Journal of Mathematical Physics, Analysis, Geometry
2021, Vol. 17, No. 3, pp. 295–325
doi: https://doi.org/10.15407/mag17.03.295

A Note on a Damped Focusing Inhomogeneous Choquard Equation

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This paper is devoted to the focusing inhomogeneous Choquard equation with linear damping:

$$i\dot{u} + \triangle u + iau = -|x|^{-\gamma}(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{on } \mathbb{R}^N,$$

where $a \geq 0$ and $0 < \gamma < \inf(N, 2+\alpha)$. Global existence and scattering are proved for sufficiently large damping. For arbitrary damping, global existence of solutions is obtained if the initial data belong to some invariant sets.

Key words: damped Choquard equation, large damping, global existence, scattering, invariant sets

Mathematical Subject Classification 2010: 35Q55

1. Introduction

This manuscript is concerned with the following damped focusing nonlinear Schrödinger problem of Choquard type with inhomogeneous nonlinear term:

$$\begin{cases} i\dot{u} + \triangle u + iau = -|x|^{-\gamma}(I_\alpha * |u|^p)|u|^{p-2}u, \\ u(0,\cdot) = \psi, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ for some $N \geq 3$, $a \geq 0$, $0 < \alpha < N$, $p > 1$ and $0 < \gamma < \inf(N, 2+\alpha)$. The Riesz potential is defined on $\mathbb{R}^N$ by

$$I_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|\cdot|^{N-\alpha}} := \frac{\mathcal{K}}{|\cdot|^{N-\alpha}}.$$

The free operator associated to the damped Schrödinger equation stands for

$$U_a(t)\psi := e^{-at}\mathcal{F}^{-1}(e^{-it|\cdot|^2})\psi, \quad \psi \in H^1(\mathbb{R}^N).$$

When $a = 0$, equation (1.1) has several origins such as quantum mechanics [13], the Hartree–Fock theory to describe an electron trapped in its own hole [15] and non-relativistic quantum theory [12]. In [20], equation (1.1) is used to describe self-gravitating matter in a programme in which quantum state reduction is understood as a gravitational phenomenon. Recently, problem (1.1)($a = 0$ and $2 \leq p < \frac{N+\alpha-\gamma}{N-2}$) has been widely studied. Indeed, in [5, 8, 10, 21, 22], the authors discussed local and global well-posedness, existence of blow-up solutions, scattering and strong instability of standing waves for some Schrödinger–Choquard equation.

Before we proceed to the discussion, it is useful to look at the most vital symmetry which is scaling symmetry. Indeed, the first equation in (1.1) enjoys the following scaling invariance:

$$u_\lambda(t) = \lambda^{\frac{\alpha+2-\gamma}{2(p-1)}} u(\lambda^2 t, \lambda\cdot), \quad \lambda > 0.$$

For a real number μ, we have

$$\|u_\lambda(t)\|_{\dot{H}^\mu} = \lambda^{\mu - \frac{N}{2} + \frac{\alpha+2-\gamma}{2(p-1)}} \|u(\lambda^2 t, \lambda\cdot)\|_{\dot{H}^\mu}.$$

So, the critical exponent is

$$s_c := \frac{N}{2} - \frac{\alpha+2-\gamma}{2(p-1)},$$

for which the $\dot{H}^\mu$ norm is unaffected by scaling. The case $s_c = 0$ corresponds to the mass critical exponent $p_* = 1 + \frac{\alpha+2-\gamma}{N}$. The energy critical case $s_c = 1$ corresponds to $p^* = 1 + \frac{\alpha+2-\gamma}{N-2}$. For smaller p, that is, $p \in]1, p^*[$, which is called the energy subcritical exponent, contracting time reduces the size of the $\dot{H}^1$ norm. This is the effect that will be exploited to build up solutions. For any solutions to (1.1), let us define the following quantities called mass and energy:

$$\begin{aligned} M(u(t)) &:= \int_{\mathbb{R}^N} |u(t)|^2 dx, \\ E(u(t)) &:= \int_{\mathbb{R}^N} \{|\nabla u(t)|^2 - \frac{1}{p}|x|^{-\gamma}(I_\alpha * |u|^p)|u(t)|^p\} dx. \end{aligned}$$

The standard damped Schrödinger equation

$$i\dot{u} + \triangle u + iau = -|u|^{2(p-1)}u \tag{1.2}$$

arises in various areas of nonlinear optics, plasma physics and fluid mechanics, see [1, 2, 9, 11, 26, 27]. In [18], M. Ohta and G. Todorova established that the Cauchy problem associated to (1.2) is well posed in the energy space and the solution is global for large damping. For other modifications of the classical equation (1.2), see also [6, 7, 24, 25]. It is thus quite natural to complete the nonlinear Choquard equation by a linear dissipative term to take into account some dissipation phenomena. This paper seems to be the first to treat the well-posedness issues for the damped inhomogeneous Schrödinger–Choquard problem (1.1).

The aim of this note is to prove that large damping prevents finite-time blow-up of solutions. Indeed, global existence and scattering are proved if the dissipation coefficient is sufficiently large. For arbitrary damping, global existence of solutions is obtained when the initial data belong to some invariant sets.

This paper is organized as follows: Section two summarizes the main results and gives some technical tools needed in the sequel. In Section three, we prove some inhomogeneous Gagliardo–Nirenberg inequality adapted to the above problem. Section four is devoted to establishing the existence of ground state for the standard stationary problem related to (1.1). In Section five, we prove that (1.1) is locally well-posed. In Section six, global existence for large damping is shown. Scattering of such global solutions is obtained in Section seven. In the last Section, without any assumption on the size of damping, we obtain global existence via some stable sets.

We close this section with some notations. We consider the Lebesgue spaces $L^r := L^r(\mathbb{R}^N)$ equipped with the norms $\|f\|_r := \|f\|_{L^r} = (\int_{\mathbb{R}^N} |f(x)|^r dx)^{\frac{1}{r}}$ if $r < \infty$, else $\|f\|_\infty := \|f\|_{L^\infty} = \sup\mathrm{ess}_{x\in\mathbb{R}^N} |f(x)|$. For the vector valued functions $\|(f_j)\|_r := \sup_j \|f_j\|_r$. When $r = 2$, let $\|f\| := \|f\|_2$. The usual inhomogeneous Sobolev space is denoted by $W^{1,r} := W^{1,r}(\mathbb{R}^N)$ and endowed with the complete norm $\|f\|_{W^{1,r}} := (\|f\|^r + \|\nabla f\|^r)^{\frac{1}{r}}$. In the case $r = 2$, we denote $H^1 := W^{1,2}$ which is equipped with $\|f\|_{H^1} := (\|f\|^2 + \|\nabla f\|^2)^{\frac{1}{2}}$. If X is an abstract space, the set of continuous functions defined on $[0, T[$ and valued in X is denoted by $C_T(X) := C([0, T), X)$, if necessary the interval of time may be closed. Also, we denote $L^q_I(X) := L^q(I, X)$ where I is an interval of $\mathbb{R}$. The set X_{rd} stands for the set of radial elements in X. Constants will be denoted by C which may vary from line to line. For simplicity, let $\int f(x)\,dx := \int_{\mathbb{R}^N} f(x)\,dx$ and $\int f(x, y)\,dx\,dy := \int\int f(x, y)\,dx\,dy$. Finally, if A and B are non-negative quantities, we write $A \lesssim B$ to denote $A \le CB$, if $A \le \varepsilon B$, we write $A = \circ(B)$ and $A \sim B$ if $A = B + \circ(B)$.

2. Main results and background

Let us introduce at first some relevant quantities. For $w > 0$ and $v \in H^1$, one denotes

$$S_w(v) : = w\|v\|^2 + \|\nabla v\|^2 - \frac{1}{p}\int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p\,dx,$$

$$K_w(v) = w\|v\|^2 + \|\nabla v\|^2 - \int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p\,dx,$$

and then defines

$$H_w(v) := (S_w - K_w)(v).$$

Also, let $B := Np - N - \alpha + \gamma$ and $A := 2p - B$, then denote

$$J(v) := \frac{\|\nabla v\|^B \|v\|^A}{\int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p\,dx}, \quad v \in H^1 - \{0\}.$$

Next, we define the so-called energy subcritical ground state solution of problem (1.1).

Definition 2.1. Any solution $\phi \in H^1 - \{0\}$ of

$$\triangle\phi - w\phi + |x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = 0, \tag{2.1}$$

which minimizes the problem

$$m_w := \inf_{v\in H^1-\{0\}}\{S_w(v) \text{ s.t } K_w(v) = 0\}, \tag{2.2}$$

is called the ground state of problem (1.1).

Also, we give the definition of admissible pairs.

Definition 2.2. A pair of real numbers (q, r), which satisfies

$$2 \le q, r \le \infty, \quad (q, r) \neq (2, \infty) \quad \text{and} \quad N\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{2}{q},$$

is said to be admissible and is denoted by $(q, r) \in \Gamma$.

To close this introduction, we consider

$$Q(X) := 2(N-2)X^2 - (3(N-2) + 2(\alpha - \gamma + 1))X + (N-2) + \alpha - \gamma.$$

Elementary computations prove that $Q(p^*) > 0$ and $Q(1 + \frac{\alpha-\gamma}{N}) < 0$. Then Q admits two distinguished real roots $p^-_{\alpha-\gamma,N} < p^+_{\alpha-\gamma,N}$ such that

$$p^-_{\alpha-\gamma,N} < 1 + \frac{\alpha-\gamma}{N} < p^+_{\alpha-\gamma,N} < p^*.$$

2.1. Main results. First, the existence of ground states to (1.1) is obtained, the question of uniqueness is not treated.

Proposition 2.3. *Let $N \ge 3$ and $1 + \frac{\alpha-\gamma}{N} < p < p^*$. Then there exists a ground state solution to (2.1) and (2.2).*

Second, the best constant of inhomogeneous Gagliardo–Nirenerg inequality related to problem (1.1) is investigated.

Proposition 2.4. *Let $N \ge 3$ and $1 + \frac{\alpha-\gamma}{N} < p < p^*$. Then*

- *there exists $C(N, p, \alpha, \gamma) > 0$ such that*

$$\forall v \in H^1, \int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p dx \le C(N, p, \alpha, \gamma)\|v\|^A\|\nabla v\|^B; \tag{2.3}$$

- *the minimization problem*

$$\frac{1}{C(N, p, \alpha, \gamma)} := \inf_{v\in H^1-\{0\}} J(v)$$

is attained in some $Q \in H^1$ satisfying $C(N, p, \alpha, \gamma) = \int |x|^{-\gamma}(I_\alpha * |Q|^p)|Q|^p dx$ and

$$-B\triangle Q + AQ - \frac{2p}{C(N, p, \alpha, \gamma)}|x|^{-\gamma}(I_\alpha * |Q|^p)|Q|^{p-2}Q = 0; \tag{2.4}$$

- *moreover, there is ϕ, a ground state solution to (2.1), such that*

$$C(N,p,\alpha,\gamma) = \frac{2p}{A}\Big(\frac{A}{B}\Big)^{\frac{B}{2}}\|\phi\|^{-2(p-1)}. \tag{2.5}$$

Let us state our third result, the Cauchy problem (1.1) is locally well-posed in the mass and energy spaces.

Theorem 2.5 (L^2-theory)**.** *Suppose $N \geq 3$, α, γ satisfy*

$$0 < \gamma < \inf(N, 2+\alpha), \quad N-2 < \alpha - \gamma,$$

and $2 \leq p < p_*$. Then, for any $\psi \in L^2$, there exist $T^* := T^*_{a,\psi} > 0$ and a unique maximal solution to problem (1.1) such that

$$u \in C_{T^*}(L^2) \cap L^q_{loc}([0,T^*), L^r) \quad \text{for any } (q,r) \in \Gamma.$$

Theorem 2.6 (H^1-theory)**.** *Suppose $N \geq 3$, α, γ satisfy*

$$0 < \gamma < \inf(N, 2+\alpha), \quad \max(\gamma+1, 2-\alpha+\gamma) < N < 4+\alpha-\gamma$$

and $2 \leq p < p^*$. Then, for any $\psi \in H^1$, there exist $T^* := T^*_{a,\psi} > 0$ and a unique maximal solution $u \in C_{T^*}(H^1)$ to problem (1.1). In addition, we have:

- *$u \in L^q_{loc}([0,T^*), W^{1,r})$ for any $(q,r) \in \Gamma$;*
- *$M(u(t)) = e^{-2at}M(\psi)$ and $\frac{d}{dt}S_w(u(t)) = -2aK_w(u(t))$ on $[0,T^*)$.*

Remark 2.7.

