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# Berezin Transforms Attached to Landau Levels on the Complex Projective Space $\mathbf{P}^n(\mathbb{C})$

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We construct coherent states for each eigenspace of a magnetic Laplacian on the complex projective $n$-space in order to apply a quantization-dequantization method. Doing so allows to define the Berezin transform for these spaces. We then establish a formula for this transform as a function of the Fubini–Study Laplacian in a closed form involving of a terminating Kampé de Fériet function. For the lowest spherical Landau level on the Riemann sphere the obtained formula reduces to the one derived by Berezin himself.

*Key words:* complex projective space, coherent states, Berezin transform, magnetic Laplacian, Fubini–Study Laplacian, Koornwinder's formula, Clebsh–Gordan type relation, Kampé de Fériet function

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## 1. Introduction

The Berezin transform was introduced and studied in [7–9] for classical complex symmetric spaces and in [10, 18, 22, 27], for the Bergman, Hardy and Bargmann–Fock spaces. We also refer to [45] for compact hermitian symmetric spaces and to [35] for hermitian line bundles over Kähler manifolds. In his influential paper [8], Berezin used the coherent states quantization together with the correspondence principle to express the transform bearing his name in the sphere and in the Lobachevsky plane through their corresponding Laplace–Beltrami operators. For quantization methods, we refer to the survey [1]. Actually, this transform sets a connection between what Berezin called the contravariant and the covariant symbols of a linear operator and turns out to be closely related to heat flows of elliptic operators [15, 18].

More generally, the Berezin transform stemming from systems of coherent states attached to a class of generalized Bergman spaces on $\mathbb{C}^n$ and on the hyperbolic complex balls $\mathbb{B}^n$ was introduced in [33] and [24] respectively. There, these spaces arise as eigenspaces of Schrödinger operators with uniform magnetic fields whose discrete spectra are the so-called Euclidean and hyperbolic Landau levels. From a complex geometrical point of view, these Schrödinger operators are Bochner Laplacians on hermitian line bundles parametrized by the magnetic field strength or the reciprocal of the Planck parameter [35].

Extensions of Berezin formulas to the magnetic realm were obtained in [33] and [24] by means of spherical Fourier transforms on the Euclidean and the hyperbolic spaces. Another expression of the Berezin transform on $\mathbb{B}^n$, involving Wilson polynomials [3], was derived in [13]. Expressing the magnetic Berezin transform as a function of the Laplace–Beltrami operator on a given complex Kähler manifold has interesting applications [19, 43], see also [35] for the so-called big Hankel operators with non analytic symbols. In particular, the Berezin transform on weighted Bergman spaces is a contraction. At the physical level, obtaining of this kind of formulas comes in the same spirit as obtaining of the famous Simon's diamagnetism inequality [41]. Moreover, since the Laplace-Beltrami operator describes a free particle, the expressions alluded to above transfer the effect of the magnetic field to the representing function. As a matter of the fact, they shed the light on the interplay between the geometry of the phase space on the one hand and the physical quantities on the other hand.

In this paper, we are interested in the complex projective $n$-space $\mathbf{P}^n(\mathbb{C})$, which is the prototype of rank-one complex Riemannian symmetric spaces of compact type. It is canonically endowed with its Fubini–Study metric and the magnetic Laplacian $\Delta_\nu$ ($\nu$ is proportional to the magnetic field strength), as introduced and studied in [26], is the Bochner Laplacian on powers of the Hopf line bundle and of its conjugate (see also [12, 36]). When acting on the space of smooth bounded functions, this operator admits a discrete spectrum consisting of eigenvalues called spherical Landau levels, and we shall be concerned with the corresponding eigenspaces $\mathcal{A}_m^\nu$, $m \in \mathbb{Z}_+$. To each of these eigenspaces, we attach coherent states and use them to define the Berezin transform. Our main result is then a formula expressing this transform as a function of the Fubini–Study Laplacian. In particular, we recover the original formula of Berezin derived in [9] for the complex projective line which corresponds in our framework to the lowest spherical Landau level ($m = 0$) on $\mathbf{P}^1(\mathbb{C})$.

