2(-1)^{n-1}\sigma_n \operatorname{sign}(n)\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}\, h_n(\xi), \quad n \in Z \setminus \{0\}, \tag{3.2}$$

where

$$h_n(\xi) = h_n(\ldots, \xi_{-1}, \xi_1, \ldots) = \sqrt{(\xi_n - \lambda_{-1})(\xi_n - \lambda_0) \prod_{k\neq n,0} \frac{(\xi_n - \lambda_{2k-1})(\xi_n - \lambda_{2k})}{(\xi_n - \xi_k)^2}}.$$

The system of Dubrovin equations (3.2) and the following first and second trace formulas:

$$p(\tau, t = \frac{\lambda_{-1} + \lambda_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(\tau, t) \right), \tag{3.3}$$

$$\begin{aligned} q(\tau,t) + 2p^2(\tau,t) = {} & \frac{(\lambda_{-1})^2 + (\lambda_0)^2}{2} \\ & + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{(\lambda_{2k-1})^2 + (\lambda_{2k})^2}{2} - \xi_k^2(\tau, t) \right) \end{aligned} \tag{3.4}$$

provide the basis for solving the inverse problem.

4. Constructing a hierarchy for the Kaup system with a self-consistent source

In this section, we present our method for constructing a hierarchy for the Kaup system with a self-consistent source using the inverse spectral theory of the quadratic pencil of Sturm–Liouville equations with a periodic potential.

We consider the system

$$\begin{cases} p_t = H_1[p,q] + G_1(x,t), \\ q_t = H_2[p,q] + G_2(x,t) \end{cases} \tag{4.1}$$

and the initial condition

$$p(x,t)|_{t=0} = p_0(x), \quad q(x,t)|_{t=0} = q_0(x). \tag{4.2}$$

Here $p = p(x,t)$ and $q = q(x,t)$ are sufficiently smooth functions that are π-periodic in x. The terms $H_1[p,q]$, $H_2[p,q]$ depend polynomially on p and q. The aim is to find the functions $H_1[p,q]$ and $H_2[p,q]$ such that the Cauchy problem (4.1), (4.2) is completely integrable in the framework of the inverse spectral problem of the quadratic pencil of Sturm–Liouville equation (2.5) with periodic potentials $p(x,t)$ and $q(x,t)$.

Let $y_n(x,t)$ be the normalized eigenfunction of the Dirichlet problem for equation (2.5) corresponding to the eigenvalue $\xi_n = \xi_n(t)$. Differentiating the identity

$$-(y_n'', y_n) + (qy_n, y_n) + 2\xi_n(py_n, y_n) - \xi_n^2 = 0$$

with respect to t, as in [37], we can get

$$2\dot{\xi}_n \left(\xi_n - \int_0^\pi p y_n^2\,dx\right) = \int_0^\pi (q_t + 2\xi_n p_t) y_n^2\,dx. \tag{4.3}$$

Substituting the expression (4.1) into (4.3), we derive

$$\begin{aligned} 2\dot{\xi}_n \left(\xi_n - \int_0^\pi p y_n^2\,dx\right) &= \int_0^\pi \{H_2[p,q] + 2\xi_n H_1[p,q]\} y_n^2\,dx \\ &\quad + \int_0^\pi (G_2 + 2\xi_n G_1) y_n^2\,dx = J_1 + J_2, \end{aligned} \tag{4.4}$$

where

$$J_1 = \int_0^\pi \{H_2[p,q] + 2\xi_n H_1[p,q]\} y_n^2\,dx, \tag{4.5}$$

$$J_2 = \int_0^\pi (G_2 + 2\xi_n G_1) y_n^2\,dx. \tag{4.6}$$

We seek the antiderivative of the integrand in (4.5) as a quadratic form of y_n and y_n', that is,

$$\{a y_n^2 + b y_n y_n' + c {y_n'}^2\}' = (H_2[p,q] + 2\xi_n H_1[p,q]) y_n^2, \tag{4.7}$$

where $a = a(x,t,\xi_n)$, $b = b(x,t,\xi_n)$, and $c = c(x,t,\xi_n)$ are independent of y_n and y_n'. Using the equality

$$y_n'' = [q + 2p\xi_n - \xi_n^2] y_n$$

from (4.7), we obtain

$$y_n^2(a' + bq + 2bp\xi_n - b\xi_n^2) + y_n y_n'[2a + b' + 2cq + 4pc\xi_n - 2c\xi_n^2] + {y_n'}^2(b + c')$$

$$= (H_2[p,q] + 2\xi_n H_1[p,q])y_n^2. \tag{4.8}$$

Comparing the left- and the right-hand sides of (4.8), we find

$$b = -c', a = \frac{1}{2}c'' + c(\xi_n^2 - 2p\xi_n - q),$$
$$H_2[p,q] + 2\xi_n H_1[p,q] = \frac{1}{2}c''' + 2c'({\xi_n}^2 - 2p\xi_n - q) - c(2p'\xi_n + q'). \tag{4.9}$$

The functions $H_1[p,q]$ and $H_2[p,q]$ in (4.9) do not depend explicitly on ξ_n. Therefore we seek for $c(x,t,\xi_n)$ in the form

$$c(x,t,\xi_n) = \sum_{k=0}^{N} c_k(x,t)\xi_n^{N-k}. \tag{4.10}$$

Putting (4.10) into (4.9), we obtain

$$\begin{aligned} H_2[p,q] + 2\xi_n H_1[p,q] &= 2c_0'\xi_n^{N+2} + [2c_1' - 4pc_0' - 2p'c_0]\xi_n^{N+1} \\ &+ \sum_{k=0}^{N-2}\left[\frac{1}{2}c_k''' - 2qc_k' - q'c_k - 4pc_{k+1}' - 2p'c_{k+1} + 2c_{k+2}'\right]\xi_n^{N-k} \\ &+ \left[\frac{1}{2}c_{N-1}''' - 2qc_{N-1}' - q'c_{N-1} - 4pc_N' - 2p'c_N\right]\xi_n + \frac{1}{2}c_N''' - 2qc_N' - q'c_N. \end{aligned}$$

Comparing the left- and the right-hand sides of the last equality, we find

$$\begin{aligned} c_0' &= 0, \\ c_1' &= p'c_0, \\ c_{k+2}' &= -\frac{1}{4}[c_k''' - 2q'c_k - 4qc_k' - 4p'c_{k+1} - 8pc_{k+1}'], \quad k = 0,\ 1,\ \ldots,\ N-2, \\ H_1[p,q] &= \frac{1}{4}c_{N-1}''' - qc_{N-1}' - \frac{1}{2}q'c_{N-1} - 2pc_N' - p'c_N, \\ H_2[p,q] &= \frac{1}{2}c_N''' - 2qc_N' - q'c_N. \end{aligned}$$

Next we consider J_2 of (4.6). As we show in Section 5 below, the choices

$$G_1(x,t) = \sum_{k=-\infty}^{\infty} \alpha_k(t)s(\pi,\lambda_k,t)(\psi^2(x,\lambda_k,t))_x,$$
$$G_2(x,t) = 2\sum_{k=-\infty}^{\infty} \alpha_k(t)s(\pi,\lambda_k,t)\left\{-p_x\psi^2(x,\lambda_k,t) + (\lambda_k - 2p)(\psi^2(x,\lambda_k,t))_x\right\}$$

allow us to determine an explicit antiderivative for the expression $G_2 + 2\xi_n G_1$ that appears in the definition of J_2.

