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On Steady Flows of Quasi-Newtonian Fluids in Orlicz–Sobolev Spaces

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The paper deals with the existence of weak solutions to steady quasi-Newtonian flows by means of the Galerkin approximations and the measure-valued solutions, namely Young measures, which turned out to be a good tool to describe the weak solutions of our problem in Orlicz spaces.

Key words: quasi-Newtonian fluid, Orlicz spaces, weak monotonicity, weak solution, Young measures

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1. Introduction and main result

Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$. In this paper, we deal with the existence of weak solutions to the following steady quasi-Newtonian viscous fluid:

$$-\operatorname{div}\sigma(x,u,Du) + u\cdot\nabla u + \nabla\pi = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $u : \Omega \to \mathbb{R}^m$ is the velocity field, $\pi : \Omega \to \mathbb{R}$ is the pressure, $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is the Cauchy stress tensor, where $\mathbb{M}^{m\times n}$ denotes the space of $m \times n$ matrices equipped with the inner product $A_{ij}B_{ij}$ with conventional summation, and f are the given body forces.

Consider first the case when the convective term $u\nabla u$ is assumed to be small and thus neglected, and σ have a polynomial growth/coercivity condition with respect to u and Du (the velocity gradient) with weak monotonicity. The problem

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.4}$$

is known to be solved by Hungerbühler in [16] for $f \in W^{-1,p'}(\Omega;\mathbb{R}^m)$ $(p' = p/(p-1))$, and Dolzmann [11] established the existence result for the measure valued function $f = \mu$ and replaced the weak derivative Du by the approximately differentiable $apDu$. We proved in [7] the existence result for (1.4) by using a different notion of monotonicity for $\sigma(\cdot) = a(x, Du) + \phi(u)$. See also [8]. In the same setting and with the additional convective term $u.\nabla u$, Arada and Sequeira [1] proved the existence of weak solutions using polynomial growth and coercivity conditions. They also used weak monotonicity assumptions on the stress tensor σ.

In [3], the present authors studied the existence of solutions for the quasilinear elliptic system (1.4) in Orlicz–Sobolev spaces by using the Young measure and mild monotonicity assumptions on σ. See also [5, 6, 9] for the unsteady case.

The first mathematical investigations on the class of systems (1.1)–(1.3) go back to O.A. Ladyzhenskaya [20, 21] and J.L. Lions [23]. They both considered the unsteady case and showed the existence of a weak solution whenever the coercivity parameter p of the nonlinear elliptic operator related to the stress tensor satisfies $p \geq \frac{3n+2}{n+2}$.

For $\sigma(x, u, Du) = T(x, Du)$ in (1.1)–(1.3), Gwiazda et al. [13] showed the existence result in the setting of Musielak–Orlicz when the source term f is equal to $\operatorname{div} F$, with $F \in \mathbb{M}^{n\times n}$ and $F \in L_{\overline{M}}(\Omega)$. The authors used the concept of Young measure to define the weak solution and they restricted the N-function to satisfy the following condition: $M(x, F) \geq c|F|^q$ for $F \in \mathbb{M}^{n\times n}$, $c > 0$ and $q \geq \frac{3n}{n+2}$. They also proved that the mapping T belongs to some class of monotone operators, namely the class $(\mathcal{S}_m)$. In [26], the author established the existence of weak solutions for steady flows of non-Newtonian incompressible fluids with the help of a general x-dependent convex function in generalized Orlicz spaces. Later, Gwiazda et al. [14] proved the existence of weak solutions to the generalized Stokes system in anisotropic Orlicz spaces.

The aim of the paper is to extend the result of [1] to a more general space where the growth and coercivity of σ are not polynomial. Consequently, the L^p-framework will not capture the described situation. For this reason, the homogeneous Orlicz–Sobolev spaces $W^1_{0,\mathrm{div}} L_M(\Omega; \mathbb{R}^m)$ are a suitable framework to explore the growth assumptions by means of a convex function, namely an N-function. Further, we extend the result of [3] to a steady quasi-Newtonian given by (1.1)–(1.3). We will prove the existence of weak solutions for problem (1.1)–(1.3) based on the results of [3, 4]. The function spaces and notations will be presented in detail in Section 2.

As mentioned above, our aim is to prove the existence result in the setting of Orlicz spaces by using the concept of Young measure as a technical tool to describe weak limits of sequences constructed by the Galerkin approximations due to Landes (cf. [22]). This approach was widely used in the calculus of variations, optimal control theory and non-linear partial differential equations.

Finally, we set the assumptions on the stress tensor σ. Consider two N-functions M and P such that P grows essentially less rapidly than M and $M, \overline{M} \in \Delta_2$ (see Section 2).

(H0) (Continuity) $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function (i.e., measurable with respect to x and continuous with respect to the last variables).

(H1) (Growth and Coercivity) There exist $\alpha, \gamma > 0$, $d_1 \in E_{\overline{M}}(\Omega)$ and $d_2 \in L^1(\Omega)$ such that

$$|\sigma(x,s,F)| \leq d_1(x) + \overline{M}^{-1}P(\gamma|s|) + \overline{M}^{-1}M(\gamma|F|),$$

$$\sigma(x,s,F) : F \geq \alpha M(|F|) - d_2(x)$$

for a.e. $x \in \Omega$ and all $(s,F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$.

(H2) (Monotonicity) σ satisfies one of the following conditions:

(a) For all $(x,s) \in \Omega \times \mathbb{R}^m$, $F \mapsto \sigma(x,s,F)$ is a C^1-function and it is monotone, i.e., for all $(x,s) \in \Omega \times \mathbb{R}^m$ and $F, G \in \mathbb{M}^{m\times n}$, we have

$$\big(\sigma(x,s,F) - \sigma(x,s,G)\big) : (F - G) \geq 0.$$

(b) There exists a function $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,s,F) = \frac{\partial W}{\partial F}(x,s,F)$, and $F \to W(x,s,F)$ is convex and C^1 for all $(x,s) \in \Omega \times \mathbb{R}^m$.

