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Exact Solutions of Nonlinear Equations in Mathematical Physics via Negative Power Expansion Method

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In this paper, a direct method called negative power expansion (NPE) method is presented and extended to construct exact solutions of nonlinear mathematical physical equations. The presented NPE method is also effective for the coupled, variable-coefficient and some other special types of equations. To illustrate the effectiveness, the $(2+1)$-dimensional dispersive long wave (DLW) equations, Maccari's equations, Tzitzeica–Dodd–Bullough (TDB) equation, Sawada–Kotera (SK) equation with variable coefficients and two lattice equations are considered. As a result, some exact solutions are obtained including traveling wave solutions, non-traveling wave solutions and semi-discrete solutions. This paper shows that the NPE method is a simple and effective method for solving nonlinear equations in mathematical physics.

Key words: exact solution, NPE method, (2+1)-dimensional DLW equations, Maccari's equations, TDB equation, SK equation with variable coefficients, lattice equations

Mathematical Subject Classification 2010: 35Q51, 35J99, 68W30

1. Introduction

Constructing exact solutions of nonlinear mathematical physical equations is of theoretical and practical significance. Since the famous Korteweg–de Vries (KdV) equation was solved in 1967 [7], a large number of exact solutions like [2–7,10,11,13–16,18–20,22–28,31,33,34,36,38,41,43] of nonlinear partial differential equations (PDEs) have been found. The exp-function method [10] proposed by He and Wu has been widely used for constructing exact solutions of nonlinear PDEs. As for the last development of this method, we would like to mention that the exp-function method [10] has been adopted to construct solitary solutions, blowup solutions and discontinuous solutions of the generalized Boussinesq equation [12], the fractal Boussinesq equation [12] and the generalized KdV–Burgers equation [9].

In the process of trying to solve the problem of "expansion of intermediate expression" caused by ansatz solution of the exp-function method [10], Zhang and Li [32], and Zhang, You and Xu [37] respectively put forward the direct algorithm of the exp-function method and the simplest exp-function method, which are extended and uniformly called in this paper the negative power expansion method or the NPE method for short. In [32, 37], the KdV equation, the (3+1)-dimensional Jimbo–Miwa (JM) and two special cases of the Mikhauilov–Novikov–Wang (MNW) equations were taken as three examples of classical, high-dimensional and high-order equations to test the validity of the primary form of the NPE method. This paper will present and extend the NPE method to the coupled, variable-coefficient and some other special types of equations, including the (2+1)-dimensional DLW equations, Maccari's equations, TDB equation, SK equation with variable coefficients and two lattice equations.

The rest of this paper is organized as follows. In Section 2, we describe the NPE method. In Section 3, we extend the NPE method to the coupled, variable-coefficient and some other special types of equations. In Section 4, the comparisons between the NPE method and the exp-function method are given. In Section 5, some conclusions and discussions are given.

2. Description of the NPE method

For the given $(m+1)$-dimensional nonlinear PDE

$$P(u, u_t, u_{x_1}, \ldots, u_{x_m}, u_{tx_1}, \ldots, u_{tx_m}, u_{tt}, u_{x_1x_1}, \ldots, u_{x_mx_m}, \cdots) = 0, \tag{2.1}$$

where P is a polynomial of the dependent variable u and its derivatives with respect to the independent variables $\{t, x_1, x_2, \cdots, x_m\}$ or P can be transformed into a polynomial after a suitable transformation of u. To determine u by the NPE method, we take the following three steps:

Step 1. Supposing that the ansatz solution of (2.1) has the form

$$u = \sum_{i=0}^{n} u_i \phi^{i-n}, \ \phi = \mathrm{e}^{\xi} + a, \tag{2.2}$$

where ξ and u_i $(u_0 \neq 0)$ are undetermined functions of $\{t, x_1, x_2, \ldots, x_m\}$, a is the embedded constant parameter, n is a nonnegative integer determined by balancing the highest order nonlinear terms and the highest order partial derivative terms in (2.1).

Step 2. Substituting (2.2) into (2.1) and collecting all the coefficients of $\phi^{-j} (j = 0, 1, 2, \ldots)$, then setting each coefficient of the same power of ϕ to zero to derive a set of over-determined PDEs for ξ or some other undetermined parameters introduced by using a necessary simplified form of ξ and u_i.

Step 3. Solving the set of over-determined PDEs derived in ***Step 2*** with the help of *Mathematica* or *Maple* to determine ξ and u_i, and finally, to determine (2.2), namely a solution of (2.1).

We note here that if $a = 1$, then the NPE method described above corresponds to its primary form [32, 37].

Theorem 2.1. *Let the highest order nonlinear term and the highest order partial derivative term in* (2.1) *be* $u_{x_1}^{(s)}$ *and* $(u_{x_1}^{(p)})^q(u_{x_1}^{(r)})^l u^h$ *respectively. Then the nonnegative integer* n *in* (2.2) *has the following formula*

$$n = \frac{s - lr - pq}{h + l + q - 1}, \tag{2.3}$$

where h, l, p, q, r *and* s *are all nonnegative integers.*

Proof. From (2.2), we have

$$u_{x_1}^{(s)} = -n(-n-1)\cdots(-n-s+1)\, u_0\, \phi^{-n-s}\, \phi'^s + \cdots, \tag{2.4}$$

and therefore determine the highest negative orders of ϕ in $u_{x_1}^{(s)}$, $(u_{x_1}^{(p)})^q$ and $(u_{x_1}^{(r)})^l$ as

$$\deg(u_{x_1}^{(s)}) = -n - s, \quad \deg[(u_{x_1}^{(p)})^q] = q(-n-p), \quad \deg[(u_{x_1}^{(r)})^l] = l(-n-r). \tag{2.5}$$

At the same time, we have $\deg(u^h) = -hn$. Thus,

$$\deg[(u_{x_1}^{(p)})^q(u_{x_1}^{(r)})^l u^h] = q(-n-p) + l(-n-r) - hn. \tag{2.6}$$

So, when balancing $u_{x_1}^{(s)}$ and $(u_{x_1}^{(p)})^q(u_{x_1}^{(r)})^l u^h$, we have

$$-n - s = q(-n-s) + l(-n-r) - hn, \tag{2.7}$$

which is namely (2.3). □

Theorem 2.1 shows that the NPE method for determining the value of n in (2.2) is different from that of the auxiliary equation methods [23, 25, 27, 33–35, 38], in which the value of n is related to the auxiliary equations. The main reason is that the expansions of the ansatz solutions for these two methods are different. The ansatz solution (2.2) is a negative power expansion of ϕ, while the corresponding ansatz solution of an auxiliary equation method is a polynomial expansion of ϕ which satisfies an auxiliary equation.