Now consider the system of equations

$$\begin{cases} p_t = \frac{1}{4}c_{N-1}''' - qc_{N-1}' - \frac{1}{2}q'c_{N-1} - 2pc_N' - p'c_N + G_1(x,t) \\ q_t = \frac{1}{2}c_N''' - 2qc_N' - q'c_N + G_2(x,t), \end{cases} \tag{4.11}$$

where the functions $c_k = c_k(x,t)$, $k = 0,1,\dots,N$, are expressed in terms of the functions $p = p(x,t)$ and $q = q(x,t)$ as follows: choose continuous functions $d_k = d_k(t)$, $k = 0,1,\dots,N$. Then the c_k are defined recursively by

$$\begin{aligned} c_0 &= d_0(t), \\ c_1 &= pc_0 + d_1(t), \\ c_k &= -\frac{1}{4}[c''_{k-2} - 2qc_{k-2} - 4pc_{k-1}] \\ &\quad + \frac{1}{2}\int_0^x [2pc'_{k-1} + qc'_{k-2}]\,dx + d_k(t), \quad k = 2,3,\dots,N. \end{aligned} \tag{4.12}$$

Varying N, we obtain the hierarchy for the Kaup system with a self-consistent source (4.11). Evaluating (4.12) up to $k = 3$ gives

$$\begin{aligned} c_0 &= d_0, \\ c_1 &= pd_0 + d_1, \\ c_2 &= \frac{1}{2}[q + 3p^2 - p^2(0,t)]d_0 + pd_1 + d_2, \\ c_3 &= -\frac{1}{4}[p'' - 6pq - 10p^3 + 2pp^2(0,t) + 2p(0,t)q(0,t) + 4p^3(0,t)]d_0 \\ &\quad + \frac{1}{2}[q + 3p^2 - p^2(0,t)]d_1 + pd_2 + d_3. \end{aligned}$$

Observe that the integral in the definition of c_k (4.12) has disappeared since one may find explicit antiderivatives. We conjecture that this is true for all c_k and we have validated this conjecture up to $k = 6$. This implies in particular that the assumption on the periodicity of c stated in Theorem 5.1 below is satisfied for all $N \leq 6$.

Clearly, (4.11), (4.12) yield a wealth of new equations that constitute a hierarchy for the Kaup system with a self-consistent source. Let us give two particular simple examples contained in this rich family of equations. We get the classical system of Kaup equations with a self-consistent source for $N = 1$, $d_0 = 2$, $d_1 = 0$:

$$\begin{cases} p_t = -6pp_x - q_x + G_1(x,t), \\ q_t = p_{xxx} - 4p_x q - 2pq_x + G_2(x,t), \end{cases}$$

and for $N = 2$, $d_0 = 4$, $d_1 = 0$, $d_2 = 2p^2(0,t)$:

$$\begin{cases} p_t = p_{xxx} - 6p_x q - 6pq_x - 30p^2 p_x + G_1(x,t), \\ q_t = q_{xxx} + 6pp_{xxx} + 18p_x p_{xx} - 6qq_x - 24pqp_x - 6p^2 q_x + G_2(x,t). \end{cases} \tag{4.13}$$

Note that for $p(x) = 0$ the last equation reduces to the Korteweg-de Vries equation with a self-consistent source. Moreover, in [19] a hierarchy of completely integrable systems was constructed for the class of rapidly decreasing functions by a different approach. Their special case (5.2) agrees with (4.13) without source terms if one identifies $Q(t,x) \equiv 2p(-t,x)$ and $U(t,x) \equiv q(-t,x)$.

5. Main result

The main result of the paper is stated in the theorem below.

Theorem 5.1. *Suppose $p(x,t)$, $q(x,t)$ and $\psi(x,\lambda_k,t)$ solve problem (2.1)–(2.5) and assume that the function c defined in (4.10) is π-periodic in x. Then the spectrum of the pencil (2.5) does not depend on t, and the spectral parameters $\xi_n(t)$, $\sigma_n(t)$, $n\in Z\setminus\{0\}$, satisfy an analogue of the system of Dubrovin equations*

$$\begin{aligned}\dot{\xi}_n(t) &= 2(-1)^n\sigma_n(t)\,\mathrm{sign}(n)\sqrt{(\xi_n(t)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(t))}\\ &\times\sqrt{(\xi_n(t)-\lambda_{-1})(\xi_n(t)-\lambda_0)\prod_{k\neq n,0}\frac{(\xi_n(t)-\lambda_{2k-1})(\xi_n(t)-\lambda_{2k})}{(\xi_n(t)-\xi_k(t))^2}}\\ &\times\left\{\sum_{k=0}^{N}c_k(0,t)\xi_n^{N-k}+\sum_{k=-\infty}^{\infty}\frac{\alpha_k(t)s(\pi,\lambda_k,t)}{\xi_n-\lambda_k}\right\}. \end{aligned}\tag{5.1}$$

The sign $\sigma_n(t)=\pm1$ *changes at each collision of the point $\xi_n(t)$ with the boundaries of its gap $[\lambda_{2n-1},\lambda_{2n}]$. Moreover, the following initial conditions are fulfilled:*

$$\xi_n(t)|_{t=0}=\xi_n^0,\quad \sigma_n(t)|_{t=0}=\sigma_n^0,\quad n\in Z\setminus\{0\},\tag{5.2}$$

where ξ_n^0, σ_n^0, $n\in Z\setminus\{0\}$ are the spectral parameters of the quadratic pencil of Sturm–Liouville equations corresponding to the coefficients $p_0(x)$ and $q_0(x)$.

Proof. Let $y_n(x,t)$ be the normalized eigenfunction of the Dirichlet problem for equation (2.5) corresponding to the eigenvalue $\xi_n=\xi_n(t)$. It is easy to see that

$$y_n(x,t)=\frac{1}{\gamma_n(t)}s(x,\xi_n(t),t),\tag{5.3}$$

where

$$\gamma_n^2(t)=\int_0^\pi s^2(x,\xi_n(t),t)\,dx.$$

Due to (4.10) and (4.12), we have

$$q_t+2\xi_np_t=\frac{1}{2}c'''+2c'(\xi_n^2-2p\xi_n-q)-c(2p'\xi_n+q')+G_2+2\xi_nG_1.\tag{5.4}$$

Substituting the expression (5.4) into (4.3), we derive the following equality:

$$\begin{aligned}2\dot{\xi}_n\left(\xi_n-\int_0^\pi py_n^2\,dx\right)&=\int_0^\pi\left\{\frac{1}{2}c'''+2c'(\xi_n^2-2p\xi_n-q)-c(2p'\xi_n+q')\right\}y_n^2\,dx\\ &+\int_0^\pi(G_2+2\xi_nG_1)y_n^2\,dx=I_1+J_2,\end{aligned}\tag{5.5}$$

where

$$I_1=\int_0^\pi\left\{\frac{1}{2}c'''+2c'(\xi_n^2-2p\xi_n-q)-c(2p'\xi_n+q')\right\}y_n^2\,dx,$$

and J_2 is given by (4.6). We now calculate I_1. Using the identity

$$qy_n = \xi_n^2 y_n + y_n'' - 2\xi_n p y_n,$$

it follows from (4.7), the discussion thereafter, and from the periodicity of $c(x,t,\xi_n)$ that

$$\begin{aligned} I_1 &= \left\{ \left[\frac{1}{2} c'' + c(\xi_n^2 - 2p\xi_n - q) \right] y_n^2 - c' y_n y_n' + c {y_n'}^2 \right\} \Big|_0^\pi \\ &= c(0,t,\xi_n)[{y_n'}^2(\pi,t) - {y_n'}^2(0,t)]. \end{aligned} \tag{5.6}$$