(c) σ is strictly monotone, i.e., σ is monotone and

$$\big(\sigma(x,s,F) - \sigma(x,s,G)\big) : (F - G) = 0 \Rightarrow F = G.$$

(d) σ is strictly M-quasimonotone, i.e.,

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,s,\lambda) - \sigma(x,s,\overline{\lambda})\big) : (\lambda - \overline{\lambda}) d\nu_x(\lambda)\, dx > 0,$$

where $\overline{\lambda} = \langle \nu_x, id\rangle$, and $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a bounded sequence in $L_M(\Omega)$ and not a Dirac measure for a.e. $x \in \Omega$.

Now, a function $u \in W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$ is said to be a weak solution of problem (1.1)–(1.3) if for all $\varphi \in W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$,

$$\int_\Omega \big(\sigma(x,u,Du) : D\varphi + u \cdot \nabla u \cdot \varphi\big)\, dx = \langle f, \varphi\rangle$$

holds, where $\langle \cdot,\cdot\rangle$ is the duality pairing of $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$ and its dual.

Our main result reads as follows:

Theorem 1.1. *If σ satisfies the conditions (H0)–(H2), then problem (1.1)–(1.3) has a weak solution for every $f \in W^{-1}_{\mathrm{div}}L_{\overline{M}}(\Omega;\mathbb{R}^m)$.*

The present paper is organized as follows. In Section 2, we recall the definition of an N-function, the spaces of Orlicz and Orlicz–Sobolev altogether with some of their properties. We end this section by recalling the definition of Young measures and some of their useful properties. Section 3 is devoted to the construction of the approximate solution by the Galerkin method. Section 4 concerns the existence of a Young measure related to the Orlicz–Sobolev space and a proof of the div-curl inequality. In the last section, the proof of the main theorem is given.

2. Preliminaries

In the first subsection we recall some definitions and well-known facts about N-functions, Orlicz and Orlicz–Sobolev spaces. For more details, we refer readers to [12, 18, 19]. The second subsection is devoted to a brief overview about Young measures. The reader not familiar with the theory of measure-valued solutions should refer to [2, 10, 15, 25] for more details.

2.1. Notation and properties of Orlicz–Sobolev spaces. Let $M : \mathbb{R}^+ := [0, +\infty) \to \mathbb{R}^+$ be an N-function, i.e., M is continuous convex, with $M(\tau) > 0$ for $\tau > 0$, $\frac{M(\tau)}{\tau} \to 0$ (respectively $+\infty$) as $\tau \to 0^+$ (respectively $\tau \to +\infty$). Thus, M admits the representation

$$M(\tau) = \int_0^\tau m(s)\, ds,$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing right continuous, with $m(0) = 0$, $m(\tau) > 0$ for $\tau > 0$ and $m(\tau) \to +\infty$ as $\tau \to +\infty$. The N-function $\overline{M}$ conjugate to M is defined by

$$\overline{M}(\tau) = \int_0^\tau \overline{m}(s)\, ds,$$

where, $\overline{m}(\tau) = \sup\{s,\, m(s) \le \tau\}$. Clearly, $\overline{\overline{M}} = M$ and one has Young's inequality

$$\tau s \le M(\tau) + \overline{M}(s)$$

for all $s, \tau > 0$. The N-function M is said to satisfy the Δ_2 condition (resp. near infinity) if there exists $k > 0$ (resp. $\tau_0 > 0$) such that

$$M(2\tau) \le k M(\tau)$$

for all $\tau \ge 0$ (resp. $\tau \ge \tau_0$).

Let Ω be an open subset of $\mathbb{R}^n$ and M be an N-function. The Orlicz class $\mathcal{L}_M(\Omega; \mathbb{R}^m)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}^m$ such that

$$\int_\Omega M(|u(x)|)\, dx < \infty.$$

The Orlicz space $L_M(\Omega; \mathbb{R}^m)$ is the set of (equivalence classes of) measurable functions $u : \Omega \to \mathbb{R}^m$ such that $\frac{u}{\beta} \in \mathcal{L}_M(\Omega; \mathbb{R}^m)$ for some $\beta > 0$. It is a Banach space under the norm

$$\|u\|_M = \inf\left\{\beta > 0 \,:\, \int_\Omega M\left(\frac{|u(x)|}{\beta}\right) dx \le 1\right\}.$$

The closure in $L_M(\Omega; \mathbb{R}^m)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega; \mathbb{R}^m)$. The equality $E_M(\Omega; \mathbb{R}^m) = L_M(\Omega; \mathbb{R}^m)$ holds if and only if M satisfies the Δ_2 condition for all τ or for τ large according to whether Ω has a finite measure or not. The dual space of $E_M(\Omega; \mathbb{R}^m)$ can be identified with $L_{\overline{M}}(\Omega;\mathbb{R}^m)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega;\mathbb{R}^m)$ is equivalent to $\|.\|_{\overline{M}}$. We recall Hölder's inequality

$$\int_\Omega |u(x)v(x)|\,dx \le 2\|u\|_M\,\|v\|_{\overline{M}}$$

for all $u \in L_M(\Omega;\mathbb{R}^m)$ and all $v \in L_{\overline{M}}(\Omega;\mathbb{R}^m)$. The space $L_M(\Omega;\mathbb{R}^m)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition for all τ or for all τ large according to whether Ω has a finite measure or not. We say that u_k converges to u for the modular convergence in $L_M(\Omega;\mathbb{R}^m)$ if for some $\beta > 0$,

$$\int_\Omega M\left(\frac{|u_k - u|}{\beta}\right) dx \to 0 \quad \text{as} \quad k \to \infty.$$

Furthermore, if $M \in \Delta_2$ (near infinity only if $|\Omega| < \infty$), then the modular convergence coincides with the norm convergence.