1. The assumption $p \geq 2$ is due to the contraction arguments used in the proof. This condition forces us to assume $p^* > 2$ which gives the condition $N-4 < \alpha-\gamma$. This restriction seems to be technical because the energy is well-defined for $1 + \frac{\alpha-\gamma}{N} < p < p^*$.

2. Non-existence of standing waves is a direct consequence of the mass decay.

Next, we show that global well-posedness of (1.1) holds for large damping.

Theorem 2.8. *Suppose $N \geq 3$, α, γ satisfy*

$$0 < \gamma < \inf(N, 2+\alpha), \quad \max(\gamma+1, 2-\alpha+\gamma) < N < 4+\alpha-\gamma,$$

and $p^+_{\alpha-\gamma,N} < p < p^*$. Assume $u \in C_{T^*}(H^1)$ to be the maximal solutions to (1.1) with the initial data $\psi \in H^1$. Then there exists a positive real number $a^* := a^*(\|\psi\|_{H^1})$ such that $T^*_{a,\psi} = \infty$ for all $a > a^*$.

Now we establish a scattering result about the global solutions given by Theorem 2.8.

Theorem 2.9. *Suppose $a > a^*$ and the assumptions of Theorem 2.8 hold. Assume $u \in C(\mathbb{R}_+, H^1)$ to be the global solutions to (1.1) with the initial data $\psi \in H^1$. Then there exists $u_+ \in H^1$ such that*

$$\lim_{t\to+\infty} \|u(t) - U_a(t)u_+\|_{H^1} = 0. \tag{2.6}$$

In addition, the scattering mapping $S : H^1 \to H^1, \psi \to u_+$ is continuous and one-to-one.

Following Ohta and Todorova [18], whatever the size of the damping is, we prove the existence of a global solution to (1.1) via some stable sets

$$A_w := \{v \in H^1 \text{ s.t } S_w(v) < m_w \text{ and } K_w(v) \geq 0\}.$$

Theorem 2.10. *Let $N \geq 3$, α, γ such that $0 < \gamma < \inf(N, 2+\alpha)$ and $1 + \frac{\alpha-\gamma}{N} < p < p^*$. Then the family $\cup_{w>0} A_w$ is invariant under the flow of (1.1), and if ψ belongs to this family, then the solution emanating from ψ is global.*

2.2. Tools We start first by using some classical Sobolev injections [16] which give a meaning to the energy. Also, various computations are done in this note.

Lemma 2.11. *Let $N \geq 3$. Then*

1. $H^1 \hookrightarrow L^q$ *for any* $q \in \left[2, \frac{2N}{N-2}\right]$;
2. *the injection* $H^1_{rd} \hookrightarrow L^q$ *is compact for any* $q \in \left(2, \frac{2N}{N-2}\right)$;
3. *for any* $r \in (1, N)$ *and* $q \in \left(r, \frac{Nr}{N-r}\right]$, *we have* $W^{1,r} \hookrightarrow L^q$;
4. *for any* $q \in \left[2, \frac{2N}{N-2}\right]$, *let* $\theta := N(\frac{1}{2} - \frac{1}{q})$, *and we have*

$$\|u\|_q \lesssim \|u\|^{1-\theta} \|\nabla u\|^\theta.$$

Recall the Hardy–Littlewood–Sobolev inequality [14].

Lemma 2.12. *Let $N \geq 3$, $0 < \lambda < N$, $1 < r, s < \infty$ and $f \in L^r$, $g \in L^s$. If $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then there exists $C_{N,s,\lambda} > 0$ such that*

$$\int \frac{f(x)g(y)}{|x-y|^\lambda}\, dx\, dy \leq C_{N,s,\lambda} \|f\|_r \|g\|_s.$$

Corollary 2.13. *Let $N \geq 3$, $0 < \alpha < N$, $1 < q, r, s < \infty$ and $f \in L^r, g \in L^s$. Then there exists $C_{N,s,\alpha} > 0$ such that*

1. *if* $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$, *then*

$$\int (I_\alpha * f)(x) g(y)\, dx\, dy \leq C_{N,s,\alpha} \|f\|_r \|g\|_s;$$

2. *if* $\frac{1}{q}+\frac{1}{r}+\frac{1}{s}=1+\frac{\alpha}{N}$, *then*

$$\|(I_\alpha * f)g\|_{q'} \le C_{N,s,\alpha}\|f\|_r\|g\|_s.$$

The second result obtained above is known as the Hardy–Littlwood–Paley inequality. The following result summarises some classical properties of the free damped Schrödinger kernel $U_a(t)$ [18].

Proposition 2.14. *We have:*

1. $U_a(t)\psi$ *is the solution to the linear problem associated to* (1.1)*;*
2. $U_a(t)\psi + i\int_0^t U_a(t-s)(I_\alpha * |\cdot|^\beta|u|^p)|x|^\beta|u|^{p-2}u\,ds$ *is the solution to* (1.1)*;*
3. $U_0(t)$ *is an isometry of* L^2.
4. $\|U_0(t)f\|_r \lesssim t^{-N(\frac{1}{2}-\frac{1}{r})}\|f\|_{r'}, 2\le r<\infty$*;*
5. $U_a(t)=e^{-at}U_0(t)$*;*
6. $U_a(t+s)=U_a(t)U_a(s)$*;*
7. $U_a(t)^* = U_{-a}(-t)$.

The Strichartz estimate from [4] is a standard tool to control the solutions of a Schrödinger equation in Lebesgue spaces.

Proposition 2.15. *Let* $N\ge 3$, $T>0$ *and* $\psi\in L^2$. *Then there exists* $C_N>0$ *such that*

$$\sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(L^r)} \le C_N\left(\|\psi\| + \inf_{(\tilde q,\tilde r)\in\Gamma}\|i\dot u+\triangle u\|_{L^{\tilde q'}_T(L^{\tilde r'})}\right). \tag{2.7}$$

Remark 2.16. The Strichartz inequality is compatible with truncations [21]. Indeed, if we have $i\dot u+\triangle u=h$ and $(q,r),(\tilde q,\tilde r),(\tilde q_1,\tilde r_1)\in\Gamma$, then there exists $C_{N,q,\tilde q}>0$ such that

$$\|u\|_{L^q_t(L^r)} \le C_{N,q,\tilde q}\left(\|\psi\| + \|h\|_{L^{\tilde q'}_t(L^{\tilde r'}(|x|<1))} + \|h\|_{L^{\tilde q_1'}_t(L^{\tilde r_1'}(|x|>1))}\right).$$

Using Proposition 2.14 and the one-dimensional Riesz potential inequality, we give some modified Strichartz estimates which will be proved in Appendix.

Proposition 2.17. *Let* $T>0$, $N\ge 3$ *and* $2<r<\frac{2N}{N-2}$. *Take* $\theta,\mu\in(1,+\infty)$ *such that*

$$N\left(\frac{1}{2}-\frac{1}{r}\right)=\frac{1}{\theta}+\frac{1}{\mu}.$$

Then there exists $C_{N,r,\theta}>0$ *such that*

$$\left\|\int_0^t U_a(t-s)f(s)\,ds\right\|_{L^\theta_T(L^r)} \le C_{N,r,\theta}\|f\|_{L^{\mu'}_T(L^{r'})}. \tag{2.8}$$

Remark 2.18. The previous inequality is also valid without any restrictions of $f(s)$. In particular, one has

$$\left\| \int_0^t U_a(t-s)f(s)\,ds \right\|_{L_T^\theta(L^r(|x|<1))} \leq C_{N,r,\theta}\|f\|_{L_T^{\mu'}(L^{r'}(|x|<1))}.$$

Similarly, for the integrals on the set $|x|>1$. Moreover, taking $\theta=\mu=q$, we obtain

$$\left\| \int_0^t U_a(t-s)f(s)\,ds \right\|_{L_T^q(L^r)} \leq C_{N,q}\|f\|_{L_T^{q'}(L^{r'})}.$$

Corollary 2.19. *Let $T>0$, $N\geq 3$, $\psi\in L^2$ and $(q,r),(\tilde{q},\tilde{r})\in\Gamma$. Then there exists $C_{N,q,\tilde{q}}>0$ such that*

$$\|u\|_{L_T^q(L^r)} \leq C_{N,q,\tilde{q}}\left(\|\psi\| + \|i\dot{u}+\triangle u+iau\|_{L_T^{\tilde{q}'}(L^{\tilde{r}'})}\right). \tag{2.9}$$

We end this section by showing the following absorption lemma [23].

Lemma 2.20. *Let $T>0$ and $X\in C_T(\mathbb{R}_+)$ such that $X(0)=0$ and*

$$X(t)\leq b+cX(t)^\theta \quad \text{on } [0,T],$$

where $1<\theta, 0<c$ and $0<b<(1-\frac{1}{\theta})(b\theta)^{-\frac{1}{\theta}}$. Then

$$X(t)\leq \frac{\theta}{\theta-1}b \quad \text{on } [0,T].$$

3. The stationary problem

In this section, we are going to prove Proposition 2.3. For this purpose, we are to establish first some auxiliary results.

3.1. Preliminary results.

Lemma 3.1. *For any $v\in H^1$, the function $\lambda\to H_w(\lambda v)$ is increasing on $\mathbb{R}_+$.*

Proof. The result is trivial since $p>1$ and

$$H_w(\lambda v):=\lambda^{2p}\frac{p-1}{p}\int |x|^{-\gamma}(I_\alpha*|v|^p)|v|^p\,dx. \qquad \square$$

Lemma 3.2. *If $v_n\in H^1-\{0\}$ such that $\lim_n\|v_n\|_{H^1}=0$, then there exists $n_0\in\mathbb{N}$ such that $K_w(v_n)>0$ for all $n\geq n_0$.*

Proof. Thanks to the interpolation inequality (2.3), one has

$$\int |x|^{-\gamma}(I_\alpha*|v_n|^p)|v_n|^p\,dx \lesssim \|v_n\|^A\|\nabla v_n\|^B \lesssim \|v_n\|_{H_1}^{2p} = \|v_n\|_{H_1}^{2(p-1)}\|v_n\|_{H_1}^2.$$

Then
$$\int |x|^{-\gamma}(I_\alpha * |v_n|^p)|v_n|^p\,dx = \circ(w\|v_n\|^2 + \|\nabla v_n\|^2).$$
It follows that
$$K_w(v_n) = w\|v_n\|^2 + \|\nabla v_n\|^2 + \circ(w\|v_n\|^2 + \|\nabla v_n\|^2) \sim w\|v_n\|^2 + \|\nabla v_n\|^2 > 0.$$
This finishes the proof. □

Lemma 3.3. *We have*
$$m_w = \inf_{v\in H^1-\{0\}}\{H_w(v) \mid K_w(v) \le 0\}.$$

Proof. Denote
$$\widetilde{m}_w = \inf_{v\in H^1-\{0\}}\{H_w(v) \mid K_w(v) \le 0\}.$$
Obviously, one has $\widetilde{m}_w \le m_w$. Conversely, take $v \in H^1 - \{0\}$ such that $K_w(v) < 0$. Since $\lim_{\lambda\to 0^+}\|\lambda v\|_{H^1} = 0$, then, by the previous lemma, there exists $\lambda_0 \in (0,1)$ such that $K_w(\lambda_0 v) > 0$. Thus, by a continuity argument, there exists $\lambda_1 \in (\lambda_0, 1)$ such that $K_w(\lambda_1 v) = 0$. Knowing that $\lambda \to H(\lambda v)$ is increasing on the interval $[0,1]$, we have
$$m_w \le S_w(\lambda_1 v) = H_w(\lambda_1 v) \le H_w(v).$$
Therefore $m_w \le \widetilde{m}_w$. □

3.2. Proof of Theorem 2.3. For ε small enough, denote
$$\left(\frac{N}{\gamma}\right)^- := \frac{N}{\gamma}\left(1 - \frac{\varepsilon}{1+\varepsilon}\right), \quad \left(\frac{N}{\gamma}\right)^+ := \frac{N}{\gamma}\left(1 + \frac{\varepsilon}{1-\varepsilon}\right)$$
and
$$r^- := \frac{2N}{N+\alpha-\gamma-\varepsilon\gamma}, \quad r^+ := \frac{2N}{N+\alpha-\gamma+\varepsilon\gamma}.$$
Taking (ϕ_n) as a minimizing sequence which is supposed to be radial decreasing according to some rearrangement argument. Namely,
$$\phi_n \in H^1_{rd} - \{0\}, K_w(\phi_n) = 0 \quad \text{and} \quad \lim_n H_w(\phi_n) = \lim_n S_w(\phi_n) = m_w. \tag{3.1}$$
Then
$$w\|\phi_n\|^2 + \|\phi_n\|^2 = \int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p\,dx.$$
So,
$$\frac{p-1}{p}\int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p\,dx = S_w(\phi_n) \to m_w.$$