Now we calculate J_2:

$$\begin{aligned} J_2 &\equiv \int_0^\pi (G_2 + 2\xi_n G_1) y_n^2 \, dx \\ &= \sum_{k=-\infty}^{\infty} \alpha_k s(\pi, \lambda_k, t) \int_0^\pi \left\{ -2p_x y_n^2 \psi_k^2 + 2(\xi_n + \lambda_k - 2p) y_n^2 (\psi_k^2)_x \right\} dx, \end{aligned} \tag{5.7}$$

where $\psi_k = \psi(x, \lambda_k, t)$. It is easy to see that

$$\begin{aligned} -2 \int_0^\pi p_x y_n^2 \psi_k^2 dx + 2 \int_0^\pi (\xi_n + \lambda_k - 2p) y_n^2 (\psi_k^2)_x \, dx \\ = \frac{1}{\xi_n - \lambda_k} [{y_n'}^2(\pi,t) - {y_n'}^2(0,t)]. \end{aligned} \tag{5.8}$$

Substituting (5.8) into (5.7), we arrive at

$$J_2 = \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t) s(\pi, \lambda_k, t)}{\xi_n - \lambda_k} [{y_n'}^2(\pi,t) - {y_n'}^2(0,t)]. \tag{5.9}$$

Equation (5.5) together with relations (4.10), (5.6), and (5.9) yield

$$\begin{aligned} 2\dot{\xi}_n \left(\xi_n - \int_0^\pi p y_n^2 dx \right) &= \left\{ \sum_{k=0}^{N} c_k(0,t) \xi_n^{N-k} \right\} [{y_n'}^2(\pi,t) - {y_n'}^2(0,t)] \\ &+ \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t) s(\pi, \lambda_k, t)}{\xi_n - \lambda_k} [{y_n'}^2(\pi,t) - {y_n'}^2(0,t)]. \end{aligned} \tag{5.10}$$

Due to (5.3), the last equality can be written as

$$\begin{aligned} 2\dot{\xi}_n(t) \left(\xi_n(t) \gamma_n^2(t) - \int_0^\pi p s^2(x, \xi_n(t), t) \, dx \right) \\ = \left\{ \sum_{k=0}^{N} c_k(0,t) \xi_n^{N-k} \right\} [s'^2(\pi, \xi_n(t), t) - 1] \\ + \left\{ \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t) s(\pi, \lambda_k, t)}{\xi_n - \lambda_k} \right\} [s'^2(\pi, \xi_n(t), t) - 1]. \end{aligned}$$

Using the equality (see [15]):

$$2\xi_n(t)\gamma_n^2(t) - 2\int_0^\pi p(x,t)s^2(x,\xi_n(t),t)\,dx = s'(\pi,\xi_n(t),t)\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda},$$

we obtain

$$\begin{aligned}\dot{\xi}_n(t)\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda} &= \left\{\sum_{k=0}^{N} c_k(0,t)\xi_n^{N-k}\right\}\left(s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)}\right)\\ &+ \left\{\sum_{k=-\infty}^{\infty}\frac{\alpha_k(t)s(\pi,\lambda_k,t)}{\xi_n-\lambda_k}\right\}\left(s'(\pi,\xi_n(t),t) - \frac{1}{s'(\pi,\xi_n(t),t)}\right). \qquad (5.11)\end{aligned}$$

Now, we get

$$\begin{aligned}&\left(s'(\pi,\xi_n(t),t) - \frac{1}{s'(x,\xi_n(t),t)}\right)\left(\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}\right)^{-1}\\ &= \left(\sigma_n(t)\sqrt{\Delta^2(\xi_n(t))-4}\right)\left(\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}\right)^{-1}\\ &= 2(-1)^n\sigma_n(t)\,\mathrm{sign}(n)\sqrt{(\xi_n(t)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(t))}\\ &\times\sqrt{(\xi_n(t)-\lambda_{-1})(\xi_n(t)-\lambda_0)\prod_{k\neq n,0}\frac{(\xi_n(t)-\lambda_{2k-1})(\xi_n(t)-\lambda_{2k})}{(\xi_n(t)-\xi_k(t))^2}}. \qquad (5.12)\end{aligned}$$

Here we also used the equality

$$\mathrm{sign}\left\{-\frac{\pi}{n}\prod_{k\neq n,0}\frac{\xi_k(t)-\xi_n(t)}{k}\right\} = (-1)^n\,\mathrm{sign}(n).$$

From (5.11) and (5.12) we conclude (5.1).

We notice that if instead of Dirichlet boundary conditions we consider periodic or anti-periodic boundary conditions, then equation (5.10) remains valid and we can deduce $\dot{\lambda}_n(t) = 0$ for all $n \in Z$. Hence, the spectrum of problem (2.5) does not depend on the parameter t, and the theorem is proved. □

6. Remarks

Remark 6.1. Theorem 5.1 provides the method for solving problem (2.1)–(2.5). It is the following one.

(i) Solving the direct spectral problem for (2.5) with $p_0(x+\tau)$ and $q_0(x+\tau)$, the spectral data λ_n, $n \in Z$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n \in Z \setminus \{0\}$ are obtained.

(ii) Using the result of Theorem 5.1, we find the solution of the Cauchy problem $\xi_n(\tau,t)\,|_{t=0} = \xi_n^0(\tau)$, $\sigma_n(\tau,t)\,\big|_{t=0} = \sigma_n^0(\tau)$, $n \in Z\setminus\{0\}$ for the Dubrovin-type system (5.1).

(iii) Finally, by using trace formulas (3.3) and (3.4) we obtain the expressions for $p(\tau, t)$ and $q(\tau, t)$. After that the Floquet solutions $\psi(x, \lambda_k, t)$ of equation (2.5) can be determined.

It is worth noting that all remarks given in authors' previous work [37] retain valid for system (2.1)–(2.3).

7. Discussion of a special example for $N = 2$ in system (4.13)

In this section, we show that system (4.13) has a global spatially π-periodic solution for any initial data p_0, q_0 that are π-periodic, belong to $C^5(R)$, and satisfy the condition given in [37]. Recall that system (4.13) was derived from the general construction by setting $N = 2$, $d_0 = 4$, $d_1 = 0$, $d_2 = 2p^2(0, t)$.