The Orlicz–Sobolev space $W^1L_M(\Omega;\mathbb{R}^m)$ is the set of all $u \in L_M(\Omega;\mathbb{R}^m)$ such that $Du \in L_M(\Omega;\mathbb{M}^{m\times n})$, where Du is a matrix-valued function in which all components are distributional partial derivatives of u. It is a Banach space endowed with the norm

$$\|u\|_{W^1L_M(\Omega;\mathbb{R}^m)} := \|u\|_{1,M} = \|u\|_M + \|Du\|_M.$$

The symbol $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)$ denotes the space of all $\mathcal{C}^\infty$-functions $u : \Omega \to \mathbb{R}^m$ with a compact support in Ω. Note that if $|\Omega| < \infty$ and M satisfies the Δ_2 condition near infinity, then

$$W_0^1L_M(\Omega;\mathbb{R}^m) = \overline{\mathcal{C}_0^\infty(\Omega;\mathbb{R}^m)}^{W^1L_M(\Omega;\mathbb{R}^m)},$$

and $W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m) = \big(W_0^1L_M(\Omega;\mathbb{R}^m)\big)^*$. Furthermore, for an N-function M, the embedding $W^1L_M(\Omega;\mathbb{R}^m) \hookrightarrow L_M(\Omega;\mathbb{R}^m)$ is continuous. As M satisfies the Δ_2 condition, we have the following Poincaré inequality: there exists $\theta > 0$ such that for all $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$,

$$\int_\Omega M(|u|)\,dx \le \theta \int_\Omega M(|Du|)\,dx. \tag{2.1}$$

Note that if $M, \overline{M} \in \Delta_2$, then the spaces $W^1L_M(\Omega;\mathbb{R}^m)$ and $W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$ are reflexive and separable.
By $W_{0,\mathrm{div}}^1L_M(\Omega;\mathbb{R}^m)$, we denote the Orlicz–Sobolev space with free divergence, i.e.,

$$W_{0,\mathrm{div}}^1L_M(\Omega;\mathbb{R}^m) = \big\{v \in W_0^1L_M(\Omega;\mathbb{R}^m)\ :\ \operatorname{div} v = 0\big\}.$$

The Orlicz–Sobolev space $W_{\mathrm{div}}^{-1}E_{\overline{M}}(\Omega;\mathbb{R}^m)$ is a dual space of $W_{0,\mathrm{div}}^1L_M(\Omega;\mathbb{R}^m)$.

2.2. A review on Young measures. In the following, $\mathcal{C}_0(\mathbb{R}^m)$ denotes the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|\cdot\|_\infty$-norm. Its dual space can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle \nu, f\rangle = \int_{\mathbb{R}^m} f(\lambda)\, d\nu(\lambda).$$

Note that $id(\lambda) = \lambda$, and thus $\langle \nu, id\rangle = \int_{\mathbb{R}^m} \lambda\, d\nu(\lambda)$.

Lemma 2.1. *Assume that the sequence* $\{w_j\}_{j\geq 1}$ *is bounded in* $L^\infty(\Omega;\mathbb{R}^m)$*. Then there exists a subsequence* $\{w_k\}_k$ *and a Borel probability measure* ν_x *on* $\mathbb{R}^m$ *for a. e.* $x \in \Omega$ *such that for a. e.* $g \in \mathcal{C}(\mathbb{R}^m)$ *we have*

$$g(w_k) \rightharpoonup^* \overline{g} \quad \text{weakly in } L^\infty(\Omega),$$

where

$$\overline{g}(w) = \int_{\mathbb{R}^m} g(\lambda)\, d\nu_x(\lambda),$$

and $\nu = \{\nu_x\}_{x\in\Omega}$ *are called the family of Young measures associated with the subsequence* $\{w_k\}_k$*.*

The following lemmas are considered as applications of the fundamental theorem on Young measures (cf. [10]) which will be needed in the sequel.

Lemma 2.2 ([17])**.** *If* $|\Omega| < \infty$ *and* ν_x *is the Young measure generated by the (whole) sequence* w_j*, then there holds*

$$w_j \to w \text{ in measure } \Leftrightarrow \nu_x = \delta_{w(x)} \quad \text{for a.e. } x \in \Omega.$$

Lemma 2.3 ([17])**.** *Let* $g : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ *be a Carathéodory function and* $w_k : \Omega \to \mathbb{R}^m$ *be a sequence of measurable functions such that* $w_k \to w$ *in measure and* Dw_k *generates the Young measure* ν_x *with* $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ *for a. e.* $x \in \Omega$*. Then*

$$\liminf_{k\to\infty} \int_\Omega g(x, w_k, Dw_k)\, dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} g(x, w, \lambda) d\nu_x(\lambda)\, dx,$$

provided that the negative part $g^-(x, w_k, Dw_k)$ *is equiintegrable.*

The next lemma describes limits points of gradients sequences by means of Young measures, which turns out to be an appropriate tool for overcoming the difficulty arising when the weak convergence does not behave as one desires.

Lemma 2.4 ([4])**.** *The following assertions hold.*

(1) *If the sequence* $\{Du_k\}$ *is bounded in* $L_M(\Omega;\mathbb{M}^{m\times n})$*, then there is a Young measure* ν_x *generated by* $\{Du_k\}$ *satisfying* $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ *and the weak* L^1*-limit of* Du_k *is* $\int_{\mathbb{M}^{m\times n}} \lambda\, d\nu_x(\lambda)$*.*

(2) ν_x *satisfies*

$$\langle \nu_x, id\rangle = Du(x) \quad \textit{for a.e.} \quad x \in \Omega.$$

Remark 2.5. Notice that a Young measure ν_x is called a W^1L_M-gradient Young measure if it is associated to a sequence of gradients $\{Dw_j\}$ such that $\{w_j\}$ is bounded in $W^1L_M(\Omega)$. It is called homogeneous if $\nu_x = \mu$ for a.e. $x \in \Omega$.