For the KdV equation [1],

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.8}$$

we can express its solution by

$$u = \frac{u_0}{(\mathrm{e}^{\xi} + a)^2} + \frac{u_1}{\mathrm{e}^{\xi} + a} + u_2, \quad \xi = kx + ct + w, \tag{2.9}$$

where k, c and w are arbitrary constants and the functions u_0, u_1 and u_2 are determined as

$$u_0 = -2k^2\mathrm{e}^{2\xi}, \quad u_1 = 2k^2\mathrm{e}^{\xi}, \quad u_2 = -\frac{k^3 + c}{6k}. \tag{2.10}$$

When $a=\pm 1$, we can write (2.9) as

$$u=\frac{k^2}{2}\operatorname{sech}^2\frac{\xi}{2}-\frac{k^3+c}{6k}, \quad u=-\frac{k^2}{2}\operatorname{csch}^2\frac{\xi}{2}-\frac{k^3+c}{6k}, \tag{2.11}$$

respectively. When $a=0$, (2.9) is a constant solution $u=-(k^3+c)/6k$. Selecting $k=\mathrm{i}\hat{k}$ and $c=\mathrm{i}\hat{c}$, here and thereafter $\mathrm{i}^2=-1$, we obtain two trigonometric function solutions from (2.11),

$$u=-\frac{\hat{k}^2}{2}\sec^2\frac{\xi}{2}+\frac{\hat{k}^3-\hat{c}}{6\hat{k}}, \quad u=-\frac{\hat{k}^2}{2}\csc^2\frac{\xi}{2}+\frac{\hat{k}^3-\hat{c}}{6\hat{k}}. \tag{2.12}$$

Similarly, for the high-order equation [37],

$$\begin{aligned} u_t=&-u_{7x}+49u_xu_{xxxx}+14uu_{5x}+84u_{xx}u_{xxx}-70u_x^3\\ &-252uu_xu_{xx}-56u^2u_{xxx}+\frac{224}{3}u^3u_x, \end{aligned} \tag{2.13}$$

which is a special case of the MNW equation [21], we have

$$u_0=\frac{3k^2}{2}\mathrm{e}^{2\xi}, \quad u_1=-\frac{3k^2}{2}\mathrm{e}^{\xi}, \quad u_2=\frac{k^2}{8}, \quad c=\frac{k^7}{48}, \tag{2.14}$$

and hence obtain a solution

$$u=\frac{3k^2\mathrm{e}^{2\xi}}{2(\mathrm{e}^{\xi}+a)^2}-\frac{3k^2\mathrm{e}^{\xi}}{2(\mathrm{e}^{\xi}+a)}+\frac{k^2}{8}, \quad \xi=kx+\frac{k^7}{48}t+w. \tag{2.15}$$

As a high-dimensional model, the (3+1)-dimensional Jimbo-Miwa (JM) equation [32],

$$u_{xxxy}+3u_yu_{xx}+3u_xu_{xy}+2u_{yt}-3u_{xz}=0, \tag{2.16}$$

expresses its ansatz solution by

$$u=\frac{u_0}{\phi}+u_1, \quad \phi=a+\mathrm{e}^{kx+\eta}, \tag{2.17}$$

where k is a constant to be determined later, u_0, u_1 and η are undetermined functions of $\{x,y,z,t\}$. We substitute (2.17) into the JM equation (2.16) and then set the coefficients of ϕ^{-j} $(j=0,1,2,\ldots,5)$ to be zeros. A set of PDEs is derived, from which we have

$$u_0=2k\mathrm{e}^{\xi}, \quad u_1=\frac{f_3'(z)y}{k}-\frac{k^3+2p}{3k^2}f_1(y,z)+\frac{1}{k}\int f_{1z}(y,z)\,\mathrm{d}y, \tag{2.18}$$

$$\eta=f_1(y,z)+f_2(z)+pt, \tag{2.19}$$

and therefore obtain a solution of the JM equation (2.16),

$$u=\frac{2k\mathrm{e}^{\xi}}{1+\mathrm{e}^{\xi}}+\frac{f_3'(z)y}{k}-\frac{k^3+2p}{3k^2}f_1(y,z)+\frac{1}{k}\int f_{1z}(y,z)\,\mathrm{d}y, \tag{2.20}$$

where $\xi=kx+f_1(y,z)+f_2(z)+pt$, $f_1(y,z)$, $f_2(z)$ are smooth functions of the indicated variables, $f_3'(z)=\mathrm{d}f_3(z)/\mathrm{d}z$, k is a non-zero constant, and p is an arbitrary constant.

3. Extensions of the NPE method to other equations

3.1. Coupled equations. We first consider the (2+1)-dimensional DLW equations [35],

$$u_{yt} + H_{xx} + \frac{1}{2}(u^2)_{xy} = 0, \tag{3.1}$$

$$H_t + (uH + u + u_{xy})_x = 0. \tag{3.2}$$

Here we denote $\deg H = m$ and $\deg u = n$. Balancing uH and u_{xy}, we have $-n-m = -n-2$, i.e, $m = 2$. At the same time, balancing H_{xx} and $(u^2)_{xy}$ yields $-m-2 = -2n-2$, i.e, $n = 1$. We suppose

$$H = \frac{H_0}{\phi^2} + \frac{H_1}{\phi} + H_2, \tag{3.3}$$

$$u = \frac{u_0}{\phi} + u_1, \tag{3.4}$$

where $\phi = e^{\xi} + a$, $\xi = kx + ly + ct + w$, k, l, c and w are undetermined constants, while H_0, H_1, H_2, u_0 and u_1 are undetermined functions. Substituting (3.3) and (3.4) into the DLW equations (3.1) and (3.2), then setting each of the coefficients of $\phi^{-j}(j = 0, 1, 2, \cdots, 4)$ to be zero, we derive two sets of PDEs:

$$\begin{aligned}
\phi^{-4}:\ & 6k^2e^{2\xi}H_0 + 3kle^{2\xi}u_0^2 = 0,\\
\phi^{-3}:\ & -2k^2e^{\xi}H_0 + 2k^2e^{2\xi}H_1 + 2cle^{2\xi}u_0 - kle^{\xi}u_0^2 + 2kle^{2\xi}u_0u_1 - 2ke^{\xi}u_0u_{0y}\\
& -4ke^{\xi}H_{0x} - 2le^{\xi}u_0u_{0x} = 0,\\
\phi^{-2}:\ & -k^2e^{\xi}H_1 - cle^{\xi}u_0 - kle^{\xi}u_0u_1 - le^{\xi}u_{0t} - ce^{\xi}u_{0y} - ke^{\xi}u_1u_{0y} - ke^{\xi}u_0u_{1y}\\
& -2ke^{\xi}H_{1x} - le^{\xi}u_1u_{0x} + u_{0y}u_{0x} - le^{\xi}u_0u_{1x} + u_0u_{0xy} + H_{0xx} = 0,\\
\phi^{-1}:\ & u_{0yt} + u_{1y}u_{0x} + u_{0y}u_{1x} + u_1u_{0xy} + u_0u_{1xy} + H_{1xx} = 0,\\
\phi^{0}:\ & u_{1yt} + u_{1y}u_{1x} + u_1u_{1xy} + H_{2xx} = 0,\\
\phi^{-4}:\ & -6k^2le^{3\xi}u_0 - 3ke^{\xi}H_0u_0 = 0,\\
\phi^{-3}:\ & -2ce^{\xi}H_0 + 6k^2le^{2\xi}u_0 - 2ke^{\xi}H_1u_0 - 2ke^{\xi}H_0u_1 + 2k^2e^{2\xi}u_{0y} + H_{0x}u_0\\
& +4kle^{2\xi}u_{0x} + H_0u_{0x} = 0,\\
\phi^{-2}:\ & -ce^{\xi}H_1 - ke^{\xi}u_0 - k^2le^{\xi}u_0 - ke^{\xi}H_2u_0 - ke^{\xi}H_1u_1 + H_{0t} - k^2e^{\xi}u_{0y}\\
& +u_1H_{0x} + u_0H_{1x} - 2kle^{\xi}u_{0x} + H_1u_{0x} + H_0u_{1x} - 2ke^{\xi}u_{0xy} - le^{\xi}u_{0xx} = 0,\\
\phi^{-1}:\ & H_{1t} + u_1H_{1x} + u_0H_{2x} + u_{0x} + H_2u_{0x} + H_1u_{1x} + u_{0xxy} = 0,\\
\phi^{0}:\ & H_{2t} + H_{2x}u_1 + u_{1x} + H_2u_1 + u_{1xxy} = 0.
\end{aligned}$$

Solving the above sets of PDEs, we have

$$H_0 = -2kle^{2\xi}, \quad H_1 = 2kle^{\xi}, \quad H_2 = -1, \quad u_0 = \pm 2ke^{\xi}, \quad u_1 = \frac{\pm k^2 - c}{k}, \tag{3.5}$$

and obtain the solutions of the DLW equations (3.1) and (3.2),

$$H = -\frac{2kle^{2\xi}}{(e^{\xi} + a)^2} + \frac{2kle^{\xi}}{e^{\xi} + a} + \frac{\pm k^2 - c}{k}, \tag{3.6}$$

$$u = \pm\frac{2k\mathrm{e}^{\xi}}{\mathrm{e}^{\xi}+a} + \frac{\pm k^2 - c}{k}, \tag{3.7}$$

where $\xi = kx + ly + ct + w$, $k \neq 0$, l, c and w are all constants.

We next consider the Maccari's equations [17],

$$\mathrm{i}Q_t + Q_{xx} + QR = 0, \tag{3.8}$$

$$R_t + R_y + (|Q|^2)_x = 0. \tag{3.9}$$

Supposing that

$$Q = u(x,y,t)\mathrm{e}^{\mathrm{i}(px+qy+ct+d)}, \tag{3.10}$$

where p, q and c are undetermined constants, d is an arbitrary constant. Substituting (3.10) into the Maccari's equations (3.8) and (3.9) yields

$$\mathrm{i}(u_t + 2pu_x) + u_{xx} - (c + k^2)u + uR = 0, \tag{3.11}$$

$$R_t + R_y + (u^2)_x = 0. \tag{3.12}$$

Letting $\xi = k(x + ly - 2kt + w)$, here k and l are undetermined constants, w denotes arbitrary constants, then we can transform (3.11) and (3.12) into

$$k^2u'' - (c + k^2)u + uR = 0, \tag{3.13}$$

$$(l - 2p)R' + (u^2)' = 0. \tag{3.14}$$

Integrating (3.14) with respect to ξ once and selecting the integration constant as zero, we have

$$R = -\frac{1}{l-2p}u^2. \tag{3.15}$$

Substituting (3.15) into (3.13) yields

$$l^2u'' - (c + k^2)u - \frac{1}{l-2p}u^3 = 0. \tag{3.16}$$

Balancing u'' and u^3, we have $-n - 2 = -3n$, i.e., $n = 1$. Thus, we suppose

$$u = \frac{u_0}{\phi} + u_1, \quad \phi = \mathrm{e}^{\xi} + a, \tag{3.17}$$

where u_0 and u_1 are undetermined function of ξ. Substituting (3.17) into (3.16) and setting each coefficient of ϕ^{-j} $(j = 0, 1, 2, 3)$ to be zero, we derive a set of ordinary differential equations (ODEs):

$$\phi^{-3}: \ 2\mathrm{e}^{2\xi}u_0 - \frac{u_0^3}{l-2p} = 0,$$

$$\phi^{-2}: \ -\mathrm{e}^{\xi}u_0 - \frac{3u_0^2u_1}{l-2p} - 2\mathrm{e}^{\xi}u_0' = 0,$$

$$\phi^{-1}: \ u_0\left(-c - k^2 - \frac{3u_1^2}{l-2p}\right) + u_0'' = 0,$$

$$\phi^0: \quad -(c+k^2)u_1 - \frac{u_1^3}{l-2p} + u_1'' = 0.$$

Solving the above set of ODEs, we have

$$u_0 = \pm\sqrt{2l-4p}\mathrm{e}^{\xi}, \quad u_1 = \mp\frac{\sqrt{2l-4p}}{2}, \quad c = -\frac{1+2k^2}{2}, \tag{3.18}$$

and hence obtain the solutions of the Maccari's equations (3.8) and (3.9):

$$Q = \pm\sqrt{2l-4p}\left(\frac{\mathrm{e}^{\xi}}{\mathrm{e}^{\xi}+a} - \frac{1}{2}\right)\mathrm{e}^{\mathrm{i}\left(px+qy-\frac{1+2k^2}{2}t+d\right)}, \tag{3.19}$$

$$R = \mp 2\left(\frac{\mathrm{e}^{\xi}}{\mathrm{e}^{\xi}+a} - \frac{1}{2}\right)^2, \tag{3.20}$$

where $\xi = k(x+ly-2kt+w)$, k, l, d and w are constants.

3.2. Special type equation. We have the following theorem for the special type model—TDB equation.

Theorem 3.1. *The TDB equation* [8],

$$u_{xt} = e^{-u} + e^{-2u}, \tag{3.21}$$

has a pair of solutions

$$u = \operatorname{arcsinh}\frac{v^{-1}-v}{2}, \tag{3.22}$$

with

$$v = \pm\frac{\mathrm{e}^{\xi}}{\mathrm{e}^{\xi}+a} - \frac{1\pm 1}{2}, \quad \xi = kx - \frac{t}{k} + w, \tag{3.23}$$

where $k \neq 0$ and w are constants.

