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Conformal Geometry of Semi-Direct Extensions of the Heisenberg Group

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We consider the general semi-direct extension $G_S = H \rtimes_S \mathbb{R}$ of the Heisenberg Lie group H, as defined in [10] by any $S \in \mathfrak{sp}(1,\mathbb{R})$, and equipped with a family of left-invariant metrics g_a ($a^2 \neq 1$). This construction is a natural generalization of the oscillator group. We completely determine the conformally Einstein examples.

Key words: Heisenberg group, semi-direct extensions, oscillator group, Bach-flat metrics, conformally Einstein metrics

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1. Introduction

In recent years there has been a growing interest in conformal geometry of pseudo-Riemannian metrics. A pseudo-Riemannian manifold (M, g) is said to be *(locally) conformally Einstein* if there exists an Einstein metric within the conformal class $[g]$ of the metric g.

Conformally Einstein pseudo-Riemannian metrics g on an n-dimensional smooth manifold M are characterized by the existence (at least locally) of a non-constant real smooth function φ on M, satisfying the equation

$$(n-2)\,\mathrm{Hes}_\varphi + \varphi\varrho = \frac{1}{n}\left\{(n-2)\Delta\varphi + \varphi\tau\right\} g, \tag{1.1}$$

where $\mathrm{Hes}_\varphi = \nabla d\varphi$ is the Hessian of φ and ϱ and τ denote the Ricci tensor and the scalar curvature of g, respectively. In this case, $\tilde{g} = \varphi^{-2} g$ is an Einstein metric conformal to g.

Written down in a system of local coordinates, the conformally Einstein equation (1.1) leads to an overdetermined system of PDE, which is in general very difficult to handle. The equation is trivial in dimension two and equivalent to conformal flatness for three-dimensional manifolds, so that remarkable solutions may occur starting from dimension four.

When looking for conformally Einstein metrics, it is natural to consider first the Bach tensor since conformally Einstein metrics are necessarily Bach-flat (see

Proposition 3.1). Denoting by W the Weyl tensor of (M^n, g), the *Bach tensor* $\mathcal{B}$ is given by

$$\mathcal{B} = \operatorname{div}_1 \operatorname{div}_4 W + \frac{n-3}{n-2} W\left[\varrho\right],$$

where, with respect to a pseudo-orthonormal basis $\{e_i\}$ with $\varepsilon_i = g(e_i, e_i)$, the tensor $W\left[\varrho\right]$ is the Ricci contraction of W defined by

$$W\left[\varrho\right](X,Y) = \sum_{i,j} \varepsilon_i \varepsilon_j W(e_i, X, Y, e_j)\varrho(e_i, e_j).$$

Thus, $\mathcal{B}$ is completely determined by local components

$$\mathcal{B}_{ij} = \nabla^k \nabla^l W_{kijl} + \frac{1}{2}\varrho^{kl} W_{kijl}. \tag{1.2}$$

The purpose of the paper is to study left-invariant conformally Einstein metrics on semi-direct extensions of the Heisenberg group. Following the argument introduced in [10] in arbitrary dimension, we denote by H the three-dimensional Heisenberg group and by $\mathfrak{h} = \operatorname{span}\{X, Y, U\}$ its Lie algebra, described by $[X, Y] = U$. Consider on $\mathbb{R}^2 = \operatorname{span}\{X, Y\}$ the complex structure J defined by $JY = X$. Denoting by $\langle z, z'\rangle$ the symplectic form on $\mathbb{R}^2$ associated to J, one has $\mathfrak{h} = \mathbb{R}^2 \times \mathbb{R}$ with Lie brackets given by

$$[(z, u), (z', u')] = (0, -\langle z, z'\rangle) = (z')^t J z$$

for all $z, z' \in \mathbb{R}^2$ and $u, u' \in \mathbb{R}$.

A real matrix $\mathcal{S} \in \mathbb{R}^{2,2}$ belongs to $\mathfrak{sp}(1, \mathbb{R})$ (the Lie algebra of the symplectic group $Sp(1, \mathbb{R})$ on $\mathbb{R}^2$) if $\mathcal{S}^t \circ J + J \circ \mathcal{S} = 0$. It follows that $\mathcal{S}$ is of the form

$$\mathcal{S} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{1.3}$$

for some real constants α, β, γ. Such matrix determines a corresponding derivation

$$[S, (z, u)] = (\mathcal{S}z, 0)$$

of $\mathfrak{h}$ and thus a one-dimensional semi-direct extension $\mathfrak{g} = \mathfrak{h} \rtimes (\mathbb{R}S)$ of $\mathfrak{h}$. The corresponding connected, simply connected Lie group is then given by

$$G = G_S = H \rtimes_S \mathbb{R} = \mathbb{C} \times \mathbb{R} \times \mathbb{R}.$$

Observe that the well-known case of the four-dimensional *oscillator algebra* (and the corresponding *oscillator group*) corresponds to the special case

$$\mathcal{S} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \quad \beta \neq 0.$$

Setting $z = (x, y), z' = (x', y')$, an arbitrary element and the product in G are respectively given by

$$(z, u, t) = \exp_G(xX + yY + uU) \exp_G(tS) = (z, u, 0)(0, 0, t) \tag{1.4}$$

and

$$(z,u,t)\cdot(z',u',t') = \left(z + e^{tS}z', u+u' - \frac{1}{2} < z, e^{tS}z' >, t+t'\right).$$

The study of the geometric properties of semi-direct extensions $G_S = H \rtimes_S \mathbb{R}$ was undertaken in [3], generalizing to G_S the well-known family of left-invariant metrics of the oscilllator group. These one-parameter family of metrics g_a, defined for any real value of a with $a^2 \neq 1$, have the form

$$\begin{aligned} &g_a(e_1,e_1) = g_a(e_4,e_4) = a, \\ &g_a(e_2,e_2) = g_a(e_3,e_3) = g_a(e_1,e_4) = g_a(e_4,e_1) = 1, \end{aligned} \tag{1.5}$$

where $U = e_1$, $X = e_2$, $Y = e_3$ and $S = e_4$. In general, they are neither conformally flat nor Einstein (see Proposition 2.4). This leads us in a natural way to investigate whether the conformal class of these left-invariant metrics admits some representatives with special conformal curvature properties. We obtain the following main result.