3. Galerkin approximation

Let $V_1 \subset V_2 \subset \cdots \subset W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\bigcup_{i\in\mathbb{N}} V_i$ is dense in $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$. The sequence (V_i) exists because $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$ is separable ($M \in \Delta_2$). We define the operator

$$T:\, W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m) \to W^{-1}_{\mathrm{div}}L_{\overline{M}}(\Omega;\mathbb{R}^m),$$
$$u \mapsto \left(w \mapsto \int_\Omega \sigma(x,u,Du) : Dw\,dx + \int_\Omega u\cdot\nabla u\cdot w\,dx - \langle f, w\rangle \right).$$

In the sequel, we will use a positive constant c which can change values from line to line.

Lemma 3.1. *For an arbitrary $u \in W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$, the functional $T(u)$ is linear and bounded.*

Proof. $T(u)$ is trivially linear for the arbitrary $u \in W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$. By the growth condition in (H1), $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m) \hookrightarrow L_M(\Omega;\mathbb{R}^m)$ and $P \ll M$, we have

$$\int_\Omega \overline{M}(|\sigma(x,u,Du)|)\,dx \le c\int_\Omega \big(\overline{M}(d_1(x)) + P(\gamma|u|) + M(\gamma|Du|)\big)\,dx < \infty.$$

Next, assume that

$$|u\otimes u| \le \overline{M}^{-1}P(|u|) + \overline{M}^{-1}M(|u|),$$

which gives

$$\int_\Omega \overline{M}(|u\otimes u|)\,dx \le c\int_\Omega \big(P(|u|) + M(|u|)\big)\,dx.$$

Then, by Hölder's inequality, it follows that

$$\begin{aligned}|\langle T(u), w\rangle| &= \left| \int_\Omega \sigma(x,u,Du) : Dw\,dx + \int_\Omega u\cdot\nabla u\cdot w\,dx - \langle f, w\rangle \right| \\ &\le 2|||\sigma(x,u,Du)|||_{\overline{M}}\,\|Dw\|_M + \int_\Omega |u\cdot\nabla u\cdot w|\,dx + 2\|f\|_{-1,\overline{M}}\,\|w\|_{1,M}.\end{aligned}$$

Since

$$\int_\Omega |u\cdot\nabla u\cdot w| dx = \int_\Omega |(u\otimes u)\cdot\nabla w|\,dx \le c\|u\otimes u\|_{\overline{M}}\|Dw\|_M,$$

we get

$$|\langle T(u), w\rangle| \le c\|w\|_{1,M}.$$

This implies that $T(u)$ is well-defined and bounded. □

Lemma 3.2. *The restriction of T to a finite linear subspace V of $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$ is continuous.*

Proof. Let r be the dimension of a subspace V of $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$, $(e_i)_{i=1}^r$ be a basis of V and let $(u_k = a^i_k e_i)$ be a sequence in V which converges to $u = a^i e_i$ in V (with conventional summation). Then (a_k) converges to a in $\mathbb{R}^r$ and $u_k \to u$ and $Du_k \to Du$ a. e. On the other hand, $\|u_k\|_M$ and $\|Du_k\|_M$ are bounded by a constant c. Thus, the continuity assumption in (H0) allows one to deduce that $\sigma(x,u_k,Du_k) : Dw \to \sigma(x,u,Du) : Dw$ a. e. Also, $(u_k \otimes u_k)\cdot\nabla w \to (u\otimes u)\cdot\nabla w$ a. e. Hence, by the growth condition in (H1), the Hölder inequality and the Vitali theorem, it follows for $w \in W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$ that

$$
\begin{aligned}
\|T(u_k) - T(u)\|_{-1,\overline{M}} &= \sup_{\|w\|_{1,M}\equiv 1} |\langle T(u_k), w\rangle - \langle T(u), w\rangle| \\
&= \sup_{\|w\|_{1,M}\equiv 1} \Big| \int_\Omega \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : Dw dx \\
&\quad + \int_\Omega (u_k\otimes u_k - u\otimes u)\cdot \nabla w dx \Big| \\
&\le c\big(\big\||\sigma(x,u_k,Du_k) - \sigma(x,u,Du)|\big\|_{\overline{M},\Omega} + \|u_k\otimes u_k - u\otimes u\|_{\overline{M},\Omega}\big) \le c. \qquad \square
\end{aligned}
$$

We fix some k and assume that the dimension of V_k is r and $e_1,\dots,e_r$ is a basis of V_k. For simplicity, we write $\sum_{1\le i\le r} a_i e_i = a_i e_i$ and define the map

$$
G : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(a_i e_i), e_1\rangle \\ \langle T(a_i e_i), e_2\rangle \\ \vdots \\ \langle T(a_i e_i), e_r\rangle \end{pmatrix}.
$$

Lemma 3.3. *G is continuous and*

$$
G(a)\cdot a \to +\infty \quad \textit{as} \quad \|a\|_{\mathbb{R}^r} \to +\infty.
$$

Proof. Let $u_j = a^i_j e_i \in V_k$, $u_0 = a^i_0 e_i \in V_k$. Since T is continuous on a finite dimensional subspace and

$$
\begin{aligned}
|(G(a_j) - G(a))_l| &= |\langle T(a^i_j e_i) - T(a^i_0 e_i), e_l\rangle| \\
&\le \|T(u_j) - T(u_0)\|_{-1,\overline{M}} . \|e_l\|_{1,M},
\end{aligned}
$$

it follows that G is continuous.