Proof. Taking the transformation

$$u = \operatorname{arcsinh}\frac{v^{-1}(x,t) - v(x,t)}{2}, \tag{3.24}$$

we transform the TDB equation (3.21) into

$$-vv_{xt} + v_x v_t - v^3 - v^4 = 0. \tag{3.25}$$

Balancing vv_{xt} and v^4, we have $-2n-2 = -4n$, i.e., $n = 1$. We suppose

$$v = \frac{v_0}{\phi} + v_1, \;\; \phi = \mathrm{e}^{\xi} + a, \tag{3.26}$$

where $\xi = kx + ct + w$, k and c are undetermined constants, w is an arbitrary constant, v_0 and v_1 are undetermined functions of $\{x,t\}$. Substituting (3.26) into

(3.25) and setting each coefficient of $\phi^{-j}(j=0,1,2,\dots,4)$ to be zero yields a set of ODEs:

$$\begin{aligned}
\phi^{-4}:&\ -cke^{2\xi}v_0^2-v_0^4=0,\\
\phi^{-3}:&\ cke^{\xi}v_0^2-v_0^3-2cke^{2\xi}v_0v_1+4v_0^3v_1=0,\\
\phi^{-2}:&\ cke^{\xi}v_0v_1-3v_0^2v_1-6v_0^2v_1^2+ke^{\xi}v_1v_{0t}-ke^{\xi}v_0v_{1t}+ce^{\xi}v_1v_{0x}+v_{0x}v_{0t}\\
&\ -ce^{\xi}v_0v_{1x}-v_0v_{0xt}=0,\\
\phi^{-1}:&\ -3v_0v_1^2-4v_0v_1^3+v_{0x}v_{1t}+v_{1x}v_{0t}-v_1v_{0xt}-v_0v_{1xt}=0,\\
\phi^{0}:&\ v_1v_{1xt}-v_{1x}v_{1t}+v_1^3+v_1^4=0.
\end{aligned}$$

Solving the above set of ODEs, we have

$$v_0=\pm e^{\xi},\quad v_1=-\frac{1\pm1}{2},\quad c=-\frac{1}{k},\tag{3.27}$$

and hence obtain the solutions (3.22). □

3.3. Variable-coefficient equation. We have the following theorem for the variable-coefficient model, namely the SK equation.

Theorem 3.2. *The SK equation [30] with the variable coefficients*

$$u_t+f(t)u^2u_x+g(t)u_xu_{xx}+h(t)uu_{xxx}+k(t)u_{xxxxx}=0,\tag{3.28}$$

has a pair of solutions

$$u=\frac{3p^2\omega(t)e^{2\xi}}{f(t)(1+e^{\xi})^2}-\frac{3p^2\omega(t)e^{\xi}}{f(t)(1+e^{\xi})}+\frac{p^2\omega(t)}{4f(t)},\tag{3.29}$$

where

$$\xi=px+p^5\int\frac{12f(t)k(t)+g(t)\omega(t)}{8f(t)}\,dt+w,\tag{3.30}$$

$$\omega(t)=-g(t)-2h(t)\pm\sqrt{\theta(t)},\quad \theta(t)=[g(t)+2h(t)]^2-40f(t)k(t),\tag{3.31}$$

p and w are constants, and the coefficient functions $f(t)$, $g(t)$, $h(t)$ and $k(t)$ satisfy the condition

$$\begin{aligned}20f^2(t)k'(t)=\omega(t)\{f'(t)[g(t)+2h(t)]-f(t)[g'(t)\\
+2h'(t)]\}+20k(t)f(t)f'(t).\end{aligned}\tag{3.32}$$

Proof. Balancing u_{xxxxx} and u_xu_{xx} yield $n=2$. We suppose

$$u=\frac{u_0}{\phi^2}+\frac{u_1}{\phi}+u_2,\quad \phi=e^{\xi}+a,\tag{3.33}$$

where $\xi=px+q(t)+w$, p and $q(t)$ are undetermined constant and function respectively, w is an arbitrary constant, u_0, u_1 and u_2 are undetermined functions of $\{x,t\}$. Substituting (3.33) into the SK equation (3.28) and setting each coefficient of $\phi^{-j}(j=0,1,2,\ldots,7)$ to be zero yield a set of PDEs. Under the constraint (3.32), from the set of PDEs we have

$$u_0=\frac{3p^2\omega(t)}{f(t)(1+\mathrm{e}^{\xi})^2},\quad u_1=-\frac{3p^2\omega(t)}{f(t)(1+\mathrm{e}^{\xi})},\quad u_2=\frac{p^2\omega(t)}{4f(t)},\tag{3.34}$$

$$q(t)=p^5\int\frac{12f(t)k(t)+g(t)\omega(t)}{8f(t)}\mathrm{d}t,\tag{3.35}$$

and hence obtain the solutions (3.29). □

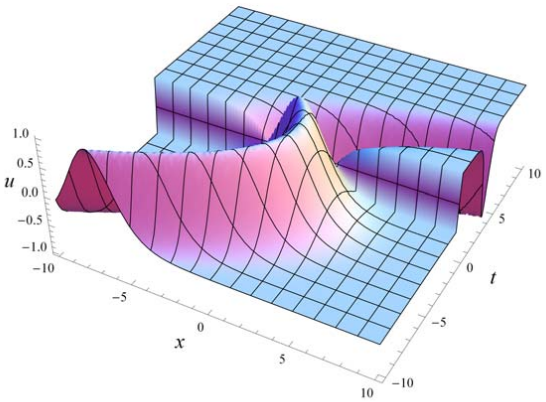

Fig. 3.1: The bright-dark bell-soliton structure of the solution (3.29).

Figure 3.1 shows a bright-dark bell-soliton structure of the solution (3.29) with "+" branch, where we select $f(t)=t$, $g(t)=-2t-4$, $h(t)=t+2$, $k(t)=-0.1t$, $a=1$, $p=1$ and $w=0$. It can be seen from Figure 3.1 that the coefficient functions $f(t)$, $g(t)$, $h(t)$ and $k(t)$ affect the propagation speed of the soliton and then the trajectory of the soliton forms a bright-dark bell spatial structure.

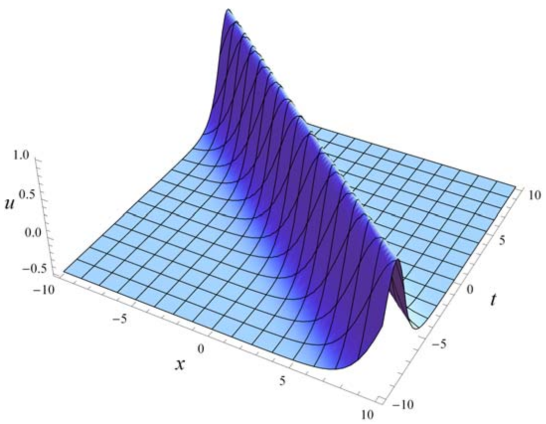

Fig. 3.2: The bright bell-soliton structure of the solution (3.29).