We should mention here that our approach for obtaining the main results is quite different from that in [5]. Indeed,

(1) We point out that in the case of the Riemann sphere ($n = 1$), the magnetic Laplacian coincides with the Hamiltonian of the Dirac monopole whose "eigenfunctions" were identified as *eigensections* by Wu and Yang [44].

(2) We introduce the Berezin transform as a result of a *quantization-dequantization* process via the construction of a suitable set of coherent states and we provide an example of such states for the case $n = 1$. We also link this construction to the definition based on reproducing kernels well known to analysts.

(3) Our argument in proving the existence of a formula expressing this transform is similar to the one used by Berezin himself [8] (case $m = 0$). Actually, we use the fact that $\mathbf{P}^n(\mathbb{C})$ is a rank-one symmetric space (the algebra of bi-invariant operators is generated by the Laplace–Beltrami operator) and that

the Berezin transform commutes with the translation operators defined by group elements of $SU(n+1)$.

(4) While preparing some tools needed for proving our formula, we refer to Koornwinder's paper [30] about the spectral function of the Fubini–Study Laplacian $\Delta_{FS}$ (see (4.15) below).

(5) We make use of a suitable Clebsh–Gordan type linearization formula due to Srivastava [42] for the product of Jacobi polynomials which provides an expansion of the product of Jacobi polynomials over polynomials of the same family. Our choice of this formula is justified by the fact that it contains less terms than in the sum resulting from the power expansion for the product of these polynomials. Consequently, we end up with a simpler formula for the Berezin transform as a function of $\Delta_{FS}$ in a *closed form* involving a terminating Kampé de Fériet function $F^{2:2,2}_{2:1,1}$. In particular, we are able to recover Berezin's original formula [9] derived in the case of the Riemann sphere and corresponding to the lowest Landau level.

The paper is organized as follows. In Section 2, we briefly review the canonical coherent states, their generalization through the Hilbertian probabilistic scheme as well as the associated quantization formalism leading to the Berezin transform whose construction is also linked with the one using reproducing kernels. In Section 3, we recall the geometrical definition of magnetic Laplacians $\Delta_\nu$ on $\mathbf{P}^n(\mathbb{C})$ and we discuss the case $n=1$ in terms of Dirac monopoles. In Section 4, we recall some notations on spherical harmonics that help us to summarize some needed results on eigenspaces of $\Delta_\nu$. The free magnetic case $\nu=0$, which corresponds to Fubini–Study Laplacian $\Delta_{FS}$, is also discussed together with its known spectral function. In Section 5, we attach a system of coherent states to each eigenspace of $\Delta_\nu$ and provide an example. Next, we introduce the corresponding Berezin transform. In Section 6, we establish a formula representing this transform as a function of $\Delta_{FS}$ in a closed form and show that we recover the original result of Berezin for $\mathbf{P}^1(\mathbb{C})$. Section 7 is devoted to some concluding remarks.

## 2. Coherent states quantization

**2.1. Canonical coherent states.** The original idea of coherent states was introduced by E. Schrödinger [38] in order to obtain quantum states in $L^2(\mathbb{R})$ that follow the classical flow associated to the harmonic oscillator Hamiltonian $\hat{H} = \frac{-\hbar^2}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}$. Namely, we have a set $\{\Psi_z \in L^2(\mathbb{R}),\ z \in \mathbb{C}\}$ labeled by elements of $\mathbb{C} \simeq T^*\mathbb{R}$ (the phase space of a particle moving on $\mathbb{R}$) given by

$$\Psi_z(x) = e^{-\frac{1}{2}\hbar z\bar{z}}\frac{1}{(\pi\hbar)^4}\exp\left(-\frac{1}{2\hbar}(\bar{z}^2 + x^2 - 2\sqrt{2}\bar{z}x)\right), \quad x \in \mathbb{R}, \tag{2.1}$$