Now take $u = a_i e_i \in V_k$. Then $\|a\|_{\mathbb{R}^r} \to +\infty$ is equivalent to $\|u\|_{1,M} \to +\infty$ and

$$
G(a)\cdot a = \langle T(a_i e_i), a_i e_i\rangle = \langle T(u), u\rangle.
$$

The coercivity condition in (H1) implies

$$
I \equiv \int_\Omega \sigma(x,u,Du) : Du\, dx \ge \alpha \int_\Omega M(|Du|)\, dx - c.
$$

Next, observe that

$$II \equiv \int_\Omega u \cdot \nabla u \cdot u \, dx = \frac{1}{2} \int_\Omega u^j \frac{\partial}{\partial x_j} |u|^2 \, dx = -\frac{1}{2} \int_\Omega \operatorname{div} u |u|^2 \, dx = 0,$$

by the condition (1.2). Finally, from Young's inequality and (2.1), we have

$$\begin{aligned} III \equiv \int_\Omega |fu| \, dx &= \frac{\alpha}{2\theta} \int_\Omega \frac{2\theta}{\alpha} |fu| \, dx \\ &\leq \frac{\alpha}{2\theta} \int_\Omega \overline{M} \left(\frac{2\theta}{\alpha} |f| \right) dx + \frac{\alpha}{2\theta} \int_\Omega M(|u|) \, dx \\ &\leq c + \frac{\alpha}{2} \int_\Omega M(|Du|) \, dx. \end{aligned}$$

Hence,

$$\begin{aligned} G(a) \cdot a = \langle T(u), u \rangle &\geq \alpha \int_\Omega M(|Du|) \, dx - \frac{\alpha}{2} \int_\Omega M(|Du|) \, dx - c \\ &= \frac{\alpha}{2} \int_\Omega M(|Du|) \, dx - c \to +\infty \quad \text{as } \|u\|_{1,M} \to +\infty, \end{aligned}$$

that is, T is coercive. □

Lemma 3.4. *For all $k \in \mathbb{N}$, there exists $u_k \in V_k$ such that*

$$\langle T(u_k), w \rangle = 0 \quad \textit{for all} \quad w \in V_k.$$

Proof. By Lemma 3.3, we have $G(a) \cdot a \to +\infty$ as $\|a\|_{\mathbb{R}^r} \to +\infty$. Then there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a) \cdot a > 0$. The usual topological argument [24] gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k \in \mathbb{N}$, there exists $u_k \in V_k$ such that $\langle T(u_k), w \rangle = 0$ for all $w \in V_k$. □

As a consequence of Lemma 3.4, the sequence (u_k) is uniformly bounded in $W^1_{0,\mathrm{div}} L_M(\Omega; \mathbb{R}^m)$. To see this, suppose that (u_k) is not uniformly bounded. Since T is coercive, then there is $R > 0$ for which $\langle T(u), u \rangle > 1$ whenever $\|u\|_{1,M} > R$. This gives a contradiction with the Galerkin approximation u_k which satisfies Lemma 3.4.

According to Lemma 2.1, there exists a Young measure ν_x generated by Du_k in $L_M(\Omega; \mathbb{M}^{m\times n})$ satisfying the properties of Lemma 2.4.

4. Div-curl inequality

The following lemma is the key step to passing to the limit in the approximating equations and proving that the weak limit u of the Galerkin approximations u_k is a solution of (1.1)–(1.3).

Lemma 4.1 (div-curl inequality)**.** *Assume that Du_k generates a Young measure ν_x. Then the following inequality holds:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x, u, \lambda) - \sigma(x, u, Du)\big) : (\lambda - Du) d\nu_x(\lambda) \, dx \leq 0. \tag{4.1}$$

Proof. We consider the sequence

$$\begin{aligned} I_k :&= \big(\sigma(x,u_k,Du_k)-\sigma(x,u,Du)\big) : (Du_k - Du) \\ &= \sigma(x,u_k,Du_k) : (Du_k - Du) - \sigma(x,u,Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}. \end{aligned}$$

Let us start with the sequence $I_{k,2}$. Since

$$\int_\Omega \overline{M}(|\sigma(x,u,Du)|)\,dx \leq c\int_\Omega \big(\overline{M}(d_1(x)) + P(\gamma|u|) + M(\gamma|Du|)\big)\,dx < \infty$$

by the growth condition in (H1) and $P \ll M$, then $\sigma \in L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$. It follows according to Lemma 2.4 that

$$\begin{aligned} \liminf_{k\to\infty}\int_\Omega I_{k,2}\,dx &= \int_\Omega\int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)d\nu_x(\lambda)\,dx \\ &= \int_\Omega \sigma(x,u,Du) : \left(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - Du\right)dx = 0. \end{aligned}$$

For the sequence $I_{k,1}$, take a measurable subset $\Omega' \subset \Omega$, and by the Hölder inequality we have

$$\int_{\Omega'} |\sigma(x,u_k,Du_k) : Du_k|\,dx \leq 2\big\||\sigma(x,u_k,Du_k)|\big\|_{\overline{M},\Omega'}\left(\int_{\Omega'} M(|Du_k|)\right)dx.$$

Since $\{u_k\}$ is bounded in $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$, then, by the growth condition in (H1) and $W^1_0L_M(\Omega) \hookrightarrow L_M(\Omega)$,

$$\int_\Omega \overline{M}(|\sigma(x,u_k,Du_k)|)\,dx \leq c\int_\Omega \overline{M}(d_1(x)) + P(\gamma|u_k|) + M(\gamma|Du_k|)\,dx < c.$$