For this data, the system of Dubrovin-type equations (5.1) for problem (2.5) with coefficients $q(x + \tau, t)$ and $p(x + \tau, t)$ takes the form

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^n \sigma_n \operatorname{sign}(n) \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \sqrt{(\xi_n - \lambda_{-1})(\xi_n - \lambda_0) \prod_{k \neq n,0} \frac{(\xi_n - \lambda_{2k-1})(\xi_n - \lambda_{2k})}{(\xi_n - \xi_k)^2}} \times \left\{ c_0(\tau, t)\xi_n^2 + c_1(\tau, t)\xi_n + c_2(\tau, t) + \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t) s(\pi, \lambda_k, t, \tau)}{\xi_n - \lambda_k} \right\} \tag{7.1}$$

and satisfies the initial condition

$$\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau), \quad n \in Z \setminus \{0\}, \tag{7.2}$$

where $\xi_n = \xi_n(\tau, t)$, $\sigma_n = \sigma_n(\tau, t)$.

Next we explain how $c_0(\tau, t), c_1(\tau, t), c_2(\tau, t)$ can be expressed through λ_k and $\xi_n(\tau, t)$. Due to (4.12), we have

$$c_0(\tau, t) = 4, \quad c_1(\tau, t) = 4p(\tau, t), \quad c_2(\tau, t) = 2[q(\tau, t) + 3p^2(\tau, t)]. \tag{7.3}$$

From [15] we take the following trace formulas for problem (2.5) with coefficients $q(x + \tau, t)$ and $p(x + \tau, t)$:

$$\Delta_m(\tau, t) = \frac{(\lambda_{-1})^m + (\lambda_0)^m}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\frac{(\lambda_{2k-1})^m + (\lambda_{2k})^m}{2} - \xi_k^m(\tau, t) \right), \quad m = 1, 2, \ldots, N. \tag{7.4}$$

Then $\Delta_m(\tau, t)$ satisfies the recursive relations:

$$\Delta_1(\tau, t) = -\frac{1}{2i}[\sigma_{1,0}(\tau, t) - \sigma_{2,0}(\tau, t)],$$

$$\Delta_m(\tau,t) = -\Bigg\{ \sum_{j=1}^{m-1} \frac{1}{(2i)^j} \left[\sigma_{1,j-1}(\tau,t) - (-1)^{j-1}\sigma_{2,j-1}(\tau,t)\right] \Delta_{m-j}(\tau,t)$$

$$+ \frac{m}{(2i)^m} \left[\sigma_{1,m-1}(\tau,t) - (-1)^{m-1}\sigma_{2,m-1}(\tau,t)\right] \Bigg\}, \quad m = 2,3,\ldots,N, \tag{7.5}$$

where

$$\sigma_{\nu,0}(\tau,t) = -\omega_\nu p(\tau,t),$$
$$\sigma_{\nu,1}(\tau,t) = \omega_\nu p'(\tau,t) + p^2(\tau,t) + q(\tau,t),$$
$$\sigma_{\nu,j+1}(\tau,t) = -\sigma'_{\nu,j}(\tau,t) - \sum_{k=0}^{j} \sigma_{\nu,j-k}(\tau,t)\sigma_{\nu,k}(\tau,t),$$
$$j = 1,2,\ldots,N, \quad \nu = 1,\ 2; \quad \omega_1 = i, \quad \omega_2 = -i.$$

The recursion leads to the following explicit formulas:

$$\Delta_1(\tau,t) = p(\tau,t),$$
$$\Delta_2(\tau,t) = 2p^2(\tau,t) + q(\tau,t),$$
$$\Delta_3(\tau,t) = -\frac{3}{4} p_{\tau\tau}(\tau,t) + 3p(\tau,t)q(\tau,t) + 4p^3(\tau,t),$$
$$\Delta_4(\tau,t) = -\frac{1}{2} q_{\tau\tau}(\tau,t) - 4p(\tau,t)p_{\tau\tau}(\tau,t) - \frac{5}{2} p_\tau^2(\tau,t)$$
$$+ 8p^2(\tau,t)q(\tau,t) + 8p^4(\tau,t) + q^2(\tau,t). \tag{7.6}$$

Taking into account (7.6) and (7.3), $c_0(\tau,t), c_1(\tau,t), c_2(\tau,t)$ can be expressed by λ_k and $\xi_n(\tau,t)$ as follows:

$$c_0(\tau,t) = 4,$$
$$c_1(\tau,t) = 4\left[\frac{\lambda_{-1}+\lambda_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2} - \xi_k(\tau,t)\right)\right],$$
$$c_2(\tau,t) = (\lambda_{-1})^2 + (\lambda_0)^2 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[(\lambda_{2k-1})^2 + (\lambda_{2k})^2 - 2\xi_k^2(\tau,t)\right]$$
$$+ 2\left[\frac{\lambda_{-1}+\lambda_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2} - \xi_k(\tau,t)\right)\right]^2. \tag{7.7}$$

The Dubrovin-type system (5.1) is then given by

$$\dot{\xi}_n = (-1)^n \sigma_n \operatorname{sign}(n) 2\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}\, h_n(\xi) g_n(\xi)$$

with

$$h_n(\xi) = \sqrt{(\xi_n - \lambda_{-1})(\xi_n - \lambda_0) \prod_{k\neq n,0} \frac{(\xi_n - \lambda_{2k-1})(\xi_n - \lambda_{2k})}{(\xi_n - \xi_k)^2}}, \tag{7.8}$$

$$
\begin{aligned}
g_n(\xi) = \Bigg\{ & 4\xi_n^2 + 4\left[\frac{\lambda_{-1}+\lambda_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2} - \xi_k\right)\right]\xi_n \\
& + (\lambda_{-1})^2 + (\lambda_0)^2 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left[(\lambda_{2k-1})^2 + (\lambda_{2k})^2 - 2\xi_k^2\right] \\
& + 2\left[\frac{\lambda_{-1}+\lambda_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{\lambda_{2k-1}+\lambda_{2k}}{2} - \xi_k\right)\right]^2 \\
& + \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)s(\pi,\lambda_k,t,\tau)}{\xi_n - \lambda_k}\Bigg\}. \qquad (7.9)
\end{aligned}
$$

Suppose for a moment that for $(\tau, t) \in R^2$ there exist $\xi_n(\tau, t)$, π-periodic in τ that satisfy not only (7.1), (7.2) but also (3.2) so that for any $t \in R$ we can reconstruct $p(\tau, t)$, $q(\tau, t)$ from trace formulas (3.3), (3.4) (see also (7.6)). We now show that such defined p and q indeed solve system (4.13) as advertised at the beginning of this section. This complements the statement of Theorem 5.1 as we then do not need to assume that a solution of (4.13) exists. Comparing the system of Dubrovin equations (5.1) and (3.2), we obtain

$$
\begin{aligned}
\frac{\partial \xi_k}{\partial t} = -\Bigg\{ & 4\xi_k^2 + 4p(\tau,t)\xi_k + 2[q(\tau,t) + 3p^2(\tau,t)] \\
& + \sum_{i=-\infty}^{\infty} \frac{\alpha_i(t)s(\pi,\lambda_i,t,\tau)}{\xi_k - \lambda_i}\Bigg\} \frac{\partial \xi_k}{\partial \tau}, \quad k \in Z \setminus \{0\}. \qquad (7.10)
\end{aligned}
$$