Thus (ϕ_n) is bounded in H^1_{rd} and by a compact Sobolev embedding, we can assume

$$\phi_n \rightharpoonup \phi \text{ in } H^1_{rd} \text{ and } \phi_n \to \phi \text{ in } L^q \text{ for any } q \in \left(2, \frac{2N}{N-2}\right).$$

Assume $\phi = 0$. Since

$$1 + \frac{\alpha}{N} = \frac{1}{\left(\frac{N}{\gamma}\right)^-} + \frac{2}{r^-} \quad \text{and} \quad 1 + \frac{\alpha}{N} = \frac{1}{\left(\frac{N}{\gamma}\right)^+} + \frac{2}{r^+},$$

then, by the Hardy–Littlewood–Sobolev inequality, one has

$$\begin{aligned}
\int |x|^{-\gamma}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx &:= \int_{|x|<1} (I_\alpha * |\phi_n|^p)|x|^{-\gamma}|\phi_n|^p\,dx \\
&\quad + \int_{|x|>1} (I_\alpha * |\phi_n|^p)|x|^{-\gamma}|\phi_n|^p\,dx \\
&\lesssim \||x|^{-\gamma}\|_{(\frac{N}{\gamma})^-(|x|<1)}\||\phi_n|^p\|^2_{r^-} + \||x|^{-\gamma}\|_{(\frac{N}{\gamma}|x|>1)^+}\||\phi_n|^p\|^2_{r+} \\
&\lesssim \||\phi_n|^p\|^2_{r^-} + \||\phi_n|^p\|^2_{r^-} \\
&= \|\phi_n\|^{2p}_{pr^-} + \|\phi_n\|^{2p}_{pr^+}.
\end{aligned}$$

Because $1 + \frac{\alpha-\gamma}{N} < p < p^*$, then $2 < pr^-$, $pr^+ < \frac{2N}{N-2}$ for some ε small enough. Taking into account a compact Sobolev embedding, we get

$$\int |x|^{-\gamma}(I_\alpha * |\phi_n|^p)|\phi_n|^p\,dx \to 0 \quad \text{as } n \to +\infty.$$

Since $K_w(\phi_n) = 0$, then $\lim_n(w\|\phi_n\|^2 + \|\nabla\phi_n\|^2) = 0$. Thanks to Lemma 3.2, we get $K_w(\phi_n) > 0$ for a large value of n. Then $\phi \neq 0$ by contradiction. Next, we have to prove that $m_w > 0$. With the lower semi-continuity of the H^1 norm, we have

$$\begin{aligned}
0 = \liminf_n K_w(\phi_n) \geq w \liminf_n \|\phi_n\|^2 + \liminf_n \|\nabla\phi_n\|^2 \\
- \int |x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^p\,dx \geq K_w(\phi).
\end{aligned}$$

In a similar way, we obtain $H_w(\phi) \leq m_w$. Furthermore, if $K_w(\phi) < 0$, then there exists $\lambda_1 \in (0,1)$ such that $K(\lambda_1\phi) = 0$. Therefore,

$$m_w \leq S_w(\lambda_1\phi) = H_w(\lambda_1\phi) \leq H_w(\phi) \leq m_w.$$

Then

$$m_w = S_w(\lambda_1\phi) = H_w(\lambda_1\phi) > 0.$$

Let $\phi := \lambda_1\phi$. Then ϕ is a minimizer for (2.2) which satisfies (3.1). Finally, we are going to prove that ϕ satisfies (2.1). There exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that $S'_w(\phi) = \mu K'_w(\phi)$. Hence,

$$-\triangle\phi + w\phi - \frac{1}{2}|x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = \mu(-\triangle\phi + w\phi - \frac{p}{2}|x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^{p-2}\phi).$$

After multiplying the previous equation with $\bar{\phi}$ and then integrating, knowing that $K_w(\phi) = 0$, it follows that

$$\mu \frac{1-p}{2} \int |x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^p \, dx = 0.$$

Consequently, $\mu = 0$ and then $S'(\phi) = 0$. Thus ϕ is a ground state to (1.1).

4. Inhomogeneous Gagliardo–Nirenberg inequality

Next we will prove Proposition 2.4.

4.1. Interpolation inequality. Using the Hardy–Littlewood–Sobolev inequality yields

$$\begin{aligned}\int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p \, dx &\lesssim \||x|^{-\gamma}\|_{\left(\frac{N}{\gamma}\right)^-(|x|<1)} \||v|^p\|_{r-}^2 + \||x|^{-\gamma}\|_{\left(\frac{N}{\gamma}\right)^+(|x|>1)} \||v|^p\|_{r+}^2 \\ &\lesssim \|v\|_{pr^-}^{2p} + \|v\|_{pr^+}^{2p}.\end{aligned}$$

Since $1 + \frac{\alpha-\gamma}{N} < p < p^*$, then $2 < pr^-, pr^+ < \frac{2N}{N-2}$. By Lemma 2.11, we obtain

$$\|v\|_{pr^-}^{2p} \lesssim (\|\nabla v\|^{\theta^-} \|v\|^{1-\theta^-})^{2p}, \quad \theta^- := N\left(\frac{1}{2} - \frac{1}{pr^-}\right)$$

and

$$\|v\|_{pr^+}^{2p} \lesssim (\|\nabla v\|^{\theta^+} \|v\|^{1-\theta^+})^{2p}, \quad \theta^+ := N\left(\frac{1}{2} - \frac{1}{pr^+}\right).$$

Making ε small enough, we obtain

$$\int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p \, dx \lesssim \|\nabla v\|^B \|v\|^A.$$

4.2. The best constant of the Gagliardo–Nirenberg inequality. First, let

$$\xi := \frac{1}{C(N,p,\alpha,\gamma)} = \inf_{v \in H^1 - \{0\}} J(v).$$

Let (v_n) be a minimizing sequence with a rearrangement argument and let (v_n) be radial decreasing. Denoting $v^{\lambda,\mu} := \lambda v(\mu \cdot)$, we have

$$\|\nabla v^{\lambda,\mu}\|^2 = \lambda^2 \mu^{2-N} \|\nabla v\|^2, \quad \|v^{\lambda,\mu}\|^2 = \lambda^2 \mu^{-N} \|v\|^2,$$
$$\int |x|^{-\gamma}(I_\alpha * |v^{\lambda,\mu}|^p)|v^{\lambda,\mu}|^p \, dx = \lambda^{2p} \mu^{-N-\alpha+\gamma} \int |x|^{-\gamma}(I_\alpha * |v|^p)|v|^p \, dx$$

Hence, $J(v^{\lambda,\mu}) = J(v)$ via some elementary computations. Now, let us define

$$\lambda_n := \frac{\|v_n\|^{\frac{N}{2}-1}}{\|\nabla v_n\|^{\frac{N}{2}}} \quad \text{and} \quad \mu_n := \frac{\|v_n\|}{\|\nabla v_n\|}.$$

So, $\psi_n := v_n^{\lambda_n,\mu_n}$ satisfies

$$\|\psi_n\| = \|\nabla\psi_n\| = 1 \quad \text{and} \quad \xi = \lim_n J(\psi_n).$$

Then, for a subsequence also denoted by (ψ_n), there exist $\psi \in H^1_{rd}$ such that $\psi_n \rightharpoonup \psi$. Now we are going to prove that

$$\int |x|^{-\gamma}(I_\alpha * |\psi_n|^p)|\psi_n|^p\,dx \to \int |x|^{-\gamma}(I_\alpha * |\psi|^p)|\psi|^p\,dx.$$

For that, let

$$I_n^1 := \int_{(|x|<1)} |x|^{-\gamma}(I_\alpha * |\psi_n|^p)|\psi_n|^p\,dx - \int_{(|x|<1)} |x|^{-\gamma}(I_\alpha * |\psi|^p)|\psi|^p\,dx$$

and

$$I_n^2 := \int_{(|x|>1)} |x|^{-\gamma}(I_\alpha * |\psi_n|^p)|\psi_n|^p\,dx - \int_{(|x|>1)} |x|^{-\gamma}(I_\alpha * |\psi|^p)|\psi|^p\,dx.$$

We have

$$\begin{aligned} I_n^1 = \int_{(|x|<1)} &|x|^{-\gamma}(I_\alpha * (|\psi_n|^p - |\psi|^p))|\psi_n|^p\,dx \\ &+ \int_{(|x|<1)} |x|^{-\gamma}(I_\alpha * |\psi|^p)(|\psi_n|^p - |\psi|^p)\,dx. \end{aligned}$$

Using the Hardy–Littlewood–Sobolev inequality, one gets

$$\begin{aligned} I_n^1 &\lesssim \||x|^{-\gamma}\|_{\left(\frac{N}{\gamma}\right)^-(|x|<1)} \||\psi_n|^p - |\psi|^p\|_{r^-} \||\psi_n|^p\|_{r^-} \\ &\quad + \||x|^{-\gamma}\|_{\left(\frac{N}{\gamma}\right)^-(|x|<1)} \||\psi|^p\|_{r^-} \||\psi_n|^p - |\psi|^p\|_{r^-} \\ &\lesssim (\||\psi_n|^p\|_{r^-} + \||\psi|^p\|_{r^-})\, \||\psi_n|^p - |\psi|^p\|_{r^-} \\ &\lesssim \left(\|\psi_n\|_{pr^-}^p + \|\psi\|_{pr^-}^p\right) \||\psi_n|^p - |\psi|^p\|_{r^-}. \end{aligned}$$

Together, the Hölder inequality, the Mean Value Theorem and the fact that the function $x \to x^{r^-}$ is convex, give us

$$\begin{aligned} \||\psi_n|^p - |\psi|^p\|_{r^-} &\lesssim \|\{|\psi_n|^{p-1} + |\psi|^{p-1}\}|\psi_n - \psi|\|_{r^-} \\ &\lesssim \|\{|\psi_n|^{(p-1)r^-} + |\psi|^{(p-1)r^-}\}|\psi_n - \psi|^{r^-}\|_1^{\frac{1}{r^-}} \\ &\lesssim \left(\|\psi_n\|_{pr^-}^{(p-1)r^-} + \|\psi\|_{pr^-}^{(p-1)r^-}\right)^{\frac{1}{r^-}} \|\psi_n - \psi\|_{pr^-}. \end{aligned}$$

It is well known that $(a+b)^\rho \le 2(a^\rho + b^\rho)$ for a, b non-negative and $0 \le \rho \le 2$. Then

$$\||\psi_n|^p - |\psi|^p\|_{r^-} \lesssim \left(\|\psi_n\|_{pr^-}^{p-1} + \|\psi\|_{pr^-}^{p-1}\right) \|\psi_n - \psi\|_{pr^-}.$$

Thus,

$$I_n^1 \lesssim \left(\|\psi_n\|_{pr^-}^p + \|\psi\|_{pr^-}^p\right)\left(\|\psi_n\|_{pr^-}^{p-1} + \|\psi\|_{pr^-}^{p-1}\right)\|\psi_n - \psi\|_{pr^-}.$$

In a similar way, one obtains

$$I_n^2 \lesssim \left(\|\psi_n\|_{pr^+}^p + \|\psi\|_{pr^+}^p\right)\left(\|\psi_n\|_{pr^+}^{p-1} + \|\psi\|_{pr^+}^{p-1}\right)\|\psi_n - \psi\|_{pr^+}.$$

Since $2 < pr^-$, $2pr^+ < \frac{2N}{N-2}$, then, by compact Sobolev injections, we have $\lim_n I_n^1 = 0$ and $\lim I_n^2 = 0$. Hence $\lim_n I_n = 0$ and when n goes to $+\infty$, we get

$$J(\psi_n) = \frac{1}{\int |x|^{-\gamma}(I_\alpha * |\psi_n|^p)|\psi_n|^p\,dx} \to \frac{1}{\int |x|^{-\gamma}(I_\alpha * |\psi|^p)|\psi|^p\,dx}.$$