Thus $\big\||\sigma(x,u_k,Du_k)|\big\|_{\overline{M},\Omega'}$ is bounded. Note that the term $\int_{\Omega'} M(|Du|)\,dx$ is arbitrarily small if the measure of Ω' is chosen small enough. Consequently, the equiintegrability of $I^-_{k,1}$ follows. Since (u_k) is uniformly bounded in $W^1_{0,\mathrm{div}}L_M(\Omega;\mathbb{R}^m)$, then $u_k \to u$ in $L_M(\Omega;\mathbb{R}^m)$ (up to a subsequence). Hence,

$$\begin{aligned} \int_\Omega M(|u_k - u|)dx &\geq \int_{\{x\in\Omega;|u_k-u|\geq\epsilon\}} M(|u_k - u|)\,dx \\ &\geq c\int_{\{x\in\Omega;|u_k-u|\geq\epsilon\}} |u_k - u|\,dx \\ &\geq c\epsilon\big|\{x\in\Omega; |u_k - u| \geq \epsilon\}\big| \end{aligned}$$

for some positive ϵ, and c is the constant of the embedding $L_M \subset L^1$. Therefore, $u_k \to u$ in measure. By virtue of Lemma 2.3, one gets

$$I := \liminf_{k\to\infty}\int_\Omega I_k\,dx = \liminf_{k\to\infty}\int_\Omega I_{k,1}\,dx$$

$$
\begin{aligned}
&= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du)\,dx \\
&\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du) d\nu_x(\lambda)\,dx.
\end{aligned}
$$

We will see next that $I \leq 0$. Define $\operatorname{dist}(u,V_k) = \inf_{v\in V_k} \|u-v\|_{1,M}$ and fix $\epsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $\operatorname{dist}(u,V_k) < \epsilon$ for all $k > k_0$, or, equivalently,

$$
\operatorname{dist}(u_k - u, V_k) = \inf_{v\in V_k} \|u_k - u - v\|_{1,M} = \inf_{w\in V_k} \|u-w\|_{1,M} = \operatorname{dist}(u,V_k) < \epsilon
$$

for any $k > k_0$. Then, for $v_k \in V_k$, we can estimate I as follows:

$$
\begin{aligned}
I &= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : (Du_k - Du)\,dx \\
&= \liminf_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : D(u_k - u - v_k) + \sigma(x,u_k,Du_k) : Dv_k\,dx \\
&\leq \liminf_{k\to\infty} \Big(2\big\||\sigma(x,u_k,Du_k)|\big\|_{\overline{M},\Omega} \|D(u_k - u - v_k)\|_{M,\Omega} \\
&\qquad + \langle f, v_k\rangle - \int_\Omega (u_k \otimes u_k).\nabla v_k\,dx \Big).
\end{aligned}
$$

The term $\big\||\sigma(x,u_k,Du_k)|\big\|_{\overline{M},\Omega}$ is uniformly bounded in k by the growth condition in (H1). On the other hand, by choosing $v_k \in V_k$ in such a way that $\|u_k - u - v_k\|_{1,M} < 2\epsilon$ for any $k > k_0$, the term $\|D(u_k - u - v_k)\|_{M,\Omega}$ is bounded by 2ϵ. Furthermore, we have

$$
\begin{aligned}
|\langle f, v_k\rangle| &= |\langle f, v_k - (u_k - u)\rangle + \langle f, u_k - u\rangle| \\
&\leq |\langle f, v_k - (u_k - u)\rangle| + |\langle f, u_k - u\rangle| \\
&\leq 2\epsilon \|f\|_{-1,\overline{M}} + o(k)
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \int_\Omega (u_k \otimes u_k)\nabla v_k dx \right| &= \left| \int_\Omega (u_k \otimes u_k) \cdot \big(\nabla(v_k - u_k + u_k)\big)\,dx \right| \\
&\leq \underbrace{\int_\Omega |(u_k \otimes u_k) \cdot \nabla u_k|\,dx}_{=0} + \int_\Omega |(u_k \otimes u_k) \cdot \nabla(v_k - u_k)|\,dx \\
&\leq \int_\Omega |(u_k \otimes u_k) \cdot \nabla(v_k - u)|\,dx + \int_\Omega |(u_k \otimes u_k) \cdot \nabla(u - u_k)|\,dx \\
&\leq 2\|u_k \otimes u_k\|_{\overline{M},\Omega} \Big[\|D(v_k - u)\|_{M,\Omega} + \|D(u - u_k)\|_{M,\Omega} \Big]. \qquad (4.2)
\end{aligned}
$$

Similarly to the proof of Lemma 3.1, we have $\|u_k \otimes u_k\|_{\overline{M}}$ is bounded since (u_k) is bounded. Hence, the right-hand side in (4.2) tends to zero as $k \to +\infty$. Since ϵ is arbitrary, this proves that $I \leq 0$. Note that

$$
\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,Du) : (\lambda - Du) d\nu_x(\lambda)\,dx = 0
$$

together with $I \leq 0$, equation (4.1) follows. □

Remark 4.2. The naming "div-curl inequality" can be explained in the following way. Suppose for a moment that $\operatorname{div}\sigma(x,u_k,Du_k)=0$ for all k and that $\sigma(x,u_k,Du_k):Du_k$ is equiintegrable. Hence, the weak limit of $\sigma(x,u_k,Du_k):Du_k$ in $L^1(\Omega)$ is given by $\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda):\lambda d\nu_x(\lambda)$. Furthermore, by the usual div-curl lemma, it follows that $\int_\Omega\sigma(x,u_k,Du_k):Du_k dx$ converges to $\int_\Omega\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda):\lambda d\nu_x(\lambda)$ and then the lemma follows with the equality.