Taking the derivative of the first trace formula (7.6) with respect to t and using equalities (7.10) and (7.4), we find

$$
\begin{aligned}
p_t &= -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\partial \xi_k}{\partial t} \\
&= 4\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \xi_k^2 \frac{\partial \xi_k}{\partial \tau} + 4p\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \xi_k \frac{\partial \xi_k}{\partial \tau} + 2\left(q + 3p^2\right)\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\partial \xi_k}{\partial \tau} \\
&\quad + \sum_{i=-\infty}^{\infty} \left\{\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\alpha_i(t)s(\pi,\lambda_i,t,\tau)}{\xi_k - \lambda_i}\frac{\partial \xi_k}{\partial \tau}\right\}. \qquad (7.11)
\end{aligned}
$$

Differentiating $\Delta_1(\tau,t)$, $\Delta_2(\tau,t)$, $\Delta_3(\tau,t)$, and $\Delta_4(\tau,t)$ with respect to τ, we obtain

$$
p_\tau(\tau,t) = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\partial \xi_k}{\partial \tau}, \qquad (7.12)
$$

$$q_\tau + 4pp_\tau = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} 2\xi_k \frac{\partial \xi_k}{\partial \tau}, \tag{7.13}$$

$$-\frac{3}{4}p_{\tau\tau\tau} + 12p^2 p_\tau + 3p_\tau q + 3pq_\tau = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} 3\xi_k^2 \frac{\partial \xi_k}{\partial \tau}, \tag{7.14}$$

$$\begin{aligned} -\frac{1}{2}q_{\tau\tau\tau} - 4pp_{\tau\tau\tau} - 9p_\tau p_{\tau\tau} + 16qpp_\tau & \\ +8p^2 q_\tau + 32p^3 p_\tau + 2qq_\tau &= -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} 4\xi_k^3 \frac{\partial \xi_k}{\partial \tau}. \end{aligned} \tag{7.15}$$

We will justify the formal derivatives (7.12)–(7.15) in Remark 9.1 of Appendix. Using equalities (7.12), (7.13), (7.14) and identities (2.6), (3.1), we arrive at

$$p_t = p_{\tau\tau\tau} - 6p_\tau q - 6pq_\tau - 30p^2 p_\tau + \sum_{k=-\infty}^{\infty} \alpha_k(t) s(\pi, \lambda_k, t)(\psi^2(\tau, \lambda_k, t))_\tau. \tag{7.16}$$

Differentiating the second trace formula of (7.6) with respect to t and using equalities (7.10), (7.4) yields

$$\begin{aligned} q_t &= -4pp_t - 2\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \xi_k \frac{\partial \xi_k}{\partial t} \\ &= -4pp_t + 8\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \xi_k^3 \frac{\partial \xi_k}{\partial \tau} + 8p\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \xi_k^2 \frac{\partial \xi_k}{\partial \tau} + 4(q+3p^2)\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \xi_k \frac{\partial \xi_k}{\partial \tau} \\ &\quad + \sum_{i=-\infty}^{\infty} \left\{ \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\xi_k \alpha_i(t) s(\pi, \lambda_i, t, \tau)}{\xi_k - \lambda_i} \frac{\partial \xi_k}{\partial \tau} \right\}. \end{aligned} \tag{7.17}$$

Substituting expressions (7.13)–(7.16) into (7.17) and taking into account (2.6) and (3.1), we derive (4.13).

We are left to prove the global existence of the functions $\xi_n(\tau, t)$ that solve both Dubrovin equations (3.2) and (7.1). In Appendix, we show that for each of the two equations the initial value problems have unique solutions that exist globally. So we may, for example, first solve for each $\tau \in R$ equation (7.1) with initial condition (7.2). The initial condition is π-periodic since there are the initial data p_0 and q_0, and we therefore have π-periodicity of $\xi_n(\tau, t)$ in τ for free. In order to see that the functions ξ_n defined in such a way also solve equation (3.2) for every value of t it suffices to show that the two flows commute. Let us define the maps E and F such that (3.2) takes the form $\xi_\tau = E(\xi)$ and (7.1) combined with (7.7), (2.6), and (3.1) reads $\xi_t = F(t, \xi)$. Note that for all $n \in Z \setminus \{0\}$ we have $F_n = -g_n E_n$, where $g_n(\xi) \equiv g_n(t, \xi)$ is defined in (7.9). With this notation the commutation of the τ-flow and the t-flow is equivalent to the identity

$$E_\xi(\xi) F(t, \xi) - F_\xi(t, \xi) E(\xi) = 0.$$

Computing the n-th component of this equation using the relation $F_n = -g_n E_n$ gives

$$\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \left[(g_n - g_l) \frac{\partial E_n}{\partial \xi_l} E_l + \frac{\partial g_n}{\partial \xi_l} E_n E_l \right] = 0.$$

Since $\frac{\partial g_n}{\partial \xi_n} = 0$ by (9.24) and (9.26), it suffices to show that for every $l \neq n$ we have

$$(g_n - g_l) \frac{\partial E_n}{\partial \xi_l} + \frac{\partial g_n}{\partial \xi_l} E_n = 0.$$

Next we observe for $l \neq n$ that $\frac{\partial E_n}{\partial \xi_l} = \frac{E_n}{\xi_n - \xi_l}$ due to (3.2) and (9.18). Hence the commutation of flows is proved if we can verify

$$\frac{g_n(t,\xi) - g_l(t,\xi)}{\xi_n - \xi_l} = -\frac{\partial g_n(t,\xi)}{\partial \xi_l} \tag{7.18}$$

for all $l \neq n$. To this end, we write $g_n = g_n^{(1)} + g_n^{(2)}$ with $g_n^{(2)}$ given by (9.21). The quadratic polynomial $g_n^{(1)}$ is then defined implicitly via (7.9). The derivative on the right-hand side of (7.18) is computed in equations (9.23) and (9.26) of Appendix. A straightforward calculation using (9.21) and (7.9) shows that (7.18) holds true. This completes the proof that the functions $\xi_n(\tau,t)$ exist globally and that system (4.13) has a global spatially π-periodic solution for any initial data p_0, q_0 that are π-periodic, belong to $C^5(R)$ and satisfy conditions of Section 2.

8. A two-gap solution for system (4.13)

In this section, we present a non-trivial solution of system (4.13) with the special autonomous choice $\alpha_k(t) = \frac{1}{\pi k^3}$ for the source term. Here we choose the spectral data such that only two gaps are open. More precisely, we choose the initial data as

$$\lambda_{-1} = -1, \quad \lambda_0 = 1, \quad \lambda_1 = 2, \quad \lambda_2 = 4, \quad \xi_1^0(0) = 2, \quad \sigma_1^0(0) = +1$$

and assume that all other gaps are closed.

In this case, system (5.1) has the same form as that considered in [37, 4.1]. According to this and Remark 6.1, it is easy to check that

$$p(\tau,t) = \frac{3 - 4\,\mathrm{sn}^2\left(-339t + 3\tau, \frac{2}{3}\right)}{1 + 2\,\mathrm{cn}^2\left(-339t + 3\tau, \frac{2}{3}\right)},$$

$$q(\tau,t) = \frac{81 - 156\,\mathrm{sn}^2\left(-339t + 3\tau, \frac{2}{3}\right) + 72\,\mathrm{sn}^4\left(-339t + 3\tau, \frac{2}{3}\right)}{\left(1 + 2\,\mathrm{cn}^2\left(-339t + 3\tau, \frac{2}{3}\right)\right)^2}$$

are the solution to system (4.13), where sn and cn denote the Jacobi elliptic functions.