Thanks to the lower semi-continuity of the H^1 norm, we get $\|\psi\| \le 1$ and $\|\nabla\psi\| \le 1$. If $\|\psi\| < 1$ or $\|\nabla\psi\| < 1$, then $\|\psi\|^A\|\nabla\psi\|^B < 1$ which implies $J(\psi) < \xi$. This contradicts the definition of ξ, and then $\|\psi\| = \|\nabla\psi\| = 1$. Consequently, $\psi_n \to \psi$ in H^1_{rd} and

$$C(N,p,\alpha,\gamma) = \frac{1}{\xi} = \frac{1}{J(\psi)} = \int |x|^{-\gamma}(I_\alpha * |\psi|^p)|\psi|^p\,dx.$$

The minimizer ψ satisfies the Euler equation

$$\partial_\eta J(\psi + \eta h)|_{\eta=0} = 0 \quad \text{for all } h \in C_0^\infty \cap H^1.$$

Then ψ satisfies the desired equation (2.4). It remains now to prove (2.5). For $\lambda, \mu \in \mathbb{R}$, let $\psi = \phi^{\lambda,\mu} := \lambda\phi(\mu\cdot)$. In equation (2.4), replacing ψ by $\phi^{\lambda,\mu}$ yields

$$\frac{B}{A}\mu^2\triangle\phi + \phi - 2\frac{\xi}{A}p\lambda^{2p-2}\mu^{-\alpha+\gamma}|x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = 0.$$

Taking

$$\mu = \left(\frac{A}{B}\right)^{\frac{1}{2}} \quad \text{and} \quad \lambda = \left(\left(\frac{A}{B}\right)^{\frac{\alpha-\gamma}{2}}\frac{A}{2p\xi}\right)^{\frac{1}{2(p-1)}},$$

we get

$$-\triangle\phi - \phi + |x|^{-\gamma}(I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = 0.$$

Since $1 = \|\psi\| = \lambda\mu^{-\frac{N}{2}}\|\phi\|$, then

$$\lambda\|\phi\| = \left(\frac{A}{B}\right)^{\frac{N}{4}},$$

$$\lambda^{2(p-1)}\|\phi\|^{2(p-1)} = \left(\frac{A}{B}\right)^{\frac{N(p-1)}{2}},$$

$$\left(\frac{A}{B}\right)^{\frac{\alpha-\gamma}{2}}\frac{A}{2p\xi}\|\phi\|^{2(p-1)} = \left(\frac{A}{B}\right)^{\frac{N(p-1)}{2}}.$$

We can deduce the following:

$$\begin{aligned}\xi &= \frac{A}{2p}\left(\frac{A}{B}\right)^{\frac{\alpha-\gamma}{2}}\left(\frac{A}{B}\right)^{-\frac{N(p-1)}{2}}\|\phi\|^{2(p-1)}\\ &= \frac{A}{2p}\left(\frac{A}{B}\right)^{-\frac{N(p-1)-\alpha+\gamma}{2}}\|\phi\|^{2(p-1)} = \frac{A}{2p}\left(\frac{A}{B}\right)^{-\frac{B}{2}}\|\phi\|^{2(p-1)}.\end{aligned}$$

In conclusion,

$$C(N,p,\alpha,\gamma) = \frac{1}{\xi} = \frac{2p}{A}\left(\frac{A}{B}\right)^{\frac{B}{2}}\|\phi\|^{-2(p-1)}.$$

5. Local well-posedness

5.1. L^2-theory. Our aim in this sub-section is to prove Theorem 2.5, namely the existence of a local solution to (1.1) in L^2. For $\psi \in L^2$ and $T > 0$, which will be fixed later, let $R := 2\sup_{(q,r)\in\Gamma}\|U_0(\cdot)\psi\|_{L^q_T(L^r)}$ and define

$$B_T(R) := \left\{ u \in \bigcap_{(q,r)\in\Gamma} L^q_T(L^r) \mid \sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(L^r)} \leq R \right\}.$$

The closed ball $B_T(R)$ is equipped with the complete distance

$$d(u,v) := \sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(L^r)}.$$

Define the function

$$\phi(u)(t) := U_0(t)\psi + i\int_0^t U_0(t-s)\left\{iau + (I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)\right\} ds.$$

For some ε small enough, let

$$\begin{aligned} \eta : &= \frac{N}{\gamma}\left(1 - \frac{\varepsilon}{\gamma+\varepsilon}\right), & \widetilde{\eta} &:= \frac{N}{\gamma}\left(1 + \frac{\varepsilon}{\gamma-\varepsilon}\right),\\ r &:= \frac{2Np}{N+\alpha-\gamma-\varepsilon}, & \widetilde{r} &:= \frac{2Np}{N+\alpha-\gamma+\varepsilon},\\ q &:= \frac{4p}{Np-N-\alpha+\gamma+\varepsilon}, & \widetilde{q} &:= \frac{4p}{Np-N-\alpha+\gamma-\varepsilon}.\end{aligned}$$

Taking $u, v \in B_T(R)$, using the Strichartz estimate (2.7) and Remark 2.16, one gets

$$\begin{aligned} d(\phi(u),\phi(v)) &\lesssim \|ia(u-v) + (I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\\ &\quad - (I_\alpha * |v|^p)|x|^{-\gamma}|v|^{p-2}v\|_{L^{q'}_T(L^{r'}(|x|<1))} + \|ia(u-v)\\ &\quad + (I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u - (I_\alpha * |v|^p)|x|^{-\gamma}|v|^{p-2}v\|_{L^{\widetilde{q}'}_T(L^{\widetilde{r}'}(|x|>1))}\\ &\lesssim a\|u-v\|_{L^{q'}_T(L^{r'}(|x|<1))} + a\|u-v\|_{L^{\widetilde{q}'}_T(L^{\widetilde{r}'}(|x|>1))}\end{aligned}$$

$$\begin{aligned}
&+ \|(I_\alpha * [|u|^p - |v|^p])|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q'}(L^{r'}(|x|<1))} \\
&+ \|(I_\alpha * |v|^p)[|x|^{-\gamma}|u|^{p-2}u - |x|^{-\gamma}|v|^{p-2}v]\|_{L_T^{q'}(L^{r'}(|x|<1))} \\
&+ \|(I_\alpha * [|u|^p - |v|^p])|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{\widetilde{q}'}(L^{\widetilde{r}'}(|x|>1))} \\
&+ \|(I_\alpha * |v|^p)[|x|^{-\gamma}|u|^{p-2}u - |x|^{-\gamma}|v|^{p-2}v]\|_{L_T^{\widetilde{q}'}(L^{\widetilde{r}'}(|x|>1))} \\
\lesssim\ & a\|u-v\|_{L_T^{q'}(L^{r'})} + a\|u-v\|_{L_T^{\widetilde{q}'}(L^{\widetilde{r}'})} + (A) + (B) + (C) + (D).
\end{aligned}$$

We have $r > 2$ then $\frac{2}{r'} > 1$, with a convexity argument, one has $\|u-v\|_{r'}^{q'} \leq \|u-v\|^{q'}$. Thus,

$$a\|u-v\|_{L_T^{q'}(L^{r'})} \leq a\left(\int_0^T \|u-v\|^{q'}\,dt\right)^{\frac{1}{q'}} \leq aT^{\frac{1}{q'}}\|u-v\|_{L_T^\infty(L^2)} \leq aT^{\frac{1}{q'}}d(u,v).$$

Similarly, we obtain $a\|u-v\|_{L_T^{\widetilde{q}'}(L^{\widetilde{r}'})} \leq aT^{\frac{1}{\widetilde{q}'}}\,d(u,v)$. By the Mean Value Theorem, we have

$$(B) \lesssim \|(I_\alpha * |v|^p)|x|^{-\gamma}(|u|^{p-2} + |v|^{p-2})|u-v|\|_{L_T^{q'}(L^{r'}(|x|<1))}.$$

The Hardy–Littlwood–Paley inequality via

$$1 + \frac{\alpha}{N} = \frac{1}{r} + \left(\frac{1}{\eta} + \frac{2p-1}{r}\right)$$

gives

$$\begin{aligned}
(B) &\lesssim \|\|v\|_r^p\||x|^{-\gamma}\|_{L^\eta(|x|<1)}(\|u\|_r^{p-2} + \|v\|_r^{p-2})\|u-v\|_r\|_{L_T^{q'}} \\
&\lesssim \|(\|u\|_r^{2(p-1)} + \|v\|_r^{2(p-1)})\|u-v\|_r\|_{L_T^{q'}}.
\end{aligned}$$

Since $p < p_*$, then there is ε small enough such that $p + \frac{\varepsilon}{N} < p_*$, which implies $Np\left(\frac{1}{2} - \frac{1}{r}\right) < 1$. So $\frac{2p}{q} < 1$ and then $(2p-1)q' < q$. Hence, there exists $\rho > 0$ such that $\frac{1}{q'} = \frac{2p-1}{q} + \frac{1}{\rho}$. Using the Hölder inequality, we get

$$(B) \lesssim T^{\frac{1}{\rho}}(\|u\|_{L^q(L^r)}^{2(p-1)} + \|v\|_{L^q(L^r)}^{2(p-1)})\|u-v\|_{L^q(L^r)} \lesssim T^{\frac{1}{\rho}}R^{2(p-1)}d(u,v).$$

Similarly,

$$\begin{aligned}
(A) &\lesssim \|(I_\alpha * [|u|^{p-1} + |v|^{p-1}]|u-v|)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q'}(L^{r'}(|x|<1))} \\
&\lesssim \|(\|u\|_r^{2(p-1)} + \|v\|_r^{2(p-1)})\|u-v\|_r\|_{L_T^{q'}} \lesssim T^{\frac{1}{\rho}}R^{2(p-1)}d(u,v).
\end{aligned}$$

Now, in order to estimate the integrals on $|x| > 1$, we make use of $\widetilde{\eta}, \widetilde{r}$ and $\widetilde{q}$ with ε small enough. Then

$$(C) := \|(I_\alpha * [|u|^p - |v|^p])|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{\widetilde{q}'}(L^{\widetilde{r}'}(|x|>1))} \lesssim T^{\frac{1}{\rho}}R^{2(p-1)}d(u,v),$$

$$
\begin{aligned}
(D) &:= \|(I_\alpha * |v|^p)[|x|^{-\gamma}|u|^{p-2}u - |x|^{-\gamma}|v|^{p-2}v]\|_{L_T^{\widetilde{q}'}(L^{\widetilde{r}'}(|x|>1))} \\
&\lesssim T^{\frac{1}{\widetilde{\rho}}} R^{2(p-1)} d(u,v).
\end{aligned}
$$

In summary,

$$
d(\Phi(u),\Phi(v)) \lesssim \left(aT^{\frac{1}{q'}} + aT^{\frac{1}{\widetilde{q}'}} + T^{\frac{1}{\rho}}R^{2(p-1)} + T^{\frac{1}{\widetilde{\rho}}}R^{2(p-1)}\right) d(u,v).
$$

Taking $v = 0$, we get

$$
\begin{aligned}
\sup_{(q,r)\in\Gamma} \|\Phi(u)\|_{L_T^q(L^r)} \leq & \sup_{(q,r)\in\Gamma} \|U_0(\cdot)\psi\|_{L_T^q(L^r)} \\
& + C\left(aT^{\frac{1}{q'}}R + aT^{\frac{1}{\widetilde{q}'}}R + T^{\frac{1}{\rho}}R^{2p-1} + T^{\frac{1}{\widetilde{\rho}}}R^{2p-1}\right) \\
& \lesssim \frac{R}{2} + C\left(aT^{\frac{1}{q'}} + aT^{\frac{1}{\widetilde{q}'}} + T^{\frac{1}{\rho}}R^{2(p-1)} + T^{\frac{1}{\widetilde{\rho}}}R^{2(p-1)}\right) R.
\end{aligned}
$$

Therefore, ϕ is a contraction of $B_T(R)$ for some $T > 0$ small enough. Its fixed point is a solution to (1.1). The rest of this sub-section is devoted to establishing the uniqueness of solutions to (1.1). Let $T > 0$ and $u, v \in \bigcap_{(q,r)\in\Gamma} L_T^q(L^r)$ be two solutions of (1.1). So, $w := u - v$ is a solution to the following Cauchy problem:

$$
\begin{cases}
i\dot{w} + \triangle w + i\gamma w = |x|^{-\gamma}(I_\alpha * |v|^p)|v|^{p-2}v - |x|^{-\gamma}(I_\alpha * |u|^p)|u|^{p-2}u, \\
w(0,\cdot) = 0.
\end{cases}
$$