Lemma 4.3. *If equation* (4.1) *holds, then* ν_x *satisfies*

$$\big(\sigma(x,u,\lambda)-\sigma(x,u,Du)\big):(\lambda-Du)=0\quad on\ \operatorname{supp}\nu_x.$$

Proof. We have

$$\int_\Omega\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,u,\lambda)-\sigma(x,u,Du)\big):(\lambda-Du)d\nu_x(\lambda)\,dx\le 0.$$

By the monotonicity of σ, the above integrand is nonnegative. Hence, it must vanish with respect to the product measure $d\nu_x(\lambda)\otimes dx$. It follows for a. e. $x\in\Omega$ that

$$\big(\sigma(x,u,\lambda)-\sigma(x,u,Du)\big):(\lambda-Du)=0\quad\text{on}\ \operatorname{supp}\nu_x.$$ □

5. Proof of Theorem 1.1

Now we can prove Theorem 1.1 by considering the conditions (a)–(d) listed in (H2).

Case(a). By ∇, we denote the derivative with respect to the third variable of σ. We claim that for a. e. $x\in\Omega$ and all $\mu\in\mathbb{M}^{m\times n}$,

$$\sigma(x,u,\lambda):\mu=\sigma(x,u,Du):\mu+\big(\nabla\sigma(x,u,Du)\mu\big):(Du-\lambda)$$

holds on $\operatorname{supp}\nu_x$. Due to the monotonicity of σ, we have for all $\tau\in\mathbb{R}$,

$$\big(\sigma(x,u,\lambda)-\sigma(x,u,Du+\tau\mu)\big):(\lambda-Du-\tau\mu)\ge 0.$$

By virtue of Lemma 4.3, we have

$$\begin{aligned}-\sigma(x,u,\lambda):\tau\mu&\ge-\sigma(x,u,Du):(\lambda-Du)+\sigma(x,u,Du+\tau\mu):(\lambda-Du-\tau\mu)\\&=\tau\big[(\nabla\sigma(x,u,Du)\mu):(\lambda-Du)-\sigma(x,u,Du):\mu\big]+o(\tau),\end{aligned}$$

where we have used the fact that

$$\sigma(x,u,Du+\tau\mu)=\sigma(x,u,Du)+\nabla\sigma(x,u,Du)\tau\mu+o(\tau).$$

Since τ is arbitrary in $\mathbb{R}$, our claim follows for all $\mu\in\operatorname{supp}\nu_x$. As $\{\sigma(x,u_k,Du_k)\}$ is bounded and equiintegrable, then its weak L^1-limit is given by

$$\overline{\sigma}=\int_{\operatorname{supp}\nu_x}\sigma(x,u,\lambda)d\nu_x(\lambda).$$

Therefore, due to our claim, one gets

$$\overline{\sigma} = \int_{\operatorname{supp}\nu_x} \sigma(x,u,Du)\,d\nu_x(\lambda) + \big(\nabla\sigma(x,u,Du)\big)^t \int_{\operatorname{supp}\nu_x} (Du-\lambda)\,d\nu_x(\lambda) = \sigma(x,u,Du).$$

It follows from the reflexivity of $L_{\overline{M}}(\Omega)$ that $\{\sigma(x,u_k,Du_k)\}$ is weakly convergent in $L_{\overline{M}}(\Omega)$ and its weak limit is $\sigma(x,u,Du)$.

Case(b). Let show that for a. e. $x\in\Omega$, $\operatorname{supp}\nu_x\subset K_x$, where

$$K_x := \{\lambda\in\mathbb{M}^{m\times n} : W(x,u,\lambda) = W(x,u,Du) + \sigma(x,u,Du) : (\lambda - Du)\}.$$

If $\lambda\in\operatorname{supp}\nu_x$, then Lemma 4.3 implies

$$(1-\tau)\big(\sigma(x,u,\lambda) - \sigma(x,u,Du)\big) : (\lambda - Du) = 0 \quad \text{for all } \tau\in[0,1]. \tag{5.1}$$

Due to the monotonicity of σ, one has

$$(1-\tau)\big(\sigma(x,u,Du+\tau(\lambda-Du)) - \sigma(x,u,\lambda)\big) : (Du-\lambda) \geq 0. \tag{5.2}$$

Subtracting (5.1) from (5.2), we get

$$(1-\tau)\big(\sigma(x,u,Du+\tau(\lambda-Du)) - \sigma(x,u,Du)\big) : (Du-\lambda) \geq 0 \quad \text{for all } \tau\in[0,1],$$

which implies by the monotonicity of σ that

$$\big(\sigma(x,u,Du+\tau(\lambda-Du)) - \sigma(x,u,Du)\big) : (\lambda-Du) = 0.$$

Thus,

$$\sigma(x,u,Du) : (\lambda-Du) = \sigma(x,u,Du+\tau(\lambda-Du)) : (\lambda-Du) \quad \text{for all } \tau\in[0,1].$$

Due to (H2)(b), we have then

$$W(x,u,\lambda) = W(x,u,Du) + \sigma(x,u,Du) : (\lambda-Du).$$

Hence $\lambda\in K_x$. The convexity of W allows one to write

$$W(x,u,\lambda) \geq W(x,u,Du) + \sigma(x,u,Du) : (\lambda-Du). \tag{5.3}$$

Put $A(\lambda)$ (respectively, $B(\lambda)$) the left- (respectively, the right-) hand side in (5.3). By the continuity and differentiability of $\lambda\mapsto A(\lambda)$, it follows that for $\rho\in\mathbb{M}^{m\times n}$ and $\tau\in\mathbb{R}$,

$$\begin{aligned} \frac{A(\lambda+\tau\rho)-A(\lambda)}{\tau} &\geq \frac{B(\lambda+\tau\rho)-B(\lambda)}{\tau} && \text{if } \tau>0,\\ \frac{A(\lambda+\tau\rho)-A(\lambda)}{\tau} &\leq \frac{B(\lambda+\tau\rho)-B(\lambda)}{\tau} && \text{if } \tau<0. \end{aligned}$$

Thus $DA = DB$, which gives

$$\sigma(x,u,\lambda) = \sigma(x,u,Du) \quad \text{for all } \lambda \in K_x \supset \operatorname{supp}\nu_x.$$