Letting $f(t)$, $g(t)$, $h(t)$ and $k(t)$ be constants, in Figure 3.2, we show a typical bright bell-soliton structure of the solution (3.29) with "+" branch by selecting $f(t) = -1$, $g(t) = -4$, $h(t) = 2$, $k(t) = 0.1$, $a = 1$, $p = 1$ and $w = 0$.

3.4. Lattice equations. For the lattice equations, because of the complexity of the iteration formulae of ansatz solutions, the steps of the NPE method for solving the continuous equations should be adjusted.

Theorem 3.3. *The lattice equation* [29],

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = (\alpha + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}), \tag{3.36}$$

has a pair of solutions

$$u_n = \pm\frac{(\mathrm{e}^d - 1)\sqrt{\beta^2 - 4\alpha\beta}}{\gamma(\mathrm{e}^d + 1)(\mathrm{e}^{\xi_n} + a)} - \frac{\beta(\mathrm{e}^d + 1) \pm (\mathrm{e}^d - 1)\sqrt{\beta^2 - 4\alpha\beta}}{2\gamma(\mathrm{e}^d + 1)}, \tag{3.37}$$

where $\xi_n = dn - \{(e^d - 1)(\beta^2 - 4\alpha\beta)/[\gamma(e^d + 1)]\}t + w$, α, β, γ, d *and* w *are constants.*

Proof. Balancing $\mathrm{d}u_n/\mathrm{d}t$ and u_n^2 yield $n = 1$. We suppose

$$u_n = \frac{u_{n,0}}{\phi_n} + u_{n,1}, \tag{3.38}$$

$$u_{n+1} = \frac{u_{n+1,0}}{\mathrm{e}^d\phi_n + a(1 - \mathrm{e}^d)} + u_{n+1,1}, \tag{3.39}$$

$$u_{n-1} = \frac{u_{n-1,0}}{\mathrm{e}^{-d}\phi_n + a(1 - \mathrm{e}^{-d})} + u_{n-1,1}, \tag{3.40}$$

where $\phi_n = \mathrm{e}^{\xi_n} - a$, $\xi_n = dn + ct + w$, d and c are undetermined constants, w is an arbitrary constant, $u_{n,0}$ and $u_{n,1}$ are undetermined functions of $\{n, t\}$. Substituting (3.38)–(3.40) into the lattice equation (3.36), replacing ϕ_n' with $\phi_n - a$ and eliminating the factors $\mathrm{e}^d\phi_n + a(1 - \mathrm{e}^d)$ and $\mathrm{e}^{-d}\phi_n + a(1 - \mathrm{e}^{-d})$ in the denominators and then setting each coefficient of $\phi_n^{2-\mu}$ $(\mu = 0, 1, 2, \ldots, 4)$ to be zero yield a set of differential-difference equations (DDEs) for $u_{n,0}$, $u_{n,1}$, k and c. Solving the set of DDEs, we have

$$u_{n,0} = \pm\frac{(\mathrm{e}^d - 1)\sqrt{\beta^2 - 4\alpha\beta}}{\gamma(\mathrm{e}^d + 1)},\quad u_{n,1} = -\frac{\beta(\mathrm{e}^d + 1) \pm (\mathrm{e}^d - 1)\sqrt{\beta^2 - 4\alpha\beta}}{2\gamma(\mathrm{e}^d + 1)}, \tag{3.41}$$

$$c = -\frac{(\mathrm{e}^d - 1)(\beta^2 - 4\alpha\beta)}{\gamma(\mathrm{e}^d + 1)}. \tag{3.42}$$

and finally arrive at the solutions (3.37). □

Theorem 3.4. *The Toda lattice equation* [39],

$$\frac{\mathrm{d}^2u_n}{\mathrm{d}t_2} = \left(\frac{\mathrm{d}u_n}{\mathrm{d}t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}), \tag{3.43}$$

has a solution

$$u_n = -\frac{ac}{e^{\xi_n} + a} + \frac{[(c^2-1)e^{2k} + 2e^k - 1]\xi_n}{c(e^k-1)^2} + c_0, \tag{3.44}$$

where $\xi_n = kn + ct + w$, c, c_0 *and* w *are constants.*

Proof. Balancing d^2u_n/dt^2 and $(du_n/dt)u_n$ yield $n = 1$. We employ (3.38)-(3.40) and substitute them into the Toda lattice equation (3.43), replacing ϕ_n' with $\phi_n - a$ and eliminating the factors $e^d\phi_n + a(1-e^d)$ and $e^{-d}\phi_n + a(1-e^{-d})$ in the denominators and then set each coefficient of $\phi_n^{2-\mu}$ $(\mu = 0, 1, 2, \ldots, 4)$ to zero to obtain a set of DDEs for $u_{n,0}$, $u_{n,1}$, k and c. Solving the set of DDEs, we have

$$u_{n,0} = -ac, \quad u_{n,1} = \frac{[(c^2-1)e^{2d} + 2e^d - 1]\xi_n}{c(e^d-1)^2} + c_0, \tag{3.45}$$

and finally arrive at the solution (3.44). □

As pointed out by Zhang et al. in [40, 42], the exact solutions with external linear functions possess a remarkable dynamical property, which is that a solitary wave does not propagate in the horizontal direction as a traditional wave. In Figure 3.3, a semi-discrete kink-soliton structure of the solution (3.44) with this characteristic is shown by selecting $a = 1$, $c = 1.06$, $d = 1$ and $c_0 = 0$.

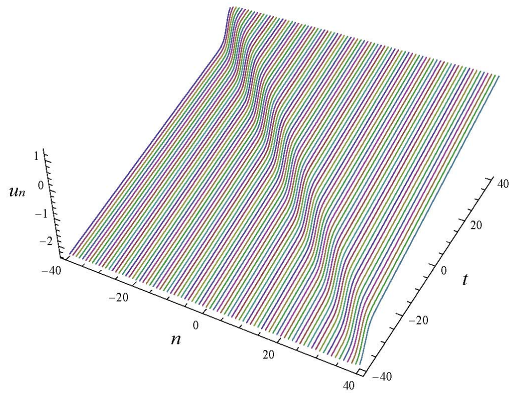

Fig. 3.3: The semi-discrete kink-soliton structure of the solution (3.44).