Theorem 1.1. *For an arbitrary $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$, let $G_S = H \rtimes_S \mathbb{R}$ denote the corresponding semi-direct extension of the Heisenberg group, and $g_a, a^2 \neq 1$, the family of left-invariant metrics given by (1.5). Then g_a is conformally Einstein if and only if $a = 0$.*

The paper is organized in the following way. In Section 2, we report some needed information concerning G_S and the family g_a of left-invariant metrics defined on it. In Section 3, we determine when such metrics are Bach-flat. This is the reason to restrict our attention to metrics g_0, and in Section 4 we prove that these metrics are indeed conformally Einstein for any $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$.

2. Preliminaries

Given a Hamiltonan matrix $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$, as described in (1.3) by some real constants α,β,γ, let us consider the corresponding semi-direct extension $G = G_S = H \rtimes_S \mathbb{R}$ of the three-dimensional Heisenberg group. Its Lie algebra $\mathfrak{g} = \mathfrak{h} \rtimes_S (\mathbb{R}S) = \operatorname{span}\{X,Y,U,S\}$ is then described by

$$[X,Y] = U, \qquad [S,X] = \alpha X + \gamma Y, \qquad [S,Y] = \beta X - \alpha Y. \tag{2.1}$$

The following explicit description was obtained in [3].

Proposition 2.1 ([3]). *Given $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$, described as in (1.3), the semi-direct extension $G = H \rtimes_S \mathbb{R}$ can be realized as the four-dimensional subgroup of* $\mathrm{GL}(4,\mathbb{R})$:

$$G_S = \{M_S(x_1,x_2,x_3,x_4) \in \mathrm{GL}(4,\mathbb{R}) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\},$$

whose group elements, by (1.4), have the form

$$M_S(x_i) = \begin{pmatrix} 1 & x_2\,w(x_4) - x_3\,u(x_4) & x_2\,z(x_4) - x_3 v\,(x_4) & 2x_1 \\ 0 & u(x_4) & v(x_4) & x_2 \\ 0 & w(x_4) & z(x_4) & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where, depending on whether $\Delta = -\det(\mathcal{S}) = \alpha^2+\beta\gamma$ *is positive, null or negative, we have:*

$$u(x_4) = \begin{cases} \cosh(\sqrt{\Delta}x_4) + \dfrac{\alpha}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ 1+\alpha x_4 & \text{if } \Delta = 0, \\ \cos(\sqrt{-\Delta}x_4) + \dfrac{\alpha}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{2.2}$$

$$v(x_4) = \begin{cases} \dfrac{\beta}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ \beta x_4 & \text{if } \Delta = 0, \\ \dfrac{\beta}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{2.3}$$

$$w(x_4) = \begin{cases} \dfrac{\gamma}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ \gamma x_4 & \text{if } \Delta = 0, \\ \dfrac{\gamma}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{2.4}$$

$$z(x_4) = \begin{cases} \cosh(\sqrt{\Delta}x_4) - \dfrac{\alpha}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}x_4) & \text{if } \Delta > 0, \\ 1-\alpha x_4 & \text{if } \Delta = 0, \\ \cos(\sqrt{-\Delta}x_4) - \dfrac{\alpha}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}x_4) & \text{if } \Delta < 0, \end{cases} \tag{2.5}$$

Observe that $u(x_4)z(x_4) - v(x_4)w(x_4) = 1$ in all the above cases. Next, let $\partial_j := \partial/\partial_{x_j}$ be the coordinate vector field corresponding to the x_j-coordinate. Then, as proved in [3],

$$\begin{aligned} &e_1 = \partial_1, \quad e_2 = \frac{x_2 w(x_4) - x_3 u(x_4)}{2}\partial_1 + u(x_4)\partial_2 + w(x_4)\partial_3, \\ &e_3 = \frac{x_2 z(x_4) - x_3 v(x_4)}{2}\partial_1 + v(x_4)\partial_2 + z(x_4)\partial_3, \quad e_4 = \partial_4 \end{aligned} \tag{2.6}$$

form a basis of left-invariant vector fields on M_S. Since the Lie brackets $[e_i, e_j]$ are completely determined by

$$[e_2, e_3] = e_1, \qquad [e_2, e_4] = -\alpha e_2 - \gamma e_3, \qquad [e_3, e_4] = -\beta e_2 + \alpha e_3, \tag{2.7}$$

the Lie algebra spanned by $\{e_1, e_2, e_3, e_4\}$ does coincide with $\mathfrak{g}_S = \mathfrak{h} \rtimes_S (\mathbb{R}S)$, setting $U = e_1$, $X = e_2$, $Y = e_3$ and $S = e_4$.

As already mentioned, for $\alpha = 0$ and $\gamma = -\beta$ (whence, $\Delta = -\beta^2 < 0$), one obtains the oscillator group, which, since its introduction in [11], has been intensively studied in several different directions (see, for example, [2], [4], [6], [7]). Generalizing the family of Lorentzian left-invariant metrics of the oscillator group, one considers on $G = G_S = H \rtimes_S \mathbb{R}$ the one-parameter family of left-invariant metrics $g_a = \langle \cdot, \cdot \rangle$, described by (1.5), for any real constant a such that $a^2 \neq 1$. In particular, g_a is Lorentzian if $a^2 < 1$ and Riemannian for $a^2 > 1$. The case with $a = 0$, $\alpha = 0$ and $\gamma = -\beta = -1$ gives the bi-invariant metric on the classic oscillator group [8].

Equations (2.6) and (1.5) yield that in global coordinates (x_1, x_2, x_3, x_4), the metric g_a is explicitly given by

$$
\begin{aligned}
g_a = {} & a dx_1^2 + \big(\frac{a}{4}x_3^2 + w^2(x_4) + z^2(x_4)\big)dx_2^2 + \big(\frac{a}{4}x_2^2 + u^2(x_4) + v^2(x_4)\big)dx_3^2 \\
& + a dx_4^2 + a x_3 dx_1 dx_2 - a x_2 dx_1 dx_3 + 2dx_1 dx_4 - \frac{1}{2}\big(a x_2 x_3 + 4u(x_4)w(x_4) \\
& + 4v(x_4)z(x_4)\big)dx_2 dx_3 + x_3 dx_2 dx_4 - x_2 dx_3 dx_4. \qquad (2.8)
\end{aligned}
$$

Remark 2.2. The description of the Lie algebra $\mathfrak{h} \rtimes (\mathbb{R}S)$ up to isomorphisms was given in the following result.