Taking $\tau \in (0,T)$, with a continuity argument we can suppose τ to be small enough such that

$$
\max\left(\|u\|_{L_\tau^{q_1}(L^{r_1})}, \|u\|_{L_\tau^{q_2}(L^{r_2})}, \|v\|_{L_\tau^{q_1}(L^{r_1})}, \|v\|_{L_\tau^{q_2}(L^{r_2})}\right) \leq 1.
$$

Arguing as previously with the Strichartz estimate (2.7), one gets

$$
\begin{aligned}
\sup_{(q,r)\in\Gamma} \|w\|_{L_\tau^q(L^r)} &\lesssim a\|w\|_{L_\tau^{q'}(L^{r'}(|x|<1))} + a\|w\|_{L_\tau^{\widetilde{q}'}(L^{\widetilde{r}'}(|x|>1))} \\
&\quad + \|(I_\alpha * |v|^p)|x|^{-\gamma}|v|^{p-2}v - (I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_\tau^{q'}(L^{r'}(|x|<1))} \\
&\quad + \|(I_\alpha * |v|^p)|x|^{-\gamma}|v|^{p-2}v - (I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_\tau^{\widetilde{q}'}(L^{\widetilde{r}'}(|x|>1))} \\
&\lesssim a\left(\tau^{\frac{1}{q'}} + \tau^{\frac{1}{\widetilde{q}'}}\right)\|w\|_{L_\tau^\infty(L^2)} + (\tau^\rho + \tau^{\widetilde{\rho}}) \\
&\quad \times \left(\|u\|_{L_\tau^q(L^r)}^{2(p-1)} + \|u\|_{L_\tau^{\widetilde{q}}(L^{\widetilde{r}})}^{2(p-1)} + \|v\|_{L_\tau^q(L^r)}^{2(p-1)} + \|v\|_{L_\tau^{\widetilde{q}}(L^{\widetilde{r}})}^{2(p-1)}\right) \\
&\quad \times \left(\|w\|_{L_\tau^q(L^r)} + \|w\|_{L_\tau^{\widetilde{q}}(L^{\widetilde{r}})}\right) \\
&\lesssim \left(a\tau^{\frac{1}{q'}} + a\tau^{\frac{1}{\widetilde{q}'}} + \tau^\rho + \tau^{\widetilde{\rho}}\right) \sup_{(q,r)\in\Gamma} \|w\|_{L_\tau^q(L^r)}.
\end{aligned}
$$

Thus, the uniqueness follows for small time τ and then on $[0,T)$ with a standard translation argument.

5.2. H^1-theory. In this subsection we have to establish that the Cauchy problem (1.1) is locally well-posed in H^1. Precisely, we are going to prove Theorem 2.6. For $T>0$ and $\psi\in H^1$, let $R:=2\sup_{(q,r)\in\Gamma}\|U_0(\cdot)\psi\|_{L^q_T(W^{1,r})}$ and define

$$B_T(R):=\left\{u\in\bigcap_{(q,r)\in\Gamma}L^q_T(W^{1,r})\mid\sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(W^{1,r})}\le R\right\},$$

which is equipped with the complete distance

$$d(u,v):=\sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(L^r)}.$$

For some $\varepsilon>0$ small enough, let us define

$$\eta_1:=\frac{N}{-\beta}\left(1-\frac{\varepsilon}{-2\beta+\varepsilon}\right),\qquad \eta_2:=\frac{N}{-\beta}\left(1+\frac{\varepsilon}{-2\beta-\varepsilon}\right),$$
$$\nu_1:=\frac{N}{-\beta+1}\left(1-\frac{\varepsilon}{2(-\beta+1)+\varepsilon}\right),\quad \nu_2:=\frac{N}{-\beta+1}\left(1+\frac{\varepsilon}{2(-\beta+1)-\varepsilon}\right),$$
$$r_1:=\frac{2Np}{N+\alpha+2\beta+2(p-1)-\varepsilon},\qquad r_2:=\frac{2Np}{N+\alpha+2\beta+2(p-1)+\varepsilon},$$

where $\nu_1>1$ and $\nu_2>1$. For all $u,v\in B_T(R)$, using the Strichartz estimate (2.7) and Remark 2.16, we obtain

$$\begin{aligned}
d(\phi(u),\phi(v))&\lesssim a\|u-v\|_{L^{q_1'}_T(L^{r_1'}(|x|<1))}+a\|u-v\|_{L^{q_2'}_T(L^{r_2'}(|x|>1))}\\
&\quad+\|(I_\alpha*[|u|^p-|v|^p])|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_1'}_T(L^{r_1'}(|x|<1))}\\
&\quad+\|(I_\alpha*|v|^p)[|x|^{-\gamma}|u|^{p-2}u-|x|^{-\gamma}|v|^{p-2}v]\|_{L^{q_1'}_T(L^{r_1'}(|x|<1))}\\
&\quad+\|(I_\alpha*[|u|^p-|v|^p])|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_2'}_T(L^{r_2'}(|x|>1))}\\
&\quad+\|(I_\alpha*|v|^p)[|x|^{-\gamma}|u|^{p-2}u-|x|^{-\gamma}|v|^{p-2}v]\|_{L^{q_2'}_T(L^{r_2'}(|x|>1))}\\
&\lesssim a\|u-v\|_{L^{q_1'}_T(L^{r_1'})}+a\|u-v\|_{L^{q_2'}_T(L^{r_2'})}+I+II+III+IV.
\end{aligned}$$

Since $\frac{2}{r_j'}>1$, then

$$a\|u-v\|_{L^{q_j'}_T(L^{r_j'})}\le aT^{\frac{1}{q_j'}}\|u-v\|_{L^\infty_T(L^2)}\le aT^{\frac{1}{q_j'}}d(u,v).$$

Let $k_j:=\frac{Nr_j}{N-r_j}$, then

$$1+\frac{\alpha}{N}=\frac{1}{r_1}+\frac{p}{k_1}+\left(\frac{1}{\eta_1}+\frac{p-2}{k_1}+\frac{1}{r_1}\right).$$

Using the Mean Value Theorem and the Hardy–Littlwood–Paley inequality, one gets

$$\begin{aligned}II &\lesssim \| \| |x|^{-\gamma} \|_{L^{\eta_1}(|x|<1)} \|v\|_{k_1}^p (\|u\|_{k_1}^{p-2} + \|v\|_{k_1}^{p-2}) \|u-v\|_{r_1} \|_{L_T^{q_1'}} \\ &\lesssim \|(\|u\|_{k_1}^{2(p-1)} + \|v\|_{k_1}^{2(p-1)}) \|u-v\|_{r_1} \|_{L_T^{q_1'}}.\end{aligned}$$

The condition $N+\alpha-\gamma-2>0$ implies $1<r_j<N$ for some ε sufficiently small. So, $W^{1,r_j} \hookrightarrow L^{k_j}$ and then

$$II \lesssim \|(\|u\|_{W^{1,r_1}}^{2(p-1)} + \|v\|_{W^{1,r_1}}^{2(p-1)}) \|u-v\|_{r_1} \|_{L_T^{q_1'}}.$$

Since $p<p^*$, then for some ε sufficiently small, we have $(2p-1)q_j' < q_j$. Hence, there exists $\rho_j > 0$ such that $\frac{1}{q_j'} = \frac{2p-1}{q_j} + \frac{1}{\rho_j}$. Using the Hölder inequality, we deduce

$$II \lesssim T^{\frac{1}{\rho_1}} (\|u\|_{L^{q_1}(W^{1,r_1})}^{2(p-1)} + \|v\|_{L^{q_1}(W^{1,r_1})}^{2(p-1)}) \|u-v\|_{L^{q_1}(L^{r_1})} \lesssim T^{\frac{1}{\rho_1}} R^{2(p-1)}\, d(u,v).$$

As above, the Hardy–Littlwood–Paley inequality gives

$$\begin{aligned}I &\lesssim \|(I_\alpha * [|u|^{p-1} + |v|^{p-1}]|u-v|)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))} \\ &\lesssim \|(\|u\|_{k_1}^{2(p-1)} + \|v\|_{k_1}^{2(p-1)}) \|u-v\|_{r_1} \|_{L_T^{q_1'}} \\ &\lesssim \|(\|u\|_{W^{1,r}}^{2(p-1)} + \|v\|_{W^{1,r}}^{2(p-1)}) \|u-v\|_{r_1} \|_{L_T^{q_1'}} \lesssim T^{\frac{1}{\rho_1}} R^{2(p-1)}\, d(u,v).\end{aligned}$$

Now, in order to estimate the integrals on $|x|>1$, we make use of η_2, r_2 and q_2 with ε small enough. Thus,

$$\begin{aligned}III &:= \|(I_\alpha * [|u|^p - |v|^p])|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))} \lesssim T^{\frac{1}{\rho_2}} R^{2(p-1)}\, d(u,v), \\ IV &:= \|(I_\alpha * |v|^p)[|x|^{-\gamma}|u|^{p-2}u - |x|^{-\gamma}|v|^{p-2}v]\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))} \\ &\lesssim T^{\frac{1}{\rho_2}} R^{2(p-1)}\, d(u,v).\end{aligned}$$

In summary, we obtain

$$d(\Phi(u),\Phi(v)) \lesssim (aT^{\frac{1}{q_1'}} + aT^{\frac{1}{q_2'}} + T^{\frac{1}{\rho_1}} R^{2(p-1)} + T^{\frac{1}{\rho_2}} R^{2(p-1)})\, d(u,v).$$

Taking $v=0$, we get

$$\begin{aligned}\sup_{(q,r)\in\Gamma} \|\Phi(u)\|_{L_T^q(L^r)} \le& \sup_{(q,r)\in\Gamma} \|U_0(\cdot)\psi\|_{L_T^q(L^r)} \\ &+ C\left(aT^{\frac{1}{q_1'}}R + aT^{\frac{1}{q_2'}}R + T^{\frac{1}{\rho_1}}R^{2p-1} + T^{\frac{1}{\rho_2}}R^{2p-1}\right).\end{aligned}$$

Now, it remains to estimate $\sup_{(q,r)\in\Gamma}\|\nabla\Phi(u)\|_{L^q_T(L^r)}$. For this purpose, let

$$V := \sup_{(q,r)\in\Gamma}\|\nabla\Phi(u)\|_{L^q_T(L^r)} - \sup_{(q,r)\in\Gamma}\|\nabla(U_0(\cdot)\psi)\|_{L^q_T(L^r)}.$$

Thanks to the Strichartz estimate (2.7), one has

$$\begin{aligned}
V &\lesssim a\|\nabla u\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))} + \|\nabla(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))}\\
&\quad + a\|\nabla u\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))} + \|\nabla(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))}\\
&\lesssim a\|\nabla u\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))} + a\|\nabla u\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))}\\
&\quad + \|(I_\alpha * |u|^{p-1}|\nabla u|)|x|^{-\gamma}|u|^{p-1}\\
&\quad + (I_\alpha * |u|^p)|x|^{\gamma-1}|u|^{p-1} + (I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}|\nabla u|\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))}\\
&\quad + \|(I_\alpha * |u|^{p-1}|\nabla u|)|x|^{-\gamma}|u|^{p-1}\\
&\quad + (I_\alpha * |u|^p)|x|^{\gamma-1}|u|^{p-1} + (I_\alpha * |u|^p)|x|^{\gamma}|u|^{p-2}|\nabla u|\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))}.
\end{aligned}$$

We have

$$a\|\nabla u\|_{L_T^{q_j'}(L^{r_j'})} \le a\left(\int_0^T \|\nabla u\|^{q_j'}\,dt\right)^{\frac{1}{q_j'}} \le aT^{\frac{1}{q_j'}}\|u\|_{L_T^\infty(\dot H^1)} \le aT^{\frac{1}{q_j'}}R.$$

Arguing as previously, by using the Hardy–Littlwood–Paley and the Hölder inequalities, we obtain

$$\begin{aligned}
V &\lesssim aT^{\frac{1}{q_1'}}R + aT^{\frac{1}{q_2'}}R + T^{\frac{1}{\rho_1}}R^{2p-1} + T^{\frac{1}{\rho_2}}R^{2p-1}\\
&\quad + \|(I_\alpha * |u|^p)|x|^{\gamma-1}|u|^{p-1}\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))}\\
&\quad + \|(I_\alpha * |u|^p)|x|^{\gamma-1}|u|^{p-1}\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))}.
\end{aligned}$$