9. Appendix: Cauchy problem for the Dubrovin equations (3.2) and (7.1)

Equation (3.2) is similar to equation (7.1) but simpler, because the factors g_n defined in (7.9) are just replaced by the constant function 1. It therefore suffices to study the existence and uniqueness of solutions of the Cauchy problem for equation (7.1). In order to simplify system (7.1), we change the variables

$$\xi_n = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1}) \sin^2 x_n(t), \quad n \in Z \setminus \{0\}. \tag{9.1}$$

After substituting (9.1), system (7.1) takes the form

$$\frac{dx_n}{dt} = (-1)^n \sigma_n(t) \operatorname{sign}\{\sin x_n(t) \cos x_n(t)\} \operatorname{sign}\{n\} h_n(\xi) g_n(t, \xi), \ n \in Z \setminus \{0\}, \tag{9.2}$$

where g_n and h_n are given by (7.9) and (7.8). Note that the t-dependence of g_n solely enters through the coefficients $\alpha_k(t)$ of the self-consistent term. Furthermore, we remark that when the variable ξ_n passes through one of the endpoints λ_{2n-1}, λ_{2n} of its band gap, both $\sigma_n(t)$ and the product $\sin x_n(t) \cos x_n(t)$ change the sign. If we choose the initial conditions

$$x_n(0) = x_n^0 = \arcsin \sqrt{\frac{\xi_n^0 - \lambda_{2n-1}}{\lambda_{2n} - \lambda_{2n-1}}}, \quad n \in Z \setminus \{0\}, \tag{9.3}$$

then $\sigma_n(t) \operatorname{sign}\{\sin x_n(t) \cos x_n(t)\} = \sigma_n(0)$. Now system (9.2) takes the form

$$\frac{dx_n}{dt} = H_n(t, x), \quad n \in Z \setminus \{0\}, \tag{9.4}$$

where $H_n(t, x) = (-1)^n \sigma_n(0) \operatorname{sign}(n) h_n(\xi) g_n(t, \xi)$. To study the solvability of the Cauchy problem (9.4), (9.3), we consider the Banach space K of sequences $\{x \in K : x = (\dots, x_{-1}, x_1, \dots),\ x \in R\}$ with the norm

$$\|x\| = \sum_{\substack{n=-\infty \\ k \neq 0}}^{\infty} |n| (\lambda_{2n} - \lambda_{2n-1}) |x_n|. \tag{9.5}$$

Denoting $H(t, x) = (\dots, H_{-1}(t, x), H_1(t, x), \dots)$, the Cauchy problem (9.4), (9.3) becomes an initial value problem for an ordinary differential equation in K:

$$\frac{dx}{dt} = H(t, x), \tag{9.6}$$

$$x(t)|_{t=0} = x^0, \quad x^0 \in K. \tag{9.7}$$

The remaining part of Appendix is devoted to showing that H is globally Lipshitz in x from which global existence and uniqueness of problem (9.6), (9.7) follows by the Picard–Lindelöf theorem.

Recall that we have assumed $p_0(x), q_0(x) \in C^5(R)$. This implies the asymptotics (see [15]):

$$\lambda_{2n-1} = n + a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \dots + \frac{a_5}{n^5} + \frac{\varepsilon_n^-}{n^5}, \tag{9.8}$$

$$\lambda_{2n} = n + a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_5}{n^5} + \frac{\varepsilon_n^+}{n^5}, \tag{9.9}$$

where $a_k, k = 0, 1, \ldots, 5$, are constants, and $(\varepsilon_n^{\pm})_n \in l_2$. Consequently, taking into consideration that $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$, we see that $\inf_{k \neq n,0} |\xi_n - \xi_k| \geq a > 0$. Using these facts, we now deduce the estimates

$$C_1|n| \leq |h_n(\xi)| \leq C_2|n|, \qquad \left|\frac{\partial h_n(\xi)}{\partial \xi_m}\right| \leq C_3|n|, \tag{9.10}$$

$$|g_n(\xi)| \leq C_4|n|^2, \qquad \left|\frac{\partial g_n(\xi)}{\partial \xi_m}\right| \leq C_5|n|\,|m|, \tag{9.11}$$

where the constants $C_k, k = 1, 2, 3, 4, 5$, are positive and do not depend on n, m, ξ, and t.

Remark 9.1. Notice that the uniform convergence of the series in (7.12)–(7.15) follows from the upper bound on h_n in (9.10) and from (9.8), (9.9).

Let us begin proving claims (9.10) and (9.11). From (7.8), we have

$$h_n^2(\xi) = (\xi_n - \lambda_{-1})(\xi_n - \lambda_0) \prod_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \left(1 + \frac{\lambda_{2k-1} - \xi_k}{\xi_k - \xi_n}\right)\left(1 + \frac{\lambda_{2k} - \xi_k}{\xi_k - \xi_n}\right).$$

It follows that

$$\begin{aligned}
\left|h_n^2(\xi)\right| &= |(\xi_n - \lambda_{-1})|\,|(\xi_n - \lambda_0)| \prod_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \left|1 + \frac{\lambda_{2k-1} - \xi_k}{\xi_k - \xi_n}\right| \left|1 + \frac{\lambda_{2k} - \xi_k}{\xi_k - \xi_n}\right| \\
&\leq |(\xi_n - \lambda_{-1})|\,|(\xi_n - \lambda_0)| \prod_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \left(1 + \left|\frac{\lambda_{2k-1} - \xi_k}{\xi_k - \xi_n}\right|\right)\left(1 + \left|\frac{\lambda_{2k} - \xi_k}{\xi_k - \xi_n}\right|\right) \\
&\leq D_0 n^2 \prod_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \left(1 + \frac{\lambda_{2k} - \lambda_{2k-1}}{a}\right)^2 \\
&\leq D_0 n^2 \prod_{k=-\infty}^{\infty} \left(1 + \frac{\lambda_{2k} - \lambda_{2k-1}}{a}\right)^2 \leq D_1 n^2.
\end{aligned}$$

This implies

$$|h_n(\xi)| \leq C_2|n|. \tag{9.12}$$

Next we derive a lower bound on $|h_n(\xi)|$. To do this, we introduce a set of indices

$$M = \left\{k \in Z : \frac{\lambda_{2k} - \lambda_{2k-1}}{a} \geq 1\right\}.$$

Observe that M is finite due to (9.8) and (9.9). Setting

$$A_n := \prod_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n} \quad \text{and} \quad B_n := \prod_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \frac{\lambda_{2k} - \xi_n}{\xi_k - \xi_n}, \tag{9.13}$$

we have

$$h_n^2(\xi) = (\xi_n - \lambda_{-1})(\xi_n - \lambda_0) A_n B_n. \tag{9.14}$$

We rewrite A_n as follows:

$$A_n = A_{n,1} A_{n,2} A_{n,3},$$

where

$$A_{n,1} := \prod_{\substack{k \in \mathbb{Z} \setminus M \\ k \neq n,0}} \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n},$$

$$A_{n,2} := \prod_{\substack{k \in M \setminus \{0\} \\ k \leq n-1}} \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n},$$

$$A_{n,3} := \prod_{\substack{k \in M \setminus \{0\} \\ k \geq n+1}} \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n}.$$