4. Comparisons between the NPE and exp-function methods

To compare the NPE method and the exp-function method [10] more precisely, we consider in this section the KdV equation (2.8) and the Burgers equation [1],

$$u_t + 2uu_x - u_{xx} = 0. \tag{4.1}$$

Employing the NPE method to solve the KdV equation (2.8), we substitute the ansatz solution (2.9) into the KdV equation (2.8) and then set each coefficient of $\phi^{-j}(j = 0, 1, 2, \ldots, 5)$ to be zero, to obtain a set of PDEs [32]:

$$\phi^{-5}: \quad -24k^3e^{3\xi}u_0 - 12ke^{\xi}u_0^2 = 0, \tag{4.2}$$

$$\phi^{-4}: \; 18k^3 e^{2\xi} u_0 - 6k^3 e^{3\xi} u_1 - 18k e^{\xi} u_0 u_1 + 18k^2 e^{2\xi} u_{0x} + 6u_0 u_{0x} = 0, \tag{4.3}$$

$$\begin{aligned}\phi^{-3}: \; & -2c e^{\xi} u_0 - 2k^3 e^{\xi} u_0 + 6k^3 e^{2\xi} u_1 - 6k e^{\xi} u_1^2 - 12k e^{\xi} u_0 u_2 - 6k^2 e^{\xi} u_{0x} \\ & + 6u_1 u_{0x} + 6k^2 e^{2\xi} u_{1x} + 6u_0 u_{1x} - 6k e^{\xi} u_{0xx} = 0, \end{aligned}\tag{4.4}$$

$$\begin{aligned}\phi^{-2}: \; & -c e^{\xi} u_1 - k^3 e^{\xi} u_1 - 6k e^{\xi} u_1 u_2 + u_{0t} + 6u_2 u_{0x} - 3k^2 e^{\xi} u_{1x} + 6u_1 u_{1x} \\ & + 6u_0 u_{2x} - 3k e^{\xi} u_{1xx} + u_{0xxx} = 0, \end{aligned}\tag{4.5}$$

$$\phi^{-1}: \; u_{1t} + 6u_2 u_{1x} + 6u_1 u_{2x} + u_{1xxx} = 0, \tag{4.6}$$

$$\phi^{0}: \; u_{2t} + 6u_2 u_{2x} + u_{2xxx} = 0. \tag{4.7}$$

Solving the above set of PDEs (4.2)–(4.7), we reach (2.10) and finally obtain the exact solution of the KdV equation (2.8),

$$u = -\frac{2k^2 e^{2\xi}}{(e^{\xi} + a)^2} + \frac{2k^2 e^{\xi}}{e^{\xi} + a} - \frac{k^3 + c}{6k}, \tag{4.8}$$

where $\xi = kx + ct + w$.

Following the steps of the exp-function method [10] for the KdV equation (2.8), we suppose

$$u = \frac{a_0 + a_1 e^{\xi} + a_2 e^{2\xi}}{b_0 + b_1 e^{\xi} + b_2 e^{2\xi}}, \quad \xi = kx + ct + w, \tag{4.9}$$

where a_0, a_1, a_2, b_0, b_1, b_2, k and c are all undetermined constants, w is an arbitrary constant. Substituting the ansatz solution (4.9) into the KdV equation (2.8) and eliminating the denominator $(b_0 + b_1 e^{\xi} + b_2 e^{2\xi})^4$, then setting each coefficient of $e^{j\xi}$ $(j = 1, 2, \ldots, 7)$ to be zero, a set of algebraic equations is derived as follows:

$$e^{\xi}: \; a_1 b_0^3 c - a_0 b_0^2 b_1 c + 6a_0 a_1 b_0^2 k - 6a_0^2 b_0 b_1 k + a_1 b_0^3 k^3 - a_0 b_0^2 b_1 k^3 = 0, \tag{4.10}$$

$$\begin{aligned} e^{2\xi}: \; & 2a_2 b_0^3 c + 2a_1 b_0^2 b_1 c - 2a_0 b_0 b_1^2 c - 2a_0 b_0^2 b_2 c + 6a_1^2 b_0^2 k + 12a_0 a_2 b_0^2 k - 6a_0^2 b_1^2 k \\ & - 12a_0^2 b_0 b_2 k + 8a_2 b_0^3 k^3 - 4a_1 b_0^2 b_1 k^3 + 4a_0 b_0 b_1^2 k^3 - 8a_0 b_0^2 b_2 k^3 = 0, \end{aligned}\tag{4.11}$$

$$\begin{aligned} e^{3\xi}: \; & 5a_2 b_0^2 b_1 c + a_1 b_0 b_1^2 c - a_0 b_1^3 c + a_1 b_0^2 b_2 c - 6a_0 b_0 b_1 b_2 c + 18a_1 a_2 b_0^2 k + 6a_1^2 b_0 b_1 k \\ & + 12a_0 a_2 b_0 b_1 k - 6a_0 a_1 b_1^2 k - 12a_0 a_1 b_0 b_2 k - 18a_0^2 b_1 b_2 k + 5a_2 b_0^2 b_1 k^3 \\ & + a_1 b_0 b_1^2 k^3 - a_0 b_1^3 k^3 - 23a_1 b_0^2 b_2 k^3 + 18a_0 b_0 b_1 b_2 k^3 = 0, \end{aligned}\tag{4.12}$$

$$\begin{aligned} e^{4\xi}: \; & 4a_2 b_0 b_1^2 c + 4a_2 b_0^2 b_2 c - 4a_0 b_1^2 b_2 c - 4a_0 b_0 b_2^2 c + 12a_2^2 b_0^2 k + 24a_1 a_2 b_0 b_1 k \\ & - 24a_0 a_1 b_1 b_2 k - 12a_0^2 b_2^2 k + 4a_2 b_0 b_1^2 k^3 - 32a2b_0^2 b2k^3 - 4a_0 b_1^2 b_2 k^3 \\ & + 32a_0 b_0 b_2^2 k^3 - 3k e^{\xi} u_{1xx} + u_{0xxx} = 0, \end{aligned}\tag{4.13}$$

$$\begin{aligned} e^{5\xi}: \; & a_2 b_1^3 c + 6a_2 b_0 b_1 b_2 c - a_1 b_1^2 b_2 c - a_1 b_0 b_2^2 c - 5a_0 b_1 b_2^2 c + 18a_2^2 b_0 b_1 k + 6a_1 a_2 b_1^2 k \\ & + 12a_1 a_2 b_0 b_2 k - 6a_1^2 b_1 b_2 k - 12a_0 a_2 b_1 b_2 k - 18a_0 a_1 b_2^2 k + a_2 b_1^3 k^3 \\ & - 18a_2 b_0 b_1 b_2 k^3 - a_1 b_1^2 b_2 k^3 + 23a_1 b_0 b_2^2 k^3 - 5a_0 b_1 b_2^2 k^3 = 0, \end{aligned}\tag{4.14}$$

$$\begin{aligned} e^{6\xi}: \; & 2a_2 b_1^2 b_2 c + 2a_2 b_0 b_2^2 c - 2a_1 b_1 b_2^2 c - 2a_0 b_2^3 c + 6a_2^2 b_1^2 k + 12a_2^2 b_0 b_2 k - 6a_1^2 b_2^2 k \\ & - 12a_0 a_2 b_2^2 k - 4a_2 b_1^2 b_2 k^3 + 8a_2 b_0 b_2^2 k^3 + 4a_1 b_1 b_2^2 k^3 - 8a_0 b_2^3 k^3 = 0, \end{aligned}\tag{4.15}$$

$$e^{7\xi}: \; a_2b_1b_2^2c - a_1b_2^3c + 6a_2^2b_1b_2k - 6a_1a_2b_2^2k + a_2b_1b_2^2k^3 - a_1b_2^3k^3 = 0. \tag{4.16}$$