The assumption $N > \gamma + 1$ implies $\nu_1 > 1$. We have $1 + \frac{\alpha}{N} = \frac{1}{r_1} + (\frac{1}{\nu_1} + \frac{2p-1}{k_1})$. Taking into account Sobolev injections, then using the Hardy–Littlwood–Paley and the Hölder inequalities, we obtain

$$\begin{aligned}
\|(I_\alpha * |u|^p)|x|^{\gamma-1}|u|^{p-1}\|_{L_T^{q_1'}(L^{r_1'}(|x|<1))} &\lesssim \|\||x|^{\gamma-1}\|_{\nu_1(|x|<1)}\|u\|_{k_1}^{2p-1}\|_{L_T^{q_1'}}\\
&\lesssim \|\|u\|_{W^{1,r_1}}^{2p-1}\|_{L_T^{q_1'}} \lesssim T^{\frac{1}{\rho_1}}R^{2p-1}.
\end{aligned}$$

Also, we use ν_2, η_2, r_2 and q_2 to control integrals on $|x| > 1$. Then

$$\|(I_\alpha * |u|^p)|x|^{\gamma-1}|u|^{p-1}\|_{L_T^{q_2'}(L^{r_2'}(|x|>1))} \lesssim T^{\frac{1}{\rho_2}}R^{2p-1}.$$

In summary, we obtain

$$\sup_{(q,r)\in\Gamma}\|\nabla\Phi(u)\|_{L^q_T(L^r)} \le \sup_{(q,r)\in\Gamma}\|U_0(\cdot)\psi\|_{L^q_T(W^{1,r})}$$

$$+C\left(aT^{\frac{1}{q_1'}}R+aT^{\frac{1}{q_2'}}R+T^{\frac{1}{\rho_1}}R^{2p-1}+T^{\frac{1}{\rho_2}}R^{2p-1}\right).$$

In conclusion, the functional Φ is a contraction of $B_T(R)$ for some $T>0$ small enough. Hence, the fixed point argument proves the existence of a unique local solution in $B_T(R)$ to the main problem (1.1). The uniqueness of maximal solutions follows by previous computations and the standard translation argument.

6. Global solutions for large damping

The aim of this section is to prove Theorem 2.8 about the existence of a global solution to (1.1) in the energy space for a large damping coefficient. Let $a>0$, and

$$u\in C_{T^*}(H^1)\cap\bigcap_{(q,r)\in\Gamma}L^q(W^{1,r})$$

be the maximal solution to (1.1). Let $\theta_j:=\frac{2q_j(p-1)}{q_j-2}$. Due to Proposition 2.14, we have

$$\begin{aligned}\|U_a(\cdot)\psi\|_{L^{\theta_j}_{T^*}(W^{1,r_j})}&\lesssim\left(\int_0^{T^*}e^{-a\theta_js}\|U_0(s)\psi\|^{\theta_j}_{W^{1,r_j}}\,ds\right)^{\frac{1}{\theta_j}}\\&\lesssim\left(\int_0^{T^*}e^{-a\theta_js}\|\psi\|^{\theta_j}_{H^1}\,ds\right)^{\frac{1}{\theta_j}}\\&\lesssim\|\psi\|_{H^1}\left(\int_0^{+\infty}e^{-a\theta_js}\,ds\right)^{\frac{1}{\theta_j}}\lesssim\frac{\|\psi\|_{H^1}}{a\theta_j}.\end{aligned}$$

We deduce

$$\|U_a(\cdot)\psi\|_{L^{\theta_j}_{T^*}(W^{1,r_j})}\lesssim\frac{\|\psi\|_{H^1}}{a\theta_j}\to0\quad\text{as }a\to+\infty.\tag{6.1}$$

Let $T\in[0,T^*)$. Then, by the Strichartz estimate (2.9) and the Hardy–Littlwood–Paley inequality, we obtain

$$\begin{aligned}\|u\|_{L^{q_1}_T(W^{1,r_1}(|x|<1))}&\lesssim\|\psi\|_{H^1}+\|(I_\alpha*|u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_1'}_T(W^{1,r_1'}(|x|<1))}\\&\lesssim\|\psi\|_{H^1}+\|\||x|^{-\gamma}\|_{\eta_1(|x|<1)}\|u\|^{2(p-1)}_{k_1}\|u\|_{r_1(|x|<1)}\\&\quad+\||x|^{-\gamma}\|_{\eta_1(|x|<1)}\|u\|^{2(p-1)}_{k_1}\|\nabla u\|_{r_1(|x|<1)}\\&\quad+\||x|^{\gamma-1}\|_{\nu_1(|x|<1)}\|u\|^{2p-1}_{k_1}\|_{L^{q_1'}_T}\lesssim\|\psi\|_{H^1}+\|\|u\|^{2p-1}_{W^{1,r_1}}\|_{L^{q_1'}_T}.\end{aligned}$$

Since $\frac{1}{q_j'}=\frac{1}{q_j}+\frac{2p-2}{\theta_j}$, then, by the Hölder inequality, we get

$$\|u\|_{L^{q_1}_T(W^{1,r_1}(|x|<1))}\lesssim\|\psi\|_{H^1}+\|u\|^{2(p-1)}_{L^{\theta_1}_T(W^{1,r_1})}\|u\|_{L^{q_1}_T(W^{1,r_1}(|x|<1))}.$$

Similarly,

$$\|u\|_{L^{q_2}_T(W^{1,r_2}(|x|>1))}\lesssim\|\psi\|_{H^1}+\|u\|^{2(p-1)}_{L^{\theta_2}_T(W^{1,r_2})}\|u\|_{L^{q_2}_T(W^{1,r_2}(|x|>1))}.$$

Taking $\mu_j \in \mathbb{R}$ such that $\frac{1}{\mu_j} = \frac{2}{q_j} - \frac{1}{\theta_j}$, one claims that $\mu_j > 1$. Indeed, the condition $\frac{1}{\mu_j} > 0$ is equivalent to $\frac{2}{q_j} > \frac{1}{2p-1}$, which is satisfied because $p^+_{\alpha,N} < p < p^*$. On the other hand, the condition $\frac{1}{\mu_j} < 1$ is equivalent to $q_j p > 2p - 1$ which is satisfied because $q_j > 2$. Now, applying the Strichartz estimate (2.8) and then using the Hardy–Littlwood–Paley inequality yield

$$\begin{aligned}
\|u\|_{L_T^{\theta_1}(W^{1,r_1}(|x|<1))} &\lesssim \|U_a(\cdot)\psi\|_{L_T^{\theta_1}(W^{1,r_1})} \\
&\quad + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{\mu_1'}(W^{1,r_1'}(|x|<1))} \\
&\lesssim \|U_a(\cdot)\psi\|_{L_{T^*}^{\theta_1}(W^{1,r_1})} + \big\|\|u\|^{2p-1}_{k_1(|x|<1)}\big\|_{L_T^{\mu_1'}} \\
&\lesssim \|U_a(\cdot)\psi\|_{L_{T^*}^{\theta_1}(W^{1,r_1})} + \big\|\|u\|^{2p-1}_{W^{1,r_1}(|x|<1)}\big\|_{L_T^{\mu'}}.
\end{aligned}$$

Since $\mu_j'(2p-1) = \theta_j$, then

$$\|u\|_{L_T^{\theta_1}(W^{1,r_1}(|x|<1))} \lesssim \|U_a(\cdot)\psi\|_{L_{T^*}^{\theta_1}(W^{1,r_1})} + \|u\|^{2p-1}_{L_T^{\theta_1}(W^{1,r_1}(|x|<1))}.$$

Taking into account (6.1) and applying Lemma 2.20 with the previous estimate for a large enough, we get

$$\|u\|_{L_T^{\theta_1}(W^{1,r_1}(|x|<1))} \lesssim \|U_a(\cdot)\psi\|_{L_{T^*}^{\theta_1}(W^{1,r_1})} \text{ on } [0,T^*). \tag{6.2}$$

Also,

$$\|u\|_{L_T^{\theta_2}(W^{1,r_1}(|x|>1))} \lesssim \|U_a(\cdot)\psi\|_{L_{T^*}^{\theta_2}(W^{1,r_2})} \text{ on } [0,T^*). \tag{6.3}$$

For any $(q,r) \in \Gamma$, by using the Strichartz estimate (2.9) and the Hardy–Littlwood–Paley iequality, one gets

$$\begin{aligned}
\|u\|_{L_T^q(W^{1,r})} &\lesssim \|\psi\|_{H^1} + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q_1'}(W^{1,r_1'}(|x|<1))} \\
&V + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L_T^{q_2'}(W^{1,r_2'}(|x|>1))} \\
&\lesssim \|\psi\|_{H^1} + \big\|\|u\|^{2(p-1)}_{k_1(|x|<1)}\|u\|_{r_1}\big\|_{L_T^{q_1'}} + \big\|\|u\|^{2(p-1)}_{k_1(|x|<1)}\|u\|_{k_1}\big\|_{L_T^{q_1'}} \\
&V + \big\|\|u\|^{2(p-1)}_{k_2(|x|>1)}\|u\|_{r_2}\big\|_{L_T^{q_2'}} + \big\|\|u\|^{2(p-1)}_{k_2(|x|>1)}\|u\|_{k_2}\big\|_{L_T^{q_2'}} \\
&\lesssim \|\psi\|_{H^1} + \|u\|^{2(p-1)}_{L_T^{\theta_1}(W^{1,r_1}(|x|<1))}\|u\|_{L_T^{q_1}(W^{1,r_1})} \\
&V + \|u\|^{2(p-1)}_{L_T^{\theta_2}(W^{1,r_2}(|x|>1))}\|u\|_{L_T^{q_2}(W^{1,r_2})}.
\end{aligned}$$

Taking into account (6.2) and (6.3), one gets

$$\begin{aligned}
&\sup_{(q,r)\in\Gamma} \|u\|_{L_T^q(W^{1,r})} \\
&\leq C\left\{\|\psi\|_{H^1} + \left(\|u\|^{2(p-1)}_{L_T^{\theta_1}(W^{1,r_1}(|x|<1))} + \|u\|^{2(p-1)}_{L_T^{\theta_2}(W^{1,r_2}(|x|>1))}\right)\sup_{(q,r)\in\Gamma}\|u\|_{L_T^q(W^{1,r})}\right\}
\end{aligned}$$

$$\leq C\left\{\|\psi\|_{H^1}+\left(\|U_a(\cdot)\psi\|^{2(p-1)}_{L^{\theta_1}_{T^*}(W^{1,r_1})}+\|U_a(\cdot)\psi\|^{2(p-1)}_{L^{\theta_2}_{T^*}(W^{1,r_2})}\right)\sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(W^{1,r})}\right\}$$
$$\leq C\left\{\|\psi\|_{H^1}+\|\psi\|^{2(p-1)}_{H^1}\left(\frac{1}{(a\theta_1)^{2(p-1)}}+\frac{1}{(a\theta_2)^{2(p-1)}}\right)\sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(W^{1,r})}\right\}.$$

For a large enough, we obtain

$$\sup_{(q,r)\in\Gamma}\|u\|_{L^q_T(W^{1,r})}\leq\frac{C\|\psi\|_{H^1}}{1-C\|\psi\|^{2(p-1)}_{H^1}\left(\frac{1}{(a\theta_1)^{2(p-1)}}+\frac{1}{(a\theta_2)^{2(p-1)}}\right)}\quad\text{on }[0,T^*).\tag{6.4}$$

Therefore, $\|u\|_{L^\infty_{T^*}(H^1)}<\infty$ and then $T^*=+\infty$. This closes the proof.