where $\hbar$ denotes the Planck parameter (we set $\hbar = 1$ for the sake of simplicity). If we denote by $\{\phi_j\}$ an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfunctions of the quantum harmonic oscillator $\hat{H}$, i.e., $\hat{H}\phi_j = j\phi_j$ (the $\phi_j$ are called number states), then the function $\Psi_z$ in (2.1) also admits an expansion over the $\{\phi_j\}$ called number states or Iwata's expansion [14]:

$$\Psi_z(x) = (\mathcal{N}(z))^{-1/2} \sum_{j=0}^{+\infty} \frac{\bar{z}^j}{\sqrt{j!}} \phi_j(x), \tag{2.2}$$

where $\mathcal{N}(z) = e^{z\bar{z}}$ is a factor ensuring the normalization condition $\langle \Psi_z, \Psi_z \rangle = 1$. But since here the coefficients $\{\frac{1}{\sqrt{j!}} z^j\}$ form an orthonormal basis for the Fock–Bargmann space of entire Gaussian square integrable functions on $\mathbb{C}$, whose reproducing kernel is known to be $e^{z\bar{w}}$, then one can interpret the quantity $\mathcal{N}(z)$ as the diagonal function of this kernel. The most important property of states (2.2) is the resolution of the identity operator. More precisely, for any $\phi \in L^2(\mathbb{R})$, the following holds:

$$\phi = \int_{\mathbb{C}} \langle \Psi_z, \phi \rangle_{L^2(\mathbb{R})} \Psi_z \, \frac{1}{\pi} \, d\sigma\,(z)\,, \tag{2.3}$$

where $\langle \Psi_z, \phi \rangle_{L^2(\mathbb{R})}$ denotes the inner product of $\Psi_z$ and $\phi$ and $d\sigma(z) \equiv dx\,dy$, $z = x + iy$, is the usual Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$.

**2.2. The Hilbertian-probabilistic scheme.** Now, one of the generalizations of coherent states, when written in the expansion form (2.2), is actually known as the Hilbertian-probabilistic scheme as explained in detail in [2, 23]. In order to use this scheme, we first need to fix some notations. Let $(X, \mu)$ be a measurable space and denote by $L^2(X, \mu)$ the space of $\mu$-square integrable functions on $X$. Let $\mathcal{A} \subset L^2(X, d\mu)$ be a closed subspace (possibly infinite-dimensional) with an orthonormal basis $\{C_j(x)\}_{j=0}^{\infty}$ and let $(\mathfrak{H}; \langle \cdot, \cdot \rangle)$ be a infinite-dimensional separable Hilbert space equipped with an orthonormal basis $\{\phi_j\}_{j=0}^{\infty}$. Let $\mathcal{D} \subseteq X$ be a subset of $X$ for which the following quantity is finite, i.e.

$$\mathcal{D} \ni x \mapsto \mathcal{N}(x) = \sum_{j=0}^{+\infty} C_j(x)\overline{C_j(x)} < +\infty. \tag{2.4}$$

Then, for $x \in \mathcal{D}$, we define the coherent state by

$$\mathfrak{H} \ni \Psi_x := (\mathcal{N}(x))^{-\frac{1}{2}} \sum_{j=0}^{+\infty} C_j(x)\phi_j. \tag{2.5}$$

These vectors are normalized, i.e., $\langle \Psi_x, \Psi_x \rangle_{\mathfrak{H}} = 1$, and provide the resolution of the identity operator of $\mathfrak{H}$ as

$$\mathbf{1}_{\mathfrak{H}} = \int_{\mathcal{D}} T_x \mathcal{N}(x)\, d\mu(x), \tag{2.6}$$

where $T_x$ is the rank-one operator defined by $T_x[\phi] = \langle \phi, \Psi_x \rangle_{\mathfrak{H}} \Psi_x$. The choice of the Hilbert space $\mathfrak{H}$ defines then a quantization of the domain $\mathcal{D} \subseteq X$ by the states (2.5) via the inclusion map $\mathcal{D} \ni x \mapsto \Psi_x \in \mathfrak{H}$. Observe that by (2.4)–(2.6) we have

$$K(x,y) = (\mathcal{N}(x))^{1/2} (\mathcal{N}(y))^{1/2} \langle \Psi_y, \Psi_x \rangle_{\mathfrak{H}} \tag{2.7}$$