1) Let $k \neq n$ and $k \notin M$. Then

$$\left| \frac{\lambda_{2k-1} - \xi_k}{\xi_k - \xi_n} \right| \leq \frac{\lambda_{2k} - \lambda_{2k-1}}{a} < 1,$$

Hence,

$$\begin{aligned} |A_{n,1}| &= \prod_{\substack{k \in \mathbb{Z} \setminus M \\ k \neq n,0}} \left| \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n} \right| = \prod_{\substack{k \in \mathbb{Z} \setminus M \\ k \neq n,0}} \left| 1 + \frac{\lambda_{2k-1} - \xi_k}{\xi_k - \xi_n} \right| \\ &\geq \prod_{\substack{k \in \mathbb{Z} \setminus M \\ k \neq n,0}} \left(1 - \left| \frac{\lambda_{2k-1} - \xi_k}{\xi_k - \xi_n} \right| \right) \geq \prod_{\substack{k \in \mathbb{Z} \setminus M \\ k \neq n,0}} \left(1 - \frac{\lambda_{2k} - \lambda_{2k-1}}{a} \right) \\ &> \prod_{k \in \mathbb{Z} \setminus M} \left(1 - \frac{\lambda_{2k} - \lambda_{2k-1}}{a} \right) = D_2. \end{aligned}$$

2) Let $k \leq n-1$ and $k \in M$. Then we obtain

$$\begin{aligned} |A_{n,2}| &= \prod_{\substack{k \in M \setminus \{0\} \\ k \leq n-1}} \left| \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n} \right| = \prod_{\substack{k \in M \setminus \{0\} \\ k \leq n-1}} \frac{\xi_n - \lambda_{2k-1}}{\xi_n - \xi_k} \\ &= \prod_{\substack{k \in M \setminus \{0\} \\ k \leq n-1}} \left(1 + \frac{\xi_k - \lambda_{2k-1}}{\xi_n - \xi_k} \right) > 1. \end{aligned}$$

3) Finally, let $k \geq n+1$ and $k \in M$. Denote $\Delta = \max_{k \in Z}(\lambda_{2k} - \lambda_{2k-1})$. First, we consider the case $k \geq n+1$, $k \in M$, $|\xi_k - \xi_n| \leq 2\Delta$. We may write

$$\prod_{\substack{k \in M^* \setminus \{0\} \\ k \geq n+1}} \left| \frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n} \right| > \prod_{\substack{k \in M \setminus \{0\} \\ k \geq n+1}} \frac{\lambda_{2k-1} - \lambda'_{2k-1}}{2\Delta} \geq \prod_{k \in M} \frac{\lambda_{2k-1} - \lambda'_{2k-1}}{2\Delta},$$

where $M^* = \{k \in M : |\xi_k - \xi_n| \leq 2\Delta\}$ and λ'_{2k-1} is chosen such that $\max\{\lambda_{2k-2}, \lambda_{2k-1} - 2\Delta\} < \lambda'_{2k-1} < \lambda_{2k-1}$. In the case $k \geq n+1$, $k \in M$, $|\xi_k - \xi_n| > 2\Delta$, one has

$$\frac{\xi_k - \lambda_{2k-1}}{\xi_k - \xi_n} < \frac{\xi_k - \lambda_{2k-1}}{2\Delta} \leq \frac{\lambda_{2k} - \lambda_{2k-1}}{2\Delta} < \frac{1}{2},$$
$$\frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n} > \frac{1}{2},$$
$$\prod_{\substack{k\in M^{**}\setminus\{0\}\\ k\geq n+1}} \left|\frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n}\right| > \prod_{\substack{k\in M^{**}\\ k\geq n+1}} \frac{1}{2} > \prod_{k\in M} \frac{1}{2},$$

where $M^{**} = \{k \in M : |\xi_k - \xi_n| > 2\Delta\}$. It follows that

$$|A_{n,3}| = \prod_{\substack{k\in M\setminus\{0\}\\ k\geq n+1}} \left|\frac{\lambda_{2k-1} - \xi_n}{\xi_k - \xi_n}\right| > \prod_{k\in M} \frac{\lambda_{2k-1} - \lambda'_{2k-1}}{4\Delta}.$$

In summary we have derived

$$|A_n| = |A_{n,1}|\,|A_{n,2}|\,|A_{n,3}| > D_3. \tag{9.15}$$

A similar reasoning yields

$$|B_n| > \tilde{D}_3. \tag{9.16}$$

Substituting (9.15) and (9.16) into (9.14), we arrive at

$$|h_n| > C_1\,|n|\,. \tag{9.17}$$

Now we estimate $\frac{\partial h_n(\xi)}{\partial \xi_m}$.

1) Let $m \neq n$. Differentiating (7.8) readily gives

$$2h_n \frac{\partial h_n(\xi)}{\partial \xi_m}$$
$$= (\xi_n - \lambda_{-1})(\xi_n - \lambda_0)\frac{-2(\lambda_{2m-1} - \xi_n)(\lambda_{2m} - \xi_n)}{(\xi_m - \xi_n)^3} \prod_{\substack{k=-\infty\\ k\neq m,n,0}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}$$
$$= -2(\xi_n - \lambda_{-1})(\xi_n - \lambda_0)\frac{1}{\xi_m - \xi_n} \prod_{\substack{k=-\infty\\ k\neq n,0}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2} = \frac{-2h_n^2}{\xi_m - \xi_n}$$

which implies

$$\frac{\partial h_n(\xi)}{\partial \xi_m} = \frac{h_n(\xi)}{\xi_n - \xi_m} \quad \text{for all } m \neq n. \tag{9.18}$$

From the last equality and (9.12), we learn

$$\left|\frac{\partial h_n(\xi)}{\partial \xi_m}\right| = \frac{|h_n(\xi)|}{|\xi_n - \xi_m|} \leq D_4|n| \quad \text{for all } m \neq n. \tag{9.19}$$

2) Let $m = n$. We introduce the notation

$$h_n^2(\xi) = \tilde{A}_n B_n,$$

where $\tilde{A}_n = (\xi_n - \lambda_{-1})(\xi_n - \lambda_0)A_n$ and A_n and B_n are the same as in (9.13). Logarithmic differentiation gives

$$\frac{\partial \tilde{A}_n}{\partial \xi_n} = \tilde{A}_n \left[\frac{1}{\xi_n - \lambda_{-1}} + \frac{1}{\xi_n - \lambda_0} + \sum_{\substack{k=-\infty \\ k \neq n,0}}^{\infty} \frac{\lambda_{2k-1} - \xi_k}{(\xi_n - \lambda_{2k-1})(\xi_n - \xi_k)} \right].$$

From this equality and taking into account the inequality $|A_n| \le D_1$, we derive the estimate

$$\left| \frac{\partial \tilde{A}_n}{\partial \xi_n} \right| \le |A_n| \, |2\xi_n - \lambda_{-1} - \lambda_0| + \left| \tilde{A}_n \right| \sum_{\substack{k=-\infty \\ k \neq n}}^{\infty} \frac{|\lambda_{2k-1} - \xi_k|}{|\xi_n - \lambda_{2k-1}| \, |\xi_n - \xi_k|} \le D_5 n^2.$$