Proposition 2.3 ([3]). *Consider an arbitrary $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$ and the corresponding derivation S of the Heisenberg Lie algebra $\mathfrak{h}$. Depending on whether $\Delta = -\det(\mathcal{S}) = \alpha^2 + \beta\gamma$ is positive, null or negative, the one-dimensional extension $\mathfrak{h} \rtimes (\mathbb{R}S)$ is isomorphic to the Lie algebra $\mathfrak{g}$ =span$\{e_1 = \tilde{U}, e_2 = \tilde{X}, e_3 = \tilde{Y}, e_4 = \tilde{S}\}$, completely described by $[e_2, e_3] = e_1$, and*

$$
\begin{array}{lllll}
\text{(I)} \ \ \textit{if} \ \ \Delta > 0: & (A) & [e_4, e_2] = \mu e_2, & [e_4, e_3] = -\mu e_3, & \mu > 0 \ \textit{or, equivalently,} \\
& (A') & [e_4, e_2] = \mu e_3, & [e_4, e_3] = \mu e_2, & \mu > 0; \\
\text{(II)} \ \ \textit{if} \ \ \Delta = 0: & (B) & [e_4, e_2] = \mu e_3, & [e_4, e_3] = 0, & \mu \geq 0; \\
\text{(III)} \ \ \textit{if} \ \ \Delta < 0: & (C) & [e_4, e_2] = \mu e_3, & [e_4, e_3] = -\mu e_2, & \mu > 0.
\end{array}
$$

We may observe that Case (C) in Proposition 2.3 is the oscillator Lie algebra. Case (A′) permits to write down the three cases of Proposition 2.3 in the unified way

$$
\mathfrak{g}: \quad [e_2, e_3] = e_1, \quad [e_4, e_2] = \mu e_3, \quad [e_4, e_3] = \varepsilon\mu e_2, \qquad \varepsilon \in \{-1, 0, 1\}, \ \mu \geq 0, \qquad (2.9)
$$

while description (A) is useful when working with explicit coordinates, as equations (2.3) and (2.4) then yield $v(x_4) = w(x_4) = 0$. In any case, in the present paper, we treat equations in full generality without making use of Proposition 2.3.

We end this section describing some curvature properties of $(G_S = H \rtimes_S \mathbb{R}, g_a)$. In particular, we identify the Einstein and conformally flat examples among left-invariant metrics g_a on $G = H \rtimes_S \mathbb{R}$. These results were obtained in [3, Section 3] taking into account Proposition 2.3, but are written down here for an arbitrary $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$.

Proposition 2.4 ([3]). *The left-invariant metric g_a on $G = H \rtimes_S \mathbb{R}$ is*

(i) *Einstein if and only if $a = 0$ and $4\alpha^2 = 1 - (\beta + \gamma)^2$. In this case, g_a is Ricci-flat;*

(ii) *Ricci-parallel if and only if $a = 0$;*

(iii) *locally conformally flat if and only if $a = 0$ and either $\beta = \gamma - 1$ or $\alpha = \beta + \gamma = 0$;*

(iv) *locally symmetric exactly in the locally conformally flat cases.*

3. Bach-flat examples

Let (M,g) denote an n-dimensional pseudo-Riemannian manifold. Its *Cotton tensor* $\mathcal{C}$ is completely determined in local coordinates by

$$\mathcal{C}_{ijk} = (\nabla_i Sc)_{jk} - (\nabla_j Sc)_{ik},$$

where $Sc = \varrho - \frac{\tau}{2(n-1)g}$ denotes the *Schouten tensor*. We have already recalled in Introduction that the vanishing of the Bach tensor $\mathcal{B} = \mathrm{div}_1\mathrm{div}_4 W + \frac{n-3}{n-2}W\,[\varrho]$ is a necessary condition for a four-dimensional pseudo-Riemannian manifold to be conformally Einstein. More precisely, the following result holds.

Proposition 3.1 ([5,9]). *Let (M,g) be a four-dimensional pseudo-Riemannian manifold such that $\tilde{g} = e^{\sigma}g$ is Einstein. Then*

1. $\mathcal{C} - W(\cdot,\cdot,\cdot,\nabla\sigma) = 0$, *and*
2. $\mathcal{B} = 0$.

Thus, it is natural to determine first the Bach-flat examples in order to restrict the list of metrics which might be conformally Einstein. We shall now obtain the following characterization, which proves the "only if" part of Theorem 1.1.

Theorem 3.2. *For an arbitrary $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$, consider the corresponding semi-direct extension $G = H \rtimes_S \mathbb{R}$ of the Heisenberg group and the family $g_a, a^2 \neq 1$, of left-invariant metrics described by (1.5). Then g_a is Bach-flat if and only if $a = 0$.*

Proof. We calculated the components of the Weyl conformal curvature tensor W with respect to the basis $\{e_i\}$ of left-invariant vector fields (see also [3]). Up to symmetries, the possibly non-vanishing components W_{ijkl} of W are given by:

$$\begin{aligned}
W_{1212} &= \frac{a^2(a^2-2\beta^2-1+\gamma^2-2\alpha^2-\gamma\beta)}{6(a^2-1)},\\
W_{1213} &= \frac{a^2\alpha(\beta-\gamma)}{2(a^2-1)},\\
W_{1224} &= \frac{a(4(\beta^2+\alpha^2)-2\gamma^2+2\gamma\beta-(a^2-1)(2+3(\beta+\gamma))}{12(a^2-1)},\\
W_{1234} &= \frac{a\alpha(a^2-1-\beta+\gamma)}{2(a^2-1)},\\
W_{1313} &= \frac{a^2(a^2+\beta^2-1-2\gamma^2-2\alpha^2-\gamma\beta)}{6(a^2-1)},\\
W_{1324} &= \frac{a\alpha(a^2-1-\beta+\gamma)}{2(a^2-1)},\\
W_{1334} &= \frac{a(4(\gamma^2+\alpha^2)-2\beta^2+2\gamma\beta-(a^2-1)(2-3(\beta+\gamma)))}{12(a^2-1)},\\
W_{1414} &= \frac{a(-2a^2+\beta^2+2+\gamma^2+4\alpha^2+2\gamma\beta)}{6},
\end{aligned}$$