7. Scattering

Our aim in this section is to prove Theorem 2.9. Namely, we prove scattering for global solutions to (1.1) given by Theorem 2.8. Thanks to Proposition 2.14, it is sufficient to show that

$$\lim_{t\to+\infty}\|U_{-a}(-t)u(t)-u_+\|_{H^1}=0.$$

For this purpose, we are going to prove that $v(t):=U_{-a}(-t)u(t)$ satisfies the Cauchy criteria in H^1. We have

$$v(t)=\psi+i\int_0^t U_a(-s)(I_\alpha*|u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\,ds.$$

Taking $\widetilde{t}>t$, using the Strichartz estimate, the Hardy–Littlwood–Paley and the Hölder inequalities, we obtain

$$\begin{aligned}\|v(t)-v(\widetilde{t})\|_{H^1}&\lesssim\|\int_{\widetilde{t}}^t U_a(-s)(I_\alpha*|u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)ds\|_{H^1}\\&\lesssim\|(I_\alpha*|u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_1'}_{[t,\widetilde{t}]}(W^{1,r_1'}(|x|<1))}\\&+\|(I_\alpha*|u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_2'}_{[t,\widetilde{t}]}(W^{1,r_2'}(|x|>1))}\\&\lesssim\|u\|^{2(p-1)}_{L^{\theta_1}_{[t,\widetilde{t}]}(W^{1,r_1}(|x|<1))}\|u\|_{L^{q_1}_{[t,\widetilde{t}]}(W^{1,r_1})}\\&+\|u\|^{2(p-1)}_{L^{\theta_2}_{[t,\widetilde{t}]}(W^{1,r_2}(|x|>1))}\|u\|_{L^{q_2}_{[t,\widetilde{t}]}(W^{1,r_2})}\end{aligned}$$

Thanks to (6.2), (6.3) and (6.4), we get

$$u\in L^{\theta_1}_{(0,+\infty)}\big(W^{1,r_1}(|x|<1)\big)\cap L^{q_1}_{(0,+\infty)}\big(W^{1,r_1}\big)\tag{7.1}$$

and

$$u\in L^{\theta_2}_{(0,+\infty)}\big(W^{1,r_2}(|x|>1)\big)\cap L^{q_2}_{(0,+\infty)}\big(W^{1,r_2}\big).\tag{7.2}$$

Therefore, the function v satisfies the Cauchy criteria in H^1 and then it is sufficient to take

$$u_+ := \lim_{t\to+\infty} v(t) = \psi + i\int_0^{+\infty} U_a(-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\,ds.$$

Now, we are to establish that the scattering mapping S is one-to-one. If $u_+ \in H^1$, we have to prove that there exists $u \in C(\mathbb{R}_+, H^1)$, the solution to the first equation of (1.1), such that (2.6) is verified. For this purpose, we use the fixed point argument at infinity. Let us define

$$\phi(u)(t) := U_a(t)u_+ + i\int_t^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\,ds.$$

Let $T > 0$ and C_N be the constant produced in the Strichartz estimate (2.9). We can define

$$\begin{aligned} B_T := \Big\{ u \in C([T,+\infty), H^1) \mid \sup_{(q,r)\in\Gamma} \|u\|_{L^q_{(T,+\infty)}(L^r)} &\le 2C_N\|u_+\|_{H^1}, \\ \|u\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))} &\le 2\|U_a(\cdot)u_+\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})}, \\ \|u\|_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2}(|x|>1))} &\le 2\|U_a(\cdot)u_+\|_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2})}, \\ \|u\|_{L^{q_1}_{(T,+\infty)}(L^{r_1}(|x|<1))} &\le 2\|U_a(\cdot)u_+\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})}, \\ \|u\|_{L^{q_2}_{(T,+\infty)}(L^{r_2}(|x|>1))} &\le 2\|U_a(\cdot)u_+\|_{L^{q_2}_{(T,+\infty)}(L^{r_2})} \Big\}. \end{aligned}$$

The set B_T is equipped with the complete [4] distance

$$d(u,v) := \sup_{(q,r)\in\Gamma} \|u-v\|_{L^q_{(T,+\infty)}(L^r)}.$$

Since

$$\lim_{T\to+\infty} \|U_a(\cdot)u_+\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))} = 0$$

and

$$\lim_{T\to+\infty} \|U_a(\cdot)u_+\|_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2}(|x|>1))} = 0,$$

then, by using the Strichartz estimate (2.8), the Hardy–Littlwood–Paley and the Hölder inequalities, for $u \in B_T$ with T large enough, we get

$$\begin{aligned} \|\phi(u)\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))} &\le \|U_a(\cdot)u_+\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})} \\ &\quad + C\|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{\mu_1'}_{(T,+\infty)}(L^{r_1'}(|x|<1))} \\ &\quad + C\|\nabla((I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u)\|_{L^{\mu_1'}_{(T,+\infty)}(L^{r_1'}(|x|<1))} \end{aligned}$$

$$
\begin{aligned}
&\le \|U_a(\cdot)u_+\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})} + C\|u\|^{2p-1}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))}\\
&\le \|U_a(\cdot)u_+\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})} + C2^{2p-1}\|U_a(\cdot)u_+\|^{2p-1}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))}\\
&\le 2\|U_a(\cdot)u_+\|_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})}.
\end{aligned}
$$

In a similar way, with T large enough, we get

$$
\|\phi(u)\|_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2}(|x|>1))} \le 2\|U_a(\cdot)u_+\|_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2})}.
$$

Furthermore,

$$
\begin{aligned}
&\|\phi(u)\|_{L^{q_1}_{(T,+\infty)}(L^{r_1}(|x|<1))}\\
&\quad\le \|U_a(\cdot)u_+\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})} + C\|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_1'}_{(T,+\infty)}(L^{r_1'}(|x|<1))}\\
&\quad\le \|U_a(\cdot)u_+\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})} + C\|u\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))}\|u\|_{L^{q_1}_{(T,+\infty)}(L^{r_1}(|x|<1))}\\
&\quad\le \|U_a(\cdot)u_+\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})} + 2C\|u\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))}\|U_a(\cdot)u_+\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})}\\
&\quad\le 2\|U_a(\cdot)u_+\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})}.
\end{aligned}
$$

Similarly, for T sufficiently large, we obtain

$$
\|\phi(u)\|_{L^{q_2}_{(T,+\infty)}(L^{r_2}(|x|>1))} \le 2\|U_a(\cdot)u_+\|_{L^{q_2}_{(T,+\infty)}(L^{r_2})}.
$$

Once again, using the Strichartz estimate (2.9) and arguing as previously, for T large enough, one obtains

$$
\begin{aligned}
&\sup_{(q,r)\in\Gamma}\|u\|_{L^q_{(T,+\infty)}(L^r)}\\
&\qquad\le C_N\Bigg(\|u_+\|_{H^1} + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_1'}_{(T,+\infty)}(W^{1,r_1'}(|x|<1))}\\
&\qquad\quad + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_2'}_{(T,+\infty)}(W^{1,r_2'}(|x|>1))}\Bigg)\\
&\qquad\le C_N(\|u_+\|_{H^1} + 2\|u\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})(|x|<1)}\|u\|_{L^{q_1}_{(T,+\infty)}(W^{1,r_1})}\\
&\qquad\quad + 2\|u\|^{2(p-1)}_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2})(|x|>1)}\|u\|_{L^{q_2}_{(T,+\infty)}(W^{1,r_2})}).
\end{aligned}
$$

By the definition of B_T, for T sufficiently large, one has

$$
\begin{aligned}
\sup_{(q,r)\in\Gamma}\|u\|_{L^q_{(T,+\infty)}(L^r)} \le C_N\|u_+\|_{H^1}\Bigg(&1 + 4C_N\|u\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})(|x|<1)}\\
&+ 4C_N\|u\|^{2(p-1)}_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2})(|x|>1)}\Bigg) \le 2C_N\|u_+\|_{H^1}.
\end{aligned}
$$

In conclusion, B_T is conserved by ϕ for T sufficiently large. It remains then to prove that ϕ is a contraction on B_T. Taking $u, v \in B_T$, using the Mean Value Theorem and computing as in Section 5, we obtain

$$\begin{aligned}
&d(\phi(u) - \phi(v))\\
&\lesssim \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u - (I_\alpha * |v|^p)|x|^{-\gamma}|v|^{p-2}v\|_{L^{q_1'}_{(T,+\infty)}(L^{r_1'}(|x|<1))}\\
&\quad + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u - (I_\alpha * |v|^p)|x|^{-\gamma}|v|^{p-2}v\|_{L^{q_2'}_{(T,+\infty)}(L^{r_2'}(|x|>1))}\\
&\lesssim \left(\|u\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))} + \|v\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1}(|x|<1))}\right)\|u-v\|_{L^{q_1}_{(T,+\infty)}(L^{r_1})}\\
&\quad + \left(\|u\|^{2(p-1)}_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2}(|x|>1))} + \|v\|^{2(p-1)}_{L^{\theta_2}_{(T,+\infty)}(W^{1,r_2}(|x|>1))}\right)\|u-v\|_{L^{q_2}_{(T,+\infty)}(L^{r_2})}\\
&\lesssim \left(\|U_a(\cdot)u_+\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})} + \|U_a(\cdot)u_+\|^{2(p-1)}_{L^{\theta_1}_{(T,+\infty)}(W^{1,r_1})}\right) d(u,v).
\end{aligned}$$

Due to (6.1), one has $\|U_a(\cdot)u_+\|_{L^{\theta_j}_{[T,+\infty)}(W^{1,r_j})} \lesssim \frac{\|u_+\|_{H^1}}{a\theta_j}$. Then

$$\lim_{T\to+\infty} \|U_a(\cdot)u_+\|_{L^{\theta_j}_{[T,+\infty)}(W^{1,r_j})} = 0.$$

Consequently, the functional ϕ defines a contraction on B_T for T large enough. Thus, for some $T_+ > 0$, ϕ admits a unique fixed point in B_{T_+} which satisfies

$$\forall t \geq T_+ \quad u(t) = U_a(t)u_+ + i\int_t^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds. \tag{7.3}$$

Now, let us define $\psi := U_a(-T_+)u(T_+)$. Since

$$\begin{aligned}
U_a(t)\psi &= U_a(t)(u_+ + i\int_{T_+}^{+\infty} U_a(-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds)\\
&= U_a(t)u_+ + i\int_{T_+}^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds,
\end{aligned}$$

then

$$\begin{aligned}
u(t) &= U_a(t)\psi - i\int_{T_+}^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds\\
&\quad + i\int_t^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds\\
&= U_a(t)\psi - i\int_{T_+}^{t} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds.
\end{aligned}$$

Hence, u resolves problem (1.1) on $(T_+, +\infty)$, and by Theorem 2.8, u is a global solution to (1.1) with the initial data ψ. In addition, from (7.3), we deduce that

$$\|u(t) - U_a(t)u_+\|_{H^1} = \left\| i\int_t^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds\right\|_{H^1}$$

$$\begin{aligned}
&\lesssim \left\| \int_{\tau}^{+\infty} U_a(\tau - s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds \right\|_{L^\infty_{(t,+\infty)}(H^1)} \\
&\lesssim \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_1'}_{(t,+\infty)}(W^{1,r_1'}(|x|<1))} \\
&\quad + \|(I_\alpha * |u|^p)|x|^{-\gamma}|u|^{p-2}u\|_{L^{q_2'}_{(t,+\infty)}(W^{1,r_2'}(|x|>1))} \\
&\lesssim \|u\|^{2(p-1)}_{L^{\theta_1}_{(t,+\infty)}(W^{1,r_1}(|x|<1))}\|u\|_{L^{q_1}_{(t,+\infty)}(W^{1,r_1})} \\
&\quad + \|u\|^{2(p-1)}_{L^{\theta_2}_{(t,+\infty)}(W^{1,r_2}(|x|>1))}\|u\|_{L^{q_2}_{(t,+\infty)}(W^{1,r_2})}.
\end{aligned}$$

Thanks to (7.1) and (7.2), we get $\lim_{t\to+\infty}\|u(t)-U_a(t)u_+\|_{H^1} = 0$. It remains to prove the uniqueness of the solution u. Let $v \in C(\mathbb{R}_+, H^1)$ be another solution to the first equation in (1.1) such that

$$\lim_{t\to+\infty} \|v(t) - U_a(t)u_+\|_{H^1} = 0.$$

With the integral formula of Duhamel, v satisfies

$$v(t) = U_a(t)u_+ + i\int_t^{+\infty} U_a(t-s)(I_\alpha * |u(s)|^p)|x|^{-\gamma}|u(s)|^{p-2}u(s)\, ds.$$

Arguing as previously yields

$$\begin{aligned}
&\sup_{(q,r)\in\Gamma} \|u - v\|_{L^q_{(t,+\infty)}(L^r)} \\
&\lesssim \left(\|u\|^{2(p-1)}_{L^{\theta_1}_{(t,+\infty)}(W^{1,r_1}(|x|<1))} + \|v\|^{2(p-1)}_{L^{\theta_1}_{(t,+\infty)}(W^{1,r_1}(|x|<1))} \right) \|u - v\|_{L^{q_1}_{(t,+\infty)}(L^{r_1})} \\
&\quad + \left(\|u\|^{2(p-1)}_{L^{\theta_2}_{(t,+\infty)}(W^{1,r_2}(|x|>1))} + \|v\|^{2(p-1)}_{L^{\theta_2}_{(t,+\infty)}(W^{1,r_2}(|x|>1))} \right) \|u - v\|_{L^{q_2}_{(t,+\infty)}(L^{r_2})} \\
&\lesssim \left(\|U_a(\cdot)u_+\|^{2(p-1)}_{L^{\theta_1}_{(t,+\infty)}(W^{1,r_1})} + \|U_a(\cdot)u_+\|^{2(p-1)}_{L^{\theta_2}_{(t,+\infty)}(W^{1,r_2})} \right) \sup_{(q,r)\in\Gamma} \|u - v\|_{L^q_{(t,+\infty)}(L^r)}.
\end{aligned}$$

The uniqueness follows as for large time,

$$\sup_{(q,r)\in\Gamma} \|u - v\|_{L^q_{(t,+\infty)}(L^r)} \le \frac{1}{2} \sup_{(q,r)\in\Gamma} \|u - v\|_{L^q_{(t,+\infty)}(L^r)}.$$

The continuity of the scattering mapping S is a consequence of previous computations.