Then solving the above set of algebraic equations (4.10)–(4.16), we have

$$a_0 = -\frac{b_0(k^3+c)}{6k}, \quad a_1 = \frac{b_1(5k^3-c)}{6k}, \quad a_2 = -\frac{b_1^2(k^3+c)}{24b_0k}, \quad b_2 = \frac{b_1^2}{4b_0}, \tag{4.17}$$

where b_0 and k are non-zero constants, b_1 and c are arbitrary constants. We thus obtain an exact solution of the KdV equation (2.8):

$$u = \frac{-\frac{b_0(k^3+c)}{6k} + \frac{b_1(5k^3-c)}{6k}e^{\xi} - \frac{b_1^2(k^3+c)}{24b_0k}e^{2\xi}}{b_0 + b_1e^{\xi} + \frac{b_1^2}{4b_0}e^{2\xi}}, \quad \xi = kx + ct + w. \tag{4.18}$$

It is easy to see that the solution (4.18) can be rewritten as

$$u = -\frac{k^3+c}{6k} + \frac{2k^2e^{\xi}}{\frac{2b_0}{b_1} + e^{\xi}} - \frac{2k^2e^{2\xi}}{(\frac{2b_0}{b_1} + e^{\xi})^2}, \quad \xi = kx + ct + w, \tag{4.19}$$

which is the obtained solution (4.8) as long as $2b_0 = a_0b_1$.

The above comparison shows that although the solution processes of the NPE method and the exp-function method [10] are similar and both can get the same solution of the KdV equation (2.8), the PDEs (4.2)–(4.7) are simpler than the algebraic equations (4.10)–(4.16). The main reason is that the NPE method collects the coefficients of $\phi^{-j} = (e^{\xi} + a)^{-j}$ $(j = 0, 1, 2, \ldots, 5)$, while the exp-function method [10] collects the coefficients of $e^{j\xi}$ $(j = 1, 2, \ldots, 7)$. Besides, both the numbers of the undetermined parameters and the equations solved in the PDEs (4.2)–(4.7) are less than those in the algebraic equations (4.10)–(4.16). This makes the calculation of solving (4.2)–(4.7) less than that of solving (4.10)–(4.16) although (4.2)–(4.7) are PDEs and (4.10)–(4.16) are algebraic equations.

Let us take the Burgers equation (4.1) as another example for comparison. In this example, we firstly take the traveling wave transformation

$$\xi = kx + ct + w \tag{4.20}$$

before solving it by the NPE and exp-function methods, here k and c are all undetermined constants while w is an arbitrary constant. Then the Burgers equation (4.1) is transformed into

$$cu' + 2kuu' - k^2u'' = 0. \tag{4.21}$$

Integrating (4.21) with respect to ξ once and setting the integration constant as A, we have

$$cu + ku^2 - k^2u' - A = 0. \tag{4.22}$$

Secondly, we use the NPE method to solve (4.22). Balancing u' and u^2 give $n = 1$. We then suppose

$$u = \frac{u_0}{\phi} + u_1, \quad \phi = e^{\xi} + a, \tag{4.23}$$

where u_0 and u_1 are undetermined functions. Using the ansatz solution (4.23), we transform (4.22) into

$$cu_1 + ku_1^2 - k^2u_1' + \frac{c^2}{4k} - \frac{k^3}{4} + \frac{cu_0}{\phi} + \frac{2ku_0u_1}{\phi} - \frac{k^2u_0'}{\phi} + \frac{k^2\mathrm{e}^{\xi}u_0}{\phi^2} + \frac{ku_0^2}{\phi^2} = 0. \quad (4.24)$$

Then setting each coefficient of $\phi^{-j}(j = 0, 1, 2)$ in (4.24) to be zero, we derive a set of ODEs as follows:

$$\phi^{-2}: \ k^2\mathrm{e}^{\xi}u_0 + ku_0^2, \quad (4.25)$$

$$\phi^{-1}: \ cu_0 + 2ku_0u_1 - k^2u_0', \quad (4.26)$$

$$\phi^{0}: \ cu_1 + ku_1^2 - k^2u_1' + \frac{c^2}{4k} - \frac{k^3}{4}. \quad (4.27)$$

Solving the above set of ODEs (4.25)–(4.27), we have

$$u_0 = -k\mathrm{e}^{\xi}, \quad u_1 = \frac{k^2 - c}{2k}, \quad A = \frac{k^4 - c^2}{4k}, \quad (4.28)$$

and hence obtain an exact solution of the Burgers equation (4.1),

$$u = -\frac{k\mathrm{e}^{\xi}}{\mathrm{e}^{\xi} + a} + \frac{k^2 - c}{2k}, \quad \xi = kx + ct + w, \quad (4.29)$$

where $k \neq 0$ and c are constants.

Finally, we solve (4.22) by using the exp-function method [10]. Supposing that

$$u = \frac{a_0 + a_1\mathrm{e}^{\xi} + a_2\mathrm{e}^{2\xi}}{b_0 + b_1\mathrm{e}^{\xi} + b_2\mathrm{e}^{2\xi}}, \quad (4.30)$$

we transform (4.22) into

$$\begin{aligned} \frac{1}{(b_0 + b_1\mathrm{e}^{\xi} + b_2\mathrm{e}^{2\xi})^2} & \Big(-Ab_0^2 + a_0b_0c - 2Ab_0b_1\mathrm{e}^{\xi} + a_1b_0c\mathrm{e}^{\xi} + a_0b_1c\mathrm{e}^{\xi} \\ & - Ab_1^2\mathrm{e}^{2\xi} - 2Ab_0b_2\mathrm{e}^{2\xi} + a_2b_0c\mathrm{e}^{2\xi} + a_1b_1c\mathrm{e}^{2\xi} + a_0b_2c\mathrm{e}^{2\xi} \\ & - 2Ab_1b_2\mathrm{e}^{3\xi} + a_2b_1c\mathrm{e}^{3\xi} + a_1b_2c\mathrm{e}^{3\xi} - Ab_2^2\mathrm{e}^{4\xi} + a_2b_2c\mathrm{e}^{4\xi} \\ & + a_0^2k + 2a_0a_1k\mathrm{e}^{\xi} + a_1^2k\mathrm{e}^{2\xi} + 2a_0a_2k\mathrm{e}^{2\xi} + 2a_1a_2k\mathrm{e}^{3\xi} \\ & + a_2^2k\mathrm{e}^{4\xi} - a_1b_0k^2\mathrm{e}^{\xi} + a_0b_1k^2\mathrm{e}^{\xi} - 2a_2b_0k^2\mathrm{e}^{2\xi} \\ & +2a_0b_2k^2\mathrm{e}^{2\xi} - a_2b_1k^2\mathrm{e}^{3\xi} + a_1b_2k^2\mathrm{e}^{3\xi}\Big) = 0. \end{aligned} \quad (4.31)$$