$$
\begin{aligned}
W_{2323} &= \frac{a(\beta^2+\gamma^2+4\alpha^2-2a^2+2+2\gamma\beta)}{6(a^2-1)},\\
W_{2424} &= \frac{a^4+a^2\beta^2-a^2-2a^2\gamma^2-2a^2\alpha^2-a^2\gamma\beta-3\beta^2}{6(a^2-1)}\\
&\qquad\qquad + \frac{+3\gamma^2+3(a^2-1)(\beta+\gamma)}{6(a^2-1)},\\
W_{2434} &= -\frac{\alpha(a^2\beta-a^2\gamma+2\gamma+2a^2-2-2\beta)}{2(a^2-1)},\\
W_{3434} &= \frac{a^4-(2\beta^2+(\gamma+3)\beta+2\alpha^2+3\gamma-\gamma^2+1)a^2}{6(a^2-1)}\\
&\qquad\qquad + \frac{+3\gamma+3\beta^2-3\gamma^2+3\beta}{6(a^2-1)}.
\end{aligned}
\tag{3.1}
$$

In order to determine the Bach tensor $\mathcal{B}$, we also need the Ricci contraction $W[\varrho]$ of W. As proved in [3], where the Ricci tensor of $(G = H \rtimes_S \mathbb{R}, g_a)$ was described for any $a^2 \neq 1$ with respect to the basis $\{e_i\}$, for g_0 one has

$$
\varrho = \begin{pmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{2}\left(1-4\alpha^2-(\beta+\gamma)^2\right)
\end{pmatrix}.
\tag{3.2}
$$

Consequently, by a standard calculation we find that the possibly non-vanishing components of $W[\varrho]$ with respect to $\{e_i\}$ are the following:

$$
\begin{aligned}
W[\varrho]_{11} &= \frac{a^3}{12(a^2-1)^2}\left\{(20\alpha^2-3a^2+4\beta\gamma+3)(\beta^2+\gamma^2)+2(a^2-1)^2\right.\\
&\quad \left. -12\alpha^2(a^2-1)-6\gamma\beta(a^2-1)+16\alpha^4-8\alpha^2\gamma\beta+4\beta^4+4\gamma^4\right\},\\
W[\varrho]_{14} &= W[\varrho]_{11} + \frac{a^3}{4(a^2-1)}\left\{(\gamma-\beta)(\gamma^2+2\gamma\beta+4\alpha^2+\beta^2)\right\},\\
W[\varrho]_{22} &= -\frac{a^2}{12(a^2-1)^2}\left\{3a^2(a^2-2)+(10\alpha^2+3\gamma^2+2\beta^2+5\gamma\beta)(1-a^2)\right.\\
&\quad +3\gamma^4+3-3\beta^3\gamma+8\alpha^2(\alpha^2+\gamma\beta)-2\beta^4+7\gamma^2(\gamma\beta+2\alpha^2)\\
&\quad \left. +3\beta^2(\gamma^2-2\alpha^2)\right\},\\
W[\varrho]_{23} &= \frac{a^2\alpha}{12(a^2-1)^2}\left\{(\beta-\gamma)(-10\gamma\beta-20\alpha^2-5(\beta^2+\gamma^2)+a^2-1)\right\},\\
W[\varrho]_{33} &= \frac{a^2}{12(a^2-1)^2}\left\{-3a^2(a^2-2)+(10\alpha^2+3\beta^2+2\gamma^2+5\gamma\beta)(a^2-1)\right.\\
&\quad +2\gamma^4-3+3\gamma^3\beta-8\alpha^2(\alpha^2+\gamma\beta)+6\alpha^2\gamma^2-7\beta^2(\beta\gamma+2\alpha^2)\\
&\quad \left. -3\beta^2(\gamma^2+\beta^2)\right\},\\
W[\varrho]_{44} &= -\frac{a}{12(a^2-1)^2}\left\{2-4a^6+10a^4-8a^2+(-24a^2+32)\alpha^2\gamma\beta\right.
\end{aligned}
$$

$$
\begin{aligned}
&- 6(a^2-1)(\beta\gamma^2+\gamma\beta^2+\gamma^2\beta^2+\gamma^3-\beta^3-4\beta\alpha^2+4\alpha^2\gamma)\\
&+ (2a^4-a^2-1)(\beta^2+\gamma^2)+(4a^4-2a^2-2)(2\alpha^2+\gamma\beta)\\
&+(3a^2-7)(\gamma^4+\beta^4)-16\alpha^4+(-32+12a^2)(\alpha^2\beta^2+\alpha^2\gamma^2)\\
&-4\gamma\beta(\gamma^2+\beta^2)\big\}\,.
\end{aligned}
$$

Therefore, following the definition of the Bach tensor $\mathcal{B}$, we obtain its components with respect to the left-invariant basis $\{e_i\}$ of vector fields. Up to symmetries, the only possibly non-vanishing components are given by:

$$
\begin{aligned}
\mathfrak{B}_{11} = a\mathfrak{B}_{14} &= \frac{a^3}{24(a^2-1)^2}\big\{(20\alpha^2+4\beta\gamma-3a^2+3)(\beta^2+\gamma^2)+20(a^2-1)^2\\
&\quad -12\alpha^2(a^2-1)-6\beta\gamma(a^2-1)+16\alpha^4-8\alpha^2\beta\gamma+4\beta^4+4\gamma^4\big\}\,,\\
\mathfrak{B}_{22} &= -\frac{a^2}{24(a^2-1)^2}\big\{(20\alpha^2+4\beta\gamma)(\gamma^2-3\beta^2)+12(a^2-1)^2\\
&\quad +(a^2-1)(\beta^2-3\gamma^2)-4\alpha^2(a^2-1)-2\beta\gamma(a^2-1)-16\alpha^4+8\alpha^2\beta\gamma\\
&\quad -20\beta^4+12\gamma^4\big\}\,,\\
\mathfrak{B}_{23} &= \frac{a^2(\beta-\gamma)\alpha}{6(a^2-1)^2}\big\{a^2-1-20\alpha^2-4\beta\gamma-8(\beta^2+\gamma^2)\big\}\,,\\
\mathfrak{B}_{33} &= -\frac{a^2}{24(a^2-1)^2}\big\{12(a^2-1)^2-4\alpha^2(a^2-1)+(a^2-1)(\gamma^2-3\beta^2)\\
&\quad -2\beta\gamma(a^2-1)-16\alpha^4+8\alpha^2\beta\gamma+12\beta^4-20\gamma^4\\
&\quad +(4\beta\gamma+20\alpha^2)(\beta^2-3\gamma^2)\big\}\,,\\
\mathfrak{B}_{44} &= \frac{a}{24(a^2-1)^2}\big\{16+4a^6+8a^4-28a^2\\
&\quad +(-5a^2+a^4+4)(\beta^2+\gamma^2+2\alpha^2+2\gamma\beta)-2\alpha^2\gamma\beta)\\
&\quad -4(3a^2-4)(\gamma^4+\beta^4+4\alpha^4+\gamma^3\beta+\beta^3\gamma+5(\alpha^2\beta^2+\alpha^2\gamma^2)\big\}\,.
\end{aligned}
$$

From the above expressions, a standard calculation easily leads to the conclusion that $\mathfrak{B}_{ij}=0$ for all indices i,j if and only if $a=0$ and this ends the proof. $\square$

Because of Proposition 3.1 and Theorem 3.2, in the family g_a of left-invariant metrics, g_0 is the only one which can admit an Einstein metric within its conformal class. Before we proceed in the next section with the treatment of the conformally Einstein equation, we observe that g_0 satisfies the Euler-Lagrange equations for all quadratic curvature functionals.

We briefly recall some basic information about quadratic curvature invariants by referring to [12] for more details. The space of quadratic curvature invariants is generated by $\{\Delta\tau, \tau^2, \|\varrho\|^2, \|R\|^2\}$. Each quadratic curvature invariant determines the corresponding quadratic curvature functional and Euler–Lagrange equations. In dimension four, because of the Gauss-Bonnet-Chern Theorem, all quadratic curvature functionals are equivalent to

$$
\mathcal{S}(g)=\int_M \tau^2\,\mathrm{dvol}_g,\qquad \mathcal{F}_t(g)=\int_M\left\{\|\varrho\|^2+t\tau^2\right\}\mathrm{dvol}_g,\quad t\in\mathbb{R}.
$$

In particular, a four-dimensional metric g is critical for all quadratic curvature functionals if and only if it is critical for $\mathcal{S}(g)$ and $\mathcal{F}_0(g) = \int_M \|\varrho\|^2 \, d\mathrm{vol}_g$ (see, for example, [1]), and so, equivalently, for $\mathcal{S}(g)$ and $\mathcal{F}_t(g)$ for a given value of t.

The Euler–Lagrange equations corresponding to functionals $\mathcal{S}$ and $\mathcal{F}_t$ are respectively given by

$$2\,\mathrm{Hes}_\tau - 2\tau\varrho + \frac{1}{2}\left(\tau^2 - \Delta\tau\right) g, \qquad \Delta\varrho + 2\left(R[\varrho] - \frac{1}{4}\|\varrho\|^2\right) + 2t\tau\left(\varrho - \frac{1}{4}\tau g\right)$$

and can be assumed as definitions of the corresponding critical metrics in the noncompact case.

We proved that four-dimensional metrics g_0 on $G = H \rtimes_S \mathbb{R}$ are Bach-flat. As such they are critical points for the functional $\int_M \|W\|^2 \, d\mathrm{vol}_g$, which is equivalent to $\mathcal{F}_{-1/3}$ [12]. Moreover, it follows at once by the description of its Ricci operator (see (4.1)) that the scalar curvature of g_0 vanishes. Therefore, g_0 is also critical for $\mathcal{S}$ and we have the following.

Corollary 3.3. *For an arbitrary $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$ and the corresponding semi-direct extension $G = H \rtimes_S \mathbb{R}$ of the Heisenberg group, the left-invariant metric g_0 is critical for all quadratic curvature functionals.*

We intend to come back to a detailed study of critical metrics g_a in future works.

4. Conformally Einstein metrics

To prove our main result we are now to prove that g_0 is indeed conformally Einstein for any given Hamiltonian matrix $\mathcal{S}$ in spite of deep differences in the curvature of g_0 for different matrices $\mathcal{S}$ (see, for example, Proposition 2.4).

Observe that, as reported in [5], the conditions listed in the above Proposition 3.1 are also *sufficient* for a four-dimensional pseudo-Riemannian manifold (M, g) to be conformally Einstein, provided that it is also *weakly generic*, in the sense that its Weyl tensor W defines an injective map from TM to $\otimes^3 T^*M$.

However, this is not the case with the metric g_0 on $G = H \rtimes_S \mathbb{R}$. In fact, taken an arbitrary vector field $X = x_1 e_1 + \cdots + x_4 e_4$, the action of W on X is given by

$$\begin{aligned} W(X) = {}& \frac{x_4}{2}(\gamma^2 - \beta^2 - \beta - \gamma)\theta_2 \otimes \theta_2 \wedge \theta_4 - x_4\alpha(\gamma - \beta - 1)\theta_2 \otimes \theta_3 \wedge \theta_4 \\ & - x_4\alpha(\gamma - \beta - 1)\theta_3 \otimes \theta_2 \wedge \theta_4 - \frac{x_4}{2}(\gamma^2 - \beta^2 - \beta - \gamma)\theta_3 \otimes \theta_3 \wedge \theta_4 \\ & - (x_3\alpha(1 - \gamma + \beta) - \frac{x_2}{2}(\beta^2 - \gamma^2 + \beta + \gamma))\theta_4 \otimes \theta_2 \wedge \theta_4 \\ & - (x_2\alpha(1 - \gamma + \beta) + \frac{x_3}{2}(\beta^2 - \gamma^2 + \beta + \gamma))\theta_4 \otimes \theta_3 \wedge \theta_4, \end{aligned}$$

where $\{\theta_i\}_{i=1}^4$ denotes the dual basis of $\{e_i\}_{i=1}^4$. It easily follows from the above expression that $W(X) = 0$ if and only if one of the following cases occurs:

(1) $x_2 = x_3 = x_4 = 0$,

(2) $\gamma = \beta + 1$,

(3) $\alpha = 0 = \beta + \gamma$.