Similarly, we deduce

$$\left| \frac{\partial B_n}{\partial \xi_n} \right| \le D_6.$$

The inequalities we have obtained so far imply the estimate

$$\left| \frac{\partial h_n^2}{\partial \xi_n} \right| \le \left| \frac{\partial \tilde{A}_n}{\partial \xi_n} \right| |B_n| + \left| \frac{\partial B_n}{\partial \xi_n} \right| \left| \tilde{A}_n \right| \le D_7 n^2.$$

Combining this estimate with (9.17) and recalling (9.19), it follows for all $n, m \in Z \setminus \{0\}$ that

$$\left| \frac{\partial h_n(\xi)}{\partial \xi_m} \right| \le C_3 |n|. \tag{9.20}$$

Let us turn our attention to $g_n(t, \xi)$. For convenience, we introduce the notation

$$g_n(t, \xi) = g_n^{(1)}(\xi) + g_n^{(2)}(t, \xi),$$

where

$$g_n^{(2)}(t, \xi) := \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t) s(\pi, \lambda_k, t, \tau)}{\xi_n - \lambda_k} = \pi \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)}{\xi_n - \lambda_k} \prod_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \frac{\xi_i - \lambda_k}{i} \tag{9.21}$$

by equation (3.1). Of course, $g_n^{(1)}$ denotes the quadratic polynomial (in ξ) that represents $g_n - g_n^{(2)}$ according to (7.9). The following estimates are immediate:

$$\left| g_n^{(1)}(\xi) \right| \le D_8 \, |n|^2 , \quad \left| g_n^{(2)}(t, \xi) \right| = O\left(\frac{1}{|n|} \right), \quad |g_n(t, \xi)| \le C_4 \, |n|^2 . \tag{9.22}$$

Next, we compute the derivatives of $g_n^{(1)}$ and $g_n^{(2)}$. For $g_n^{(1)}$, they read

$$\frac{\partial g_n^{(1)}(\xi)}{\partial \xi_m} = -4\xi_n - 4\xi_m - 4 \left[\frac{\lambda_{-1} + \lambda_0}{2} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left(\frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k \right) \right] \tag{9.23}$$

for $m \neq n$ and

$$\frac{\partial g_n^{(1)}(\xi)}{\partial \xi_n} = 0. \tag{9.24}$$

Thus we easily have for all m and n:

$$\left| \frac{\partial g_n^{(1)}(\xi)}{\partial \xi_m} \right| \leq D_9 |n||m|. \tag{9.25}$$

From (9.21), we calculate

$$\begin{aligned}
&\frac{\partial g_n^{(2)}(t,\xi)}{\partial \xi_n} = 0 \quad \text{and} \\
&\frac{\partial g_n^{(2)}(t,\xi)}{\partial \xi_m} = \pi \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)}{(\xi_n - \lambda_k)(\xi_m - \lambda_k)} \prod_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \frac{\xi_i - \lambda_k}{i} \quad \text{for } m \neq n.
\end{aligned} \tag{9.26}$$

Due to (9.26) and (9.25), we obtain

$$\left| \frac{\partial g_n^{(2)}(t,\xi)}{\partial \xi_m} \right| = O\left(\frac{1}{|nm|} \right) \text{ and } \left| \frac{\partial g_n(t,\xi)}{\partial \xi_m} \right| \leq C_5 |n||m| \text{ for all } m \text{ and } n. \tag{9.27}$$

By the construction, all D_k, $k = 1, 2, \ldots, 9$, and C_k, $k = 1, 2, \ldots, 5$, are positive constants independent of n, m, t, and ξ. We have therefore completed the proof of statements (9.10) and (9.11). With these estimates it follows for $f_n(t,\xi) := h_n(\xi) g_n(t,\xi)$ that

$$\left| \frac{\partial f_n(t,\xi)}{\partial \xi_m} \right| \leq C|n|^3 |m| \quad \text{for all } m \text{ and } n. \tag{9.28}$$

Denote $F(s) := f_n(t, \eta + s(\xi - \eta))$ for $s \in [0, 1]$. By Lagrange's theorem there exists $s^* \in (0, 1)$ with $F(1) - F(0) = F'(s^*)$ and, consequently,

$$f_n(t,\xi) - f_n(t,\eta) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{\partial\, f_n(t,\theta)}{\partial \xi_m} (\xi_m - \eta_m),$$

where $\theta = \eta + s^*(\xi - \eta)$. Keeping (9.4), (9.28) and (9.5) in view, we have

$$\begin{aligned}
|H_n(t,x) - H_n(t,y)| = |f_n(t,\xi) - f_n(t,\eta)| &\leq \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left| \frac{\partial\, f_n(t,\theta)}{\partial \xi_m} \right| |\xi_m - \eta_m| \\
&\leq \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} C\,|n|^3\,|m|\,|\lambda_{2m} - \lambda_{2m-1}|\,\left|\sin^2 x_m - \sin^2 y_m\right| \\
&\leq C\,|n|^3 \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} |m|\,|\lambda_{2m} - \lambda_{2m-1}|\,|x_m - y_m|
\end{aligned}$$

$$= C\,|n|^3\,\|x-y\|\,. \tag{9.29}$$

Finally, this implies by (9.8) and (9.9) that

$$\begin{aligned}\|H(t,x)-H(t,y)\| &= \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |n|\,|\lambda_{2n}-\lambda_{2n-1}|\,|H_n(t,x)-H_n(t,y)| \\ &\le C \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |\lambda_{2n}-\lambda_{2n-1}|\,|n|^4\,\|x-y\| \\ &= \left\{ C \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{|\varepsilon_n^+-\varepsilon_n^-|}{|n|} \right\} \|x-y\| = L\,\|x-y\|\end{aligned}$$

with

$$L := C \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{|\varepsilon_n^+-\varepsilon_n^-|}{|n|} < \infty,$$

because $(|\varepsilon_n^+-\varepsilon_n^-|)_n \in l_2$. This shows that $H(t,x)$ is indeed Lipshitz continuous in x with global Lipshitz constant L and the theorem of Picard–Lindelöf can be applied as claimed above.

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Про побудову і інтегрування ієрархії для системи Каупа із самоузгодженим джерелом в класі періодичних функцій

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У цій статті ми виводимо багату ієрархію для системи Каупа із самоузгодженим джерелом в класі періодичних функцій. Ми обговорюємо повну інтегровність побудованих систем, яка заснована на трансформуванні у спектральні дані асоційованого квадратичного пучка рівнянь Штурма—Ліувілля з періодичними коефіцієнтами. Зокрема, одержано рівняння типу Дубровіна для часової еволюції спектральних даних для розв'язків будь-якої системи в ієрархії. Крім того, на прикладі окремої системи з ієрархії ми демонструємо переваги інтегровності, доводячи існування глобальних розв'язків для задачі Коші та надаючи явний розв'язок.

Ключові слова: система рівнянь Каупа, ієрархія, самоузгоджене джерело, квадратичний пучок рівнянь Штурма–Ліувілля, обернена спектральна задача, формули слідів, періодичний потенціал