Hence,

$$\begin{aligned} \overline{\sigma} &= \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)\, d\nu_x(\lambda) \\ &= \int_{\operatorname{supp}\nu_x} \sigma(x,u,Du)\, d\nu_x(\lambda) = \sigma(x,u,Du). \end{aligned} \tag{5.4}$$

Now consider the Carathéodory function $g(x,u,\rho) = |\sigma(x,u,\rho) - \overline{\sigma}(x)|$. The sequence $g_k(x) := g(x,u_k(x),Du_k(x))$ is equiintegrable. Then

$$g_k \rightharpoonup \overline{g} \quad \text{weakly in } L^1(\Omega),$$

where

$$\begin{aligned} \overline{g}(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |\sigma(x,s,\lambda) - \overline{\sigma}(x)|\, d\delta_{u(x)}(s)\otimes d\nu_x(\lambda) \\ &= \int_{\mathbb{M}^{m\times n}} |\sigma(x,u,\lambda) - \overline{\sigma}(x)|\, d\nu_x(\lambda) = 0 \end{aligned}$$

by (5.4). As $g_k \geq 0$, then $g_k \to 0$ in $L^1(\Omega)$. Thus, for $v \in W^1_{0,\mathrm{div}} L_M(\Omega;\mathbb{R}^m)$,

$$\int_\Omega \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : Dv\, dx \to 0 \quad \text{as} \quad k\to\infty.$$

Case(c). Due to the strict monotonicity and Lemma 4.3, it follows that ν_x is a Dirac measure. Assume that $\nu_x = \delta_{\varphi(x)}$. Then

$$\varphi(x) = \int_{\mathbb{M}^{m\times n}} \lambda\, d\delta_{\varphi(x)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda\, d\nu_x(\lambda) = Du(x).$$

Thus $\nu_x = \delta_{Du(x)}$. According to Lemma 2.2, we have

$$Du_k \to Du \quad \text{in measure as} \quad k\to\infty,$$

which implies $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ a. e. Since $\sigma(x,u_k,Du_k)$ is bounded and equiintegrable (by the growth condition in (H1)), it follows by the Vitali theorem that $\sigma(x,u_k,Du_k)$ converges to $\sigma(x,u,Du)$ in $L^1(\Omega)$.

Case(d). Suppose that ν_x is not a Dirac mass on $\Omega' \subset \Omega$ of positive measure. On the one hand, due to the assumption (d), we have for a.e. $x \in \Omega'$,

$$\begin{aligned} &\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda\, d\nu_x(\lambda) \\ &\qquad > \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \overline{\lambda}\, d\nu_x(\lambda) + \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\, d\nu_x(\lambda) \\ &\qquad = \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du\, d\nu_x(\lambda), \end{aligned}$$

where we have used $\overline{\lambda} = Du(x)$ and

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\, d\nu_x(\lambda) = \sigma(x,u,\overline{\lambda}) : \int_{\mathbb{M}^{m\times n}} \lambda\, d\nu_x(\lambda) - \sigma(x,u,\overline{\lambda}) : \overline{\lambda} \int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda) = 0.$$

On the other hand, by virtue of Lemma 4.1, we deduce

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du\, d\nu_x(\lambda)\, dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)\, dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Du\, d\nu_x(\lambda)\, dx,$$

which is a contradiction. Consequently, $\nu_x = \delta_{Du(x)}$ for a.e. $x \in \Omega$. We follow then the proof of the **Case (c)**.

In conclusion, let $v \in W^1_{0,\mathrm{div}} L_M(\Omega;\mathbb{R}^m)$. Since $\bigcup_{i\in\mathbb{N}} V_i$ is dense in $W^1_{0,\mathrm{div}} L_M(\Omega;\mathbb{R}^m)$, there exists a sequence $v_k \in \bigcup_{i\in\mathbb{N}} V_i$ such that $v_k \to v$ in $W^1_{0,\mathrm{div}} L_M(\Omega;\mathbb{R}^m)$ as $k \to \infty$. We have

$$\begin{aligned}\langle T(u_k), v_k\rangle - \langle T(u), v\rangle &= \int_\Omega \sigma(x,u_k,Du_k) : Dv_k\, dx + \int_\Omega (u_k\cdot\nabla u_k)v_k\, dx - \langle f, v_k\rangle \\ &\quad - \int_\Omega \sigma(x,u,Du) : Dv dx - \int_\Omega (u\cdot\nabla u)v dx + \langle f, v\rangle \\ &= \int_\Omega \sigma(x,u_k,Du_k) : (Dv_k - Dv)\, dx \\ &\quad + \int_\Omega \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : Dv\, dx \\ &\quad + \int_\Omega \big(u_k\cdot\nabla u_k)v_k - (u.\nabla u)v\big)\, dx - \langle f, v_k - v\rangle.\end{aligned}$$

According to all cases in (H2) and $u_k\cdot\nabla u_k \to u\cdot\nabla u$, the right-hand side of the above equality tends to zero as k tends to infinity. By virtue of Lemma 3.4, it follows that

$$\langle T(u), v\rangle = 0 \quad \text{for all} \quad v \in W^1_{0,\mathrm{div}} L_M(\Omega;\mathbb{R}^m).$$

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Про стаціонарні течії квазіньютонівських рідин у просторах Орлича–Соболева

Farah Balaadich and Elhoussine Azroul

Стаття присвячена дослідженню існуванню слабких розв'язків для стаціонарних квазіньютонівських течій за допомогою наближень Гальоркіна і розв'язків у просторах мір, а саме, мір Янга, які виявилися хорошим інструментом для опису слабких розв'язків нашої задачі в просторах Орлича.

Ключові слова: квазіньютонівська рідина, простори Орлича, слабка монотонність, слабкий розв'язок, міри Янга