In Cases (2) and (3) the space is locally conformally flat (Proposition 2.4). Case (1) shows that excluding these trivial cases, the Weyl tensor is not weakly generic.

We shall now complete the proof of our main result. For any $\mathcal{S} \in \mathfrak{sp}(1, \mathbb{R})$, we consider $G = H \rtimes_S \mathbb{R}$ and its left-invariant metric g_0. We then solve the first equation in Proposition 3.1, which will restrict the possible functions φ such that $\tilde{g} = \varphi^{-2} g_0$ can be conformally Einstein.

We start by describing the Schouten and Cotton tensors of $(G = H \rtimes_S \mathbb{R}, g_0)$. From (3.2), taking into account (1.5), we easily deduce that the Ricci operator Q of $(G = H \rtimes_S \mathbb{R}, g_0)$ is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}\left(1 - 4\alpha^2 - (\beta+\gamma)^2\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

In particular, the scalar curvature is $\tau = 0$ and so, $Sc = \varrho$. By Proposition 2.4, (ii), we have that $\nabla \varrho = 0$. In particular, it is totally symmetric. Therefore, $\mathcal{C} = 0$ and condition 1 of Proposition 3.1 reduces to $W(\cdot, \cdot, \cdot, \nabla\sigma) = 0$.

In coordinates (x_1, x_2, x_3, x_4), the equation $W(\cdot, \cdot, \cdot, \nabla\sigma) = 0$ translates into the system of PDEs formed by the following independent equations:

$$\begin{aligned}
&\partial_1\sigma \big[(vw^2 - vz^2 - 2wzu)w'' + (uw^2 - uz^2 + 2vwz)z'' + (w^2+z^2)(zu'' - wv'') \\
&\quad + \big((uz^2 - uw^2 - 2vwz)w' + (vw^2 - vz^2 - 2uwz)z' \\
&\quad + \ (w^2+z^2)(zv' + wu')\big)\,(wu' + zv' - uw' - vz' + 1)\big] = 0, \\
&\partial_1\sigma \big[(u^2z - v^2z - 2uvw)u'' + (u^2w - v^2w + 2uvz)v'' + (u^2+v^2)(vw'' - uz'') \\
&\quad + \big((v^2w - u^2w - 2uvz)u' + (u^2z - v^2z - 2uvw)v' \\
&\quad + (u^2+v^2)(uw' + vz')\big)\,(u'w + v'z - uw' - vz' + 1)\big] = 0, \\
&\partial_1\sigma \big[(w^2+z^2)(vu'' - uv'') - (u^2+v^2)(zw'' + wz'') + ((w^2+z^2)(uu' + vv') \\
&\quad - (u^2+v^2)(ww' + zz'))(u'w + v'z - uw' - vz' + 1)\big] = 0, \\
&\partial_1\sigma \big[u''(x_3w(v^2-u^2) - 2x_3uvz + x_2u(w^2+z^2)) - w''(u^2+v^2)(x_2w - x_3u) \\
&\quad + v''(x_3z(u^2-v^2) - 2x_3uvw + x_2v(w^2+z^2)) - z''(u^2+v^2)(x_2z - x_3v) \\
&\quad + \big(u'(2x_3uvw + x_3z(v^2-u^2) - x_2v(w^2+z^2)) \\
&\quad + v'(-2x_3uvz + x_3w(v^2-u^2) + x_2u(w^2+z^2)) \\
&\quad + (u^2+v^2)(w'(x_2z - x_3v) - z'(x_2w - x_3u))\big)\,(u'w + v'z - uw' - vz' + 1)\big] \\
&\quad + 2\partial_2\sigma \big[u''(u^2w - v^2w + 2uvz) - v''(u^2z - v^2z - 2uvw) \\
&\quad - (u^2+v^2)(w''u + z''v)\,\big(u'(u^2z - v^2z - 2uvw) + v'(u^2w - v^2w + 2uvz) \\
&\quad + (u^2+v^2)(w'v - z'u)\big)\,(u'w + v'z - uw' - vz' + 1)\big] \\
&\quad + 2\partial_3\sigma \big[(w^2+z^2)(uu'' + vv'') - (u^2+v^2)(ww'' + zz'')
\end{aligned}$$

$$
\begin{aligned}
&\quad + \big((w^2+z^2)(uv'-vu') \\
&\quad + (u^2+v^2)(w'z-z'w)\big)(u'w+v'z-uw'-vz'+1)\big] = 0, \\
&\partial_1\sigma\,\big[w''(x_3w(u^2+v^2)-2x_2vwz+x_2u(z^2-w^2)) + u''(w^2+z^2)(x_2w-x_3u) \\
&\quad + z''(x_3z(u^2+v^2)-2x_2uwz+x_2v(w^2-z^2)) + v''(w^2+z^2)(x_2z-x_3v) \\
&\quad + \big(w'(2x_2uwz+x_2v(z^2-w^2)-x_3z(u^2+v^2)) \\
&\quad + (-2x_2wvz+x_3w(u^2+v^2)+x_2u(z^2-w^2))z' \\
&\quad + (w^2+z^2)(u'(x_3v-x_2z)+v'(x_2w-x_3u))\big)(u'w+v'z-uw'-vz'+1)\big] \\
&\quad + 2\partial_2\sigma\,\big[(w^2+z^2)(uu''+vv'')-(u^2+v^2)(ww''+zz'') \\
&\quad + \big((u^2+v^2)(zw'-wz')+(w^2+z^2)(v'u-vu')\big)(u'w+v'z-uw'-vz'+1)\big] \\
&\quad + 2\partial_3\sigma\,\big[(w''(uz^2-uw^2-2vwz)-z''(vz^2-vw^2+2uwz) \\
&\quad + (w^2+z^2)(wu''+zv'')+(w'(vz^2-vw^2+2uwz)+z'(uz^2-uw^2-2vwz) \\
&\quad + (w^2+z^2)(v'w-u'z))(u'w+v'z-uw'-vz'+1)\big] = 0.
\end{aligned}
$$

It is easy to check that for generic values of the variable x_4, the functions of u, v, w, z and their derivatives occurring in the above equations do not vanish. Therefore, these equations yield $\partial_i\sigma = 0$ for $i = 1, 2, 3$, that is, σ (and so, $\varphi = \sqrt{e^{-\sigma}}$) only depends on the variable x_4.