Then, eliminating the denominator $(b_0 + b_1\mathrm{e}^{\xi} + b_2\mathrm{e}^{2\xi})^2$ and setting each coefficient of $\mathrm{e}^{j\xi}$ $(j = 0, 1, 2, \dots, 4)$ to be zero, we derive a set of algebraic equations as follows:

$$\mathrm{e}^{0}: \ -Ab_0^2 + a_0b_0c + a_0^2k, \quad (4.32)$$

$$\mathrm{e}^{\xi}: \ -2Ab_0b_1 + a_1b_0c + a_0b_1c + 2a_0a_1k - a_1b_0k^2 + a_0b_1k^2 = 0, \quad (4.33)$$

$$\begin{aligned}
e^{2\xi}: &\quad -Ab_1^2 - 2Ab_0b_2 + a_2b_0c + a_1b_1c + a_0b_2c + a_1^2k + 2a_0a_2k \\
&\quad - 2a_2b_0k^2 + 2a_0b_2k^2 = 0, && (4.34)\\
e^{3\xi}: &\quad -2Ab_1b_2 + a2b1c + a_1b_2c + 2a_1a_2k - a2b1k^2 + a1b2k^2 = 0, && (4.35)\\
e^{4\xi}: &\quad -Ab_2^2 + a_2b_2c + a_2^2k = 0. && (4.36)
\end{aligned}$$

Solving the above set of algebraic equations (4.32)–(4.36) yields

$$a_0 = \frac{(c-k^2)[4a_1b_1ck + 4a_1^2k^2 + b_1^2(c^2-k^4)]}{8b_2k^5}, \qquad a_2 = -\frac{b_2(k^2+c)}{2k}, \tag{4.37}$$

$$b_0 = -\frac{4a_1b_1ck + 4a_1^2k^2 - b_1^2(c^2-k^4)}{4b_2k^4}, \qquad A = \frac{k^4-c^2}{4k}, \tag{4.38}$$

where a_1, b_1, b_2 and $k \neq 0$ are constants. Thus we obtain an exact solution of the Burgers equation (4.1),

$$u = \frac{\frac{(c-k^2)[4a_1b_1ck+4a_1^2k^2+b_1^2(c^2-k^4)]}{8b_2k^5} + a_1e^{\xi} - \frac{b_2(k^2+c)}{2k}e^{2\xi}}{-\frac{4a_1b_1ck+4a_1^2k^2-b_1^2(c^2-k^4)}{4b_2k^4} + b_1e^{\xi} + b_2e^{2\xi}}, \tag{4.39}$$

where $\xi = kx + ct + w$. The solution (4.39) can be further simplified to

$$u = \frac{k^2-c}{2k} - \frac{ke^{\xi}}{\frac{b_1c+2a_1k+b_1k^2}{2b_2k^2} + e^{\xi}}, \quad \xi = kx + ct + w. \tag{4.40}$$

Obviously, if we let $2ab_2k^2 = b_1c + 2a_1k + b_1k^2$, then the solution (4.29) becomes the solution (4.40).

5. Conclusions and discussions

In summary, we have presented and extended the NPE method to the (2+1)-dimensional DLW equations, Maccari's equations, the TDB equation, the SK equation with variable coefficients and two lattice equations. As a result, some exact solutions, including traveling wave solutions, non-traveling wave solutions and semi-discrete solutions, are obtained.

The NPE method does not need to go through the traveling wave transformation process. Its advantages mainly lie in the following:

1) compared with the exp-function method [10], there are fewer parameters to be determined, the speed of "expansion of intermediate expression" in the calculation process is slow;

2) compared with the Painlevé truncated expansion method [1], the process of analyzing resonance points is not required for the NPE method and the ansatz solution of the NPE method is not an infinite expansion but a simple and specific expression $\phi = e^{\xi} + a$ in advance, which makes the derivatives of ϕ with respect to ξ be e^{ξ}, and thus reduces the complexity of calculation caused by the undetermined ϕ in the process of calculation of the Painlevé truncated expansion method;

3) compared with the homogeneous balance method [43], the assumed expansion form of the ansatz solution of the NPE method is easy to be drawn up by the balancing process, and there is no need to go through the homogeneous balance method to find the logarithmic function in most cases in advance for determining the assumed expansion of the ansatz solution;

4) compared with the auxiliary equation method [34], the NPE method does not involve the auxiliary equation that the auxiliary equation method needs to use in balancing the expansion order of the ansatz solution or in the process of substituting the ansatz solution into the equation, and the final solution of the NPE method does not require any special solutions of the auxiliary equation.

As for disadvantages of the NPE method, there are two of them:

1) there are relatively fewer types of solutions obtained due to the hypothetical form of ϕ in advance, although the obtained solutions expressed by exponential functions can be transformed into hyperbolic function solutions and trigonometric function solutions by appropriate deformations, other types of solutions like Jacobi elliptic function solutions can not be constructed;

2) the undetermined coefficients embedded in the ansatz solutions are all functions of independent variables, which to some extent increase computational complexity of solving the transformed equations of the given equations as the transformed equations are generally PDEs (ODEs) or DDEs but not algebraic equations.

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Точні розв'язки нелінійних рівнянь в математичній фізиці за методом негативного розширення потужності

Bo Xu and Sheng Zhang

У статті представлено прямий метод, що називається методом негативного розширення потужності (НРП), який застосовано для побудови точних розв'язків нелінійних рівнянь математичної фізики. Запропонований метод (НРП) є також ефективним для зв'язаних рівнянь, рівнянь зі змінним коефіцієнтом та деяких інших спеціальних видів рівнянь. Щоб показати ефективність даного методу, було розглянуто $(2+1)$-вимірне дисперсійне рівняння для довгої хвилі, рівняння Маккарі, рівняння Цицейки–Додда–Буллоу, рівняння Савада–Котера зі змінними коефіцієнтами та два рівняння решітки. У результаті одержано точні розв'язки, включаючи розв'язки рівняння біжної хвилі, рівняння небіжної хвилі та напівдискретні розв'язки. У статті показано, що метод НРП — це простий та ефективний спосіб розв'язку нелінійних рівнянь в математичній фізиці.

Ключові слова: точний розв'язок, метод НРП, $(2+1)$-вимірне дисперсійне рівняння для довгої хвилі, рівняння Маккарі, рівняння Цицейки–Додда–Буллоу, рівняння Савада–Котера зі змінними коефіцієнтами, рівняння решітки