8. Global solutions via invariant sets

In this section, we are going to prove Theorem 2.10. Namely, any solution to (1.1) is global if the initial data belong to some invariant sets. For this purpose, we need to prove at first some auxiliary results.

8.1. Intermediate results. Let us start by showing the continuity of the functional S_w on H^1.

Lemma 8.1. *Let $1+\frac{\alpha-\gamma}{N}<p<p^*$. The functional $u\to S_w(u)$ satisfies the local Lipshitz condition on H^1,*

$$|S_w(u)-S_w(v)|\lesssim\left(\|u\|_{H^1}+\|v\|_{H^1}+\|u\|_{H^1}^{p-1}+\|v\|_{H^1}^{p-1}\right)\|u-v\|_{H^1},$$

on bounded subsets of H^1.

Proof. Let

$$(I):=\int(I_\alpha*|u|^p)|x|^{-\gamma}|u|^p dx-\int(I_\alpha*|v|^p)|x|^{-\gamma}|v|^p\,dx.$$

We have

$$(I)=\int(I_\alpha*(|u|^p-|v|^p))|x|^{-\gamma}|u|^p\,dx+\int(I_\alpha*|v|^p)|x|^{-\gamma}(|u|^p-|v|^p)\,dx.$$

Using the Hardy–Littlewood–Sobolev inequality, we get

$$\begin{aligned}I&\lesssim\||x|^{-\gamma}\|_{(\frac{N}{\gamma})^-(|x|<1)}\left(\||u|^p\|_{r^-}+\||v|^p\|_{r^-}\right)\||u|^p-|v|^p\|_{r^-}\\&\quad+\||x|^{-\gamma}\|_{(\frac{N}{\gamma})^+(|x|>1)}\left(\||u|^p\|_{r^+}+\||v|^p\|_{r^+}\right)\||u|^p-|v|^p\|_{r^+}\\&\lesssim\left(\|u\|_{pr^-}^p+\|v\|_{pr^-}^p\right)\||u|^p-|v|^p\|_{r^-}+\left(\|u\|_{pr^+}^p+\|v\|_{pr^+}^p\right)\||u|^p-|v|^p\|_{r^+}.\end{aligned}$$

Using to the Mean Value Theorem and taking the function $x\to x^{r^-}$ to be convex, we have

$$\begin{aligned}\||u|^p-|v|^p\|_{r^-}&\lesssim\left\|\left(|u|^{p-1}+|v|^{p-1}\right)|u-v|\right\|_{r^-}\\&\lesssim\left\|\left(|u|^{(p-1)r^-}+|v|^{(p-1)r^-}\right)|u-v|^{r^-}\right\|_1^{\frac{1}{r^-}}\\&\lesssim\left(\|u\|_{pr^-}^{(p-1)r^-}+\|v\|_{pr^-}^{(p-1)r^-}\right)^{\frac{1}{r^-}}\|u-v\|_{pr^-}\\&\lesssim\left(\|u\|_{pr^-}^{p-1}+\|v\|_{pr^-}^{p-1}\right)\|u-v\|_{pr^-}\\&\lesssim\left(\|u\|_{H^1}^{p-1}+\|v\|_{H^1}^{p-1}\right)\|u-v\|_{H^1}.\end{aligned}$$

Similarly,

$$\||u|^p-|v|^p\|_{r^+}\lesssim\left(\|u\|_{H^1}^{p-1}+\|v\|_{H^1}^{p-1}\right)\|u-v\|_{H^1}.$$

This finishes the proof by combining the previous inequalities. □

Remark 8.2. The action K_w is also continuous on H^1.

Lemma 8.3. *For any $w>0$, A_w is an open subset in H^1.*

Proof. We have $S_w(u) \leq (1+w)\|u\|_{H^1}^2$, then $S_w(u) < m_w$ for $\|u\|_{H^1}$ small enough. Due to the Gagliardo–Nirenberg inequality, there exists $C > 0$ such that

$$\int |x|^{-\gamma}(I_\alpha * |u|^p)|u|^p \, dx \leq C\|u\|^A \|\nabla u\|^B.$$

Therefore,

$$K_w(u) \geq \min(1,w)\|u\|_{H^1}^2 - C\|u\|^A\|\nabla u\|^B \geq \|u\|_{H^1}^2 \left(\min(1,w) - C\|u\|_{H^1}^{2(p-1)}\right).$$

Consequently, $K_w(u) \geq 0$ for $\|u\|_{H^1}$ small enough. From the above, it follows that A_w contains an open small ball $B(0,\delta)$ of H^1 centered on 0 and of radius $\delta > 0$. According to the definition of m, if $S_w(u) < m$ and $K_w(u) = 0$, then $u = 0$. Hence,

$$A_w = B(0,\delta) \cup \{u \in H^1 \mid S_w(u) < m \text{ and } K_w(u) > 0\}.$$

By Lemma 8.1, the set A_w is open in H^1. □

Lemma 8.4. *The set A_w is invariant under the flow of* (1.1).

Proof. Let $\psi \in A_w$ and $u \in C_{T^*}(H^1)$ be the maximal solution to (1.1) emanating from ψ. Since A_w is open in H^1, then $u(t) \in A_w$ for a small value of $t \geq 0$. Suppose there exists $t_0 \in (0,T^*)$ such that $u(t) \in A_w$ if $t \in [0,t_0)$ and $u(t_0)$ does not belong to A_w. Since $\frac{d}{dt}S_w(u(t)) = -2aK_w(u(t))$, then $S_w(u(t))$ is non-increasing on $[0,t_0)$. By Lemma 8.1, the function $t \to S_w(u(t))$ is continuous on $[0,T^*)$. Thus,

$$\forall t \in [0,t_0] \quad S_w(u(t)) \leq S_w(\psi) < m_w. \tag{8.1}$$

Knowing that $K_w(u(t_0)) < 0$, with a continuity argument, there exists an instant of time $t_1 \in (0,t_0)$ such that $K_w(u(t_1)) = 0$. By the definition of m, one has $m_w \leq S(u(t_1))$. This finishes the proof by a contradiction with (8.1). □

8.2. Proof of Theorem 2.10. Let $w > 0$, $\psi \in A_w$ and $u \in C_{T^*}(H^1)$ be the maximal solution with the initial data ψ. By the previous lemmas, we have $S_w(u(t)) < m_w$ and $K_w(u(t)) \geq 0$ for all $t \in [0,T^*)$. Then,

$$\frac{p-1}{p}(w\|u(t)\|^2 + \|\nabla u\|^2) = S_w(u(t)) - \frac{1}{p}K_w(u(t)) \leq S_w(u(t)) \leq m_w.$$

Hence, $\sup_{t\in[0,T^*)} \|u(t)\|_{H^1}^2 < \infty$ and then $T^* = \infty$. □

9. Appendix

Recall the so-called Riesz potential inequality.

Lemma 9.1. *Let $d \geq 1, q > 1, 0 < \alpha < \frac{d}{q}$ and $\frac{1}{r} = \frac{1}{q} - \frac{\alpha}{d}$. Then $I_\alpha : L^q \to L^r$ is a bounded operator. Precisely, there exists $C_{d,\alpha,q} > 0$ such that*

$$\|I_\alpha * f\|_r \leq C_{d,\alpha,q}\|f\|_q.$$

9.1. Proof of Proposition 2.17. Elementary computations give

$$0 < r < \frac{2N}{N-2} \Leftrightarrow 0 < N(\frac{1}{2} - \frac{1}{r}) < 1.$$

Then,

$$\begin{aligned}
\left\| \int_0^t U_a(t-s) f(s)\, ds \right\|_{L_T^\theta(L^r(|x|<1))} &\lesssim \left\| \int_0^t e^{-a(t-s)} U_0(t-s) f(s)\, ds \right\|_{L_T^\theta(L^r(|x|<1))} \\
&\lesssim \left\| \int_0^T \frac{1}{|t-s|^{\frac{1}{\theta}+\frac{1}{\mu}}} \|f(s)\|_{r'(|x|<1)}\, ds \right\|_{L_T^\theta} \\
&\lesssim \left\| \int_0^T I_{\frac{1}{\mu'}-\frac{1}{\theta}} * \|f(s)\|_{r'(|x|<1)}\, ds \right\|_{L_T^\theta}.
\end{aligned}$$

Applying Lemma 9.1 with $d = 1$ and taking into account $\frac{1}{\theta} + \frac{1}{d}(\frac{1}{\mu'} - \frac{1}{\theta}) = \frac{1}{\mu'}$, we obtain

$$\left\| \int_0^t U_a(t-s) f(s)\, ds \right\|_{L_T^\theta(L^r(|x|<1))} \lesssim \|f\|_{L_T^{\mu'}(L^{r'}(|x|<1))}.$$

Similarly for the integrals on $|x| < 1$. □

9.2. Proof of Corollary 2.19. If $i\dot{u} + \triangle u + iau = f$ with the data ψ, then, by using Proposition 2.14 and the standard Strichartz estimates, one gets

$$\begin{aligned}
\|u\|_{L_T^q(L^r)} &\lesssim \|U_a(t)\psi\|_{L_T^q(L^r)} + \left\| \int_0^t U_a(t-s) f(s)\, ds \right\|_{L_T^q(L^r)} \\
&\lesssim \left\| e^{-at} U_0(t)\psi \right\|_{L_T^q(L^r)} + \left\| \int_0^t e^{-a(t-s)} U_0(t-s) f(s)\, ds \right\|_{L_T^q(L^r)} \\
&\lesssim \left\| e^{-at} U_0(t)\psi \right\|_{L_T^q(L^r)} + \left\| \int_0^t U_0(t-s)(e^{-a(t-s)} f(s))\, ds \right\|_{L_T^q(L^r)} \\
&\lesssim \left\| e^{-at} U_0(t)\psi \right\|_{L_T^q(L^r)} + \left\| e^{-a(t-s)} f(s) \right\|_{L_T^{\tilde{q}'}(L^{\tilde{r}'})} \\
&\lesssim \left\| e^{-at} \|U_0(t)\psi\|_r \right\|_{L_T^q} + \left\| e^{-a(t-s)} \|f(s)\|_{\tilde{r}'} \right\|_{L_T^{\tilde{q}'}} \\
&\lesssim \left\| \|U_0(t)\psi\|_r \right\|_{L_T^q} + \left\| \|f\|_{\tilde{r}'} \right\|_{L_T^{\tilde{q}'}} \\
&\lesssim \left\| U_0(t)\psi \right\|_{L_T^q(L^r)} + \|f\|_{L_T^{\tilde{q}'}(L^{\tilde{r}'})} \lesssim \|\psi\| + \|f\|_{L_T^{\tilde{q}'}(L^{\tilde{r}'})}.
\end{aligned}$$

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Received August 27, 2020, revised November 5, 2020.

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Нотатки щодо демпфованого фокусувального неоднорідного рівняння Шокарда

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Стаття присвячена фокусувальному неоднорідному рівнянню Шокарда з лінійним демпфуванням:

$$i\dot{u} + \triangle u + iau = -|x|^{-\gamma}(I_\alpha * |u|^p)|u|^{p-2}u,$$

де $a \geq 0$ і $0 < \gamma < \inf(N, 2+\alpha)$. Глобальне існування і розсіювання доведені для відносно великого демпфування. Для довільного демпфування одержано глобальне існування, коли початкові дані належать до певних інваріантних множин.

Ключові слова: демпфоване рівняння Шокарда, велике демпфування, глобальне існування, розсіювання, інваріантні множини