For such a function $\varphi = \varphi(x_4)$, the conformally Einstein equation (1.1) reduces to the following ODE:

$$
\begin{aligned}
4\,(uz-wv)^2\,\varphi'' + \varphi\Big(&\big(2\,uz^2-2vwz\big)\,u'' + \big(-2uwz+2vw^2\big)\,v'' \\
&+ \big(-2uvz+2v^2w\big)\,w'' + \big(2u^2z-2vwu\big)\,z'' \\
&+ \big((z'v+uw'-wu'-v'z)^2-1\big)\Big) = 0,
\end{aligned} \tag{4.2}
$$

where we omitted the variable x_4, and u, v, z, w are again the smooth functions describing the typical group element of $G_S = H \rtimes_S \mathbb{R}$ as in Proposition 2.1.

It is a remarkable fact that replacing u, v, w, z from equations (2.2)-(2.5) for the different possibilities of $\Delta = \alpha^2 + \beta\gamma$ (namely: positive, null or negative), the above equation (4.2) in all cases becomes

$$
\varphi''(x_4) = \frac{4\alpha^2 + (\beta+\gamma)^2 - 1}{4}\varphi(x_4),
$$

which admits solutions for all values of parameters α, β, γ describing $\mathcal{S}$. This leads to the following result, which implies the "if" part of Theorem 1.1 and completes its proof.

Theorem 4.1. *For an arbitrary $\mathcal{S} \in \mathfrak{sp}(1,\mathbb{R})$, consider the corresponding semi-direct extension $G = H \rtimes_S \mathbb{R}$ of the Heisenberg group. Then the left-invariant metric g_0, as described by (1.5) with $a = 0$, is conformally Einstein. More precisely, setting*

$$
k := \frac{4\alpha^2 + (\beta+\gamma)^2 - 1}{4},
$$

we have that $\tilde{g} = \varphi^{-2}g_0$ is an Einstein metric (defined in the dense open subset where $\varphi \neq 0$), where

(a) *if* $k>0$, *then* $\varphi(x_4)=c_1\sinh\left(\sqrt{k}\,x_4\right)+c_2\cosh\left(\sqrt{k}\,x_4\right)$;

(b) *if* $k<0$, *then* $\varphi(x_4)=c_1\sin\left(\sqrt{|k|}\,x_4\right)+c_2\cos\left(\sqrt{|k|}\,x_4\right)$;

(c) *if* $k=0$, *then* $\varphi(x_4)=c_1x_4+c_2$;

for arbitrary real constants c_1, c_2.

5. Concluding remarks

We see from Theorem 4.1 that whatever the value of $k=\frac{4\alpha^2+(\beta+\gamma)^2-1}{4}$ (and so, for any $\mathcal{S}\in\mathfrak{sp}(1,\mathbb{R})$), there exists a two-parameter family of Einstein metrics which are conformal to the left-invariant Lorentzian metric g_0 on $G_S=H\rtimes_S\mathbb{R}$.

According to Proposition 2.4, case (c) in the above Theorem 4.1 corresponds to the case where g_0 is an Einstein metric. In fact, setting $c_1=c_2-1=0$, we get $\varphi(x_4)=1$, that is, $\tilde{g}=g_0$ itself is Einstein.

Apart from case (c) , the only further trivial cases in Theorem 4.1 occur when g_0 is locally conformally flat. By case (iii) of Proposition 2.4, this happens if and only if either $\gamma=\beta-1$ or $\alpha=\beta+\gamma=0$ (the case of the oscillator group).

For $\alpha=\beta+\gamma=0$, one has $k=-\frac{1}{4}<0$, so that this special case is included in case (b) of Theorem 4.1. On the other hand, if $\gamma=\beta+1$, then we find $k=\alpha^2+\beta^2-\beta$, which may have any sign depending on the values of α and β.

Coherently with the classification of the conformally flat cases, a straightforward calculation shows that $\tilde{g}=\varphi^{-2}g_0$, with φ as described in Theorem 4.1, is flat if and only if either $\gamma=\beta+1$ or $\alpha=\beta+\gamma=0$.

Whenever g_0 is neither Einstein nor conformally flat, the remaining solutions described in Theorem 4.1 are proper examples of conformally Einstein metrics. A direct calculation yields that in all these cases, $\tilde{\varrho}=0$, i.e., $\tilde{g}$ is Ricci-flat.

To complete the study of conformal properties of left-invariant metrics g_a on $G=H\rtimes_S\mathbb{R}$, we also checked when these metrics are *(locally) conformally symmetric*, that is, satisfy $\nabla W=0$. However, no proper examples occur. In fact, up to symmetries, the possibly non-vanishing components of the tensor ∇W with respect to the left-invariant basis $\{e_i\}$ are given by:

$$
\begin{aligned}
&(\nabla_{e_1}W)_{1212}=\frac{a^3\alpha(\beta-\gamma)}{2(a^2-1)},\\
&(\nabla_{e_1}W)_{1213}=\frac{a^3(\beta^2-\gamma^2)}{4(a^2-1)},\\
&(\nabla_{e_1}W)_{1224}=-(\nabla_{e_1}W)_{1334}=\frac{a^2\alpha(a^2-1-\beta+\gamma)}{2(a^2-1)},\\
&(\nabla_{e_1}W)_{1234}=(\nabla_{e_1}W)_{1324}=\frac{a^2((a^2-1)(\beta+\gamma)+\gamma^2-\beta^2)}{4(a^2-1)},\\
&(\nabla_{e_1}W)_{1313}=-\frac{a^3\alpha(\beta-\gamma)}{2(a^2-1)},
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{e_1}W)_{2424} &= -(\nabla_{e_1}W)_{3434} = -\frac{a\alpha((a^2-2)(\beta-\gamma)+2(a^2-1))}{2(a^2-1)},\\
(\nabla_{e_1}W)_{2434} &= -\frac{a((a^2-2)(\beta^2-\gamma^2)+2(a^2-1)(\gamma+\beta))}{4(a^2-1)},\\
(\nabla_{e_2}W)_{1223} &= \frac{a^2(2(a^2-1)+\gamma^2-\beta^2)}{8(a^2-1)},\\
(\nabla_{e_2}W)_{1314} &= \frac{a^2(-4\beta\gamma^2+3(a^2-1)(\beta+\gamma)-2\gamma\beta^2-2\gamma^3-8\alpha^2\gamma)}{8(a^2-1)},\\
(\nabla_{e_2}W)_{1323} &= \frac{a^2\alpha(\beta-\gamma)}{4(a^2-1)},\\
(\nabla_{e_2}W)_{1424} &= -\frac{a\alpha(3(a^2-1)+(a^2-1)(\beta-\gamma)-(\gamma+\beta)^2-4\alpha^2)}{4(a^2-1)},\\
(\nabla_{e_2}W)_{1434} &= \frac{a((a^2-1)(-3(\beta+\gamma)+\gamma^2-\beta^2)+2-4a^2)}{8(a^2-1)}\\
&\qquad\qquad + \frac{a(2a^4+2\gamma^3+8\alpha^2\gamma+4\beta\gamma^2+2\gamma\beta^2)}{8(a^2-1)},\\
(\nabla_{e_2}W)_{2324} &= -\frac{a((a^2-1)(3(\beta+\gamma)+2)-4\beta\gamma^2)}{8(a^2-1)}\\
&\qquad\qquad + \frac{a(-2\gamma(\gamma^2+\beta^2)-8\alpha^2\gamma+\gamma^2-\beta^2)}{8(a^2-1)},\\
(\nabla_{e_2}W)_{2334} &= \frac{\alpha(3(a^2-1)-(\beta+\gamma)^2-4\alpha^2+\gamma-\beta)}{4(a^2-1)},\\
(\nabla_{e_3}W)_{1214} &= \frac{a^2(-2\beta(\beta+\gamma)^2-8\beta\alpha^2+3(a^2-1)(\gamma+\beta))}{8(a^2-1)},\\
(\nabla_{e_3}W)_{1223} &= \frac{a^2\alpha(\beta-\gamma)}{4(a^2-1)},\\
(\nabla_{e_3}W)_{1314} &= -\frac{a^2\alpha(-4\alpha^2+3(a^2-1)-(\gamma+\beta)^2)}{4(a^2-1)},\\
(\nabla_{e_3}W)_{1323} &= \frac{a^2(2(a^2-1)-\gamma^2+\beta^2)}{8(a^2-1)},\\
(\nabla_{e_3}W)_{1424} &= -\frac{a(-8\beta\alpha^2+2-4a^2+2a^4)}{8(a^2-1)}\\
&\qquad\qquad + \frac{a((a^2-1)(3(\beta+\gamma)+\beta^2-\gamma^2)-2\beta(\gamma+\beta)^2)}{8(a^2-1)},\\
(\nabla_{e_3}W)_{1434} &= \frac{a\alpha((a^2-1)(\beta-\gamma+3)-(\gamma+\beta)^2-4\alpha^2)}{4(a^2-1)},\\
(\nabla_{e_3}W)_{2324} &= \frac{a\alpha(3(a^2-1)+\gamma-\beta-4\alpha^2-(\gamma+\beta)^2)}{4(a^2-1)},\\
(\nabla_{e_3}W)_{2334} &= \frac{a((a^2-1)(3(\beta+\gamma)-2)-8\beta\alpha^2+\gamma^2-\beta^2-2\beta(\beta+\gamma)^2)}{8(a^2-1)},
\end{aligned}
$$

$$
\begin{aligned}
(\nabla_{e_4}W)_{1212} &= -(\nabla_{e_4}W)_{1313} = \frac{a^2\alpha(\beta+1-\gamma)(\beta-\gamma)}{2(a^2-1)},\\
(\nabla_{e_4}W)_{1213} &= \frac{a^2(\beta+1-\gamma)(\beta^2-\gamma^2)}{4(a^2-1)},\\
(\nabla_{e_4}W)_{1224} &= -(\nabla_{e_4}W)_{1334} = \frac{a\alpha(\beta+1-\gamma)(a^2-1-\beta+\gamma)}{2(a^2-1)},\\
(\nabla_{e_4}W)_{1234} &= (\nabla_{e_4}W)_{1324} = \frac{a(\beta+1-\gamma)((a^2-1)(\beta+\gamma)+\gamma^2-\beta^2)}{4(a^2-1)},\\
(\nabla_{e_4}W)_{2424} &= -(\nabla_{e_4}W)_{3434} = -\frac{\alpha(\beta+1-\gamma)((a^2-2)(\beta-\gamma)+2(a^2-1))}{2(a^2-1)},\\
(\nabla_{e_4}W)_{2434} &= -\frac{(\beta+1-\gamma)((a^2-2)(\beta^2-\gamma^2)+2(a^2-1)(\gamma+\beta))}{4(a^2-1)}.
\end{aligned}
$$

Taking into account Proposition 2.4, (iii) and (iv), we obtain the following result by a straightforward calculation.

Proposition 5.1. *For a left-invariant metric $g_a, a^2 \neq 1$ on $G = H \rtimes_S \mathbb{R}$, the following properties are equivalent:*

(i) *g_a is locally conformally flat,*

(ii) *g_a is locally symmetric,*

(iii) *g_a is locally conformally symmetric,*

(iv) *$a = 0$ and either $\beta = \gamma - 1$ or $\alpha = \beta + \gamma = 0$.*

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Конформна геометрія напівпрямих розширень групи Гейзенберга

Giovanni Calvaruso and Amirhesam Zaeim

Ми розглядаємо загальне напівпряме розширення $G_S = H \rtimes_S \mathbb{R}$ гейзенбергової групи Лі H за означенням, даним в [10], по довільній $S \in \mathfrak{sp}(1,\mathbb{R})$ споряджене сім'єю ліво-інваріантних метрик g_a $(a^2 \neq 1)$. Ця побудова є природнім узагальненням осциляторної групи. Ми повністю визначаємо конформно-ейнштейннові приклади.

Ключові слова: група Гейзенберга, напівпряме розширення, осциляторна група, Бах-пласка метрика, конформно-ейнштейнова метрика