for all $x, y \in X$, meaning that (2.7) defines the reproducing kernel of $\mathcal{A}$ and that $\mathcal{N}(x) = K(x,x)$. We also note that the map $V : \mathfrak{H} \to \mathcal{A}$ defined by

$$V[\phi](x) = (\mathcal{N}(x))^{1/2} \langle \phi, \Psi_x \rangle_{\mathfrak{H}} \tag{2.8}$$

is an isometry from $\mathfrak{H}$ into $\mathcal{A}$ which sends $\phi_j$ to $C_j$. It is called a coherent state transform. In this respect, the property (2.6) bridges classical and quantum mechanics in the sense that every operator acting on $\mathfrak{H}$ or any vector lying there can be decomposed over the phase space $X$. The Klauder–Berezin coherent states quantization consists in associating to a classical observable (that is, a function on $X$ with specific properties) the operator-valued integral

$$P_f := \int_{\mathcal{D}} T_x\, f(x) \mathcal{N}(x)\, d\mu(x). \tag{2.9}$$

The map $f \mapsto P_f$ is not one-to-one in general and given an operator $P$ on $\mathfrak{H}$, any function $f$ such that $P = P_f$ is called a contravariant symbol for $P$. On the other hand, the mean value $\langle P[\Psi_x], \Psi_x \rangle_{\mathfrak{H}}$ of $P$ with respect to the coherent state $\Psi_x$ is referred to as the lower or covariant symbol of $P$. Consequently, we can associate to a classical observable $f$ the mean value $\langle P_f[\Psi_x], \Psi_x \rangle_{\mathfrak{H}}$. Doing so leads to the Berezin transform of $f$ defined by

$$f \longmapsto B[f](x) := \langle P_f[\Psi_x], \Psi_x \rangle_{\mathfrak{H}}, \quad x \in \mathcal{D} \subseteq X. \tag{2.10}$$

**2.3. Link with the Berezin transform defined via reproducing kernels.** It would be helpful to make a connection with the usual definition of the Berezin transform for a classical symmetric domain $X \subset \mathbb{C}^n$. For this, let $\mathcal{A}$ be a closed subspace of $L^2(X, \mu)$ consisting of continuous functions and possessing a reproducing kernel $K(\cdot, \cdot)$ as above. The Berezin symbol $\sigma(S)$ of a bounded linear operator $S$ on $\mathcal{A}$ is the function given by $\sigma(S)(x) = \langle S e_x, e_x \rangle$, where

$$e_x(\cdot) := K(\cdot, x)(K(x,x))^{-\frac{1}{2}} \in \mathcal{A}. \tag{2.11}$$

The Toeplitz operator $T_f$ with the symbol $f \in L^\infty(X)$ is the operator defined on $\mathcal{A}$ by $T_f[\Phi] = \mathrm{Pr}(f\Phi)$, $\Phi \in \mathcal{A}$, where Pr is the orthogonal projection from $L^2(X, \mu)$ onto $\mathcal{A}$. The Berezin transform associated to $\mathcal{A}$ is defined to be the positive self-adjoint operator $B := \sigma T$ which turns out to be a bounded operator on $L^2(X, K(x,x)\, d\mu(x))$. Now, with the above notations, one can make appeal to the transform (2.8) to check that

$$T_f[V[\varphi]] = V[P_f[\varphi]] \tag{2.12}$$

is satisfied for every $\varphi \in \mathfrak{H}$ and $f \in L^\infty(X)$. This means that $V^{-1} T_f V = P_f$ so that we can identify $P_f$ with $T_f$. Next, taking into account (2.7), (2.8), and (2.11) and recalling that $V$ is an isometric map, one can verify that

$$\sigma(T_f)(x) = \langle T_f[V[\Psi_x]], V[\Psi_x] \rangle_{L^2(X)} = \langle P_f[\Psi_x], \Psi_x \rangle_{\mathfrak{H}} \tag{2.13}$$

which says that we can identify acting on $T_f$ by $\sigma$ with $\langle P_f[\Psi_x], \Psi_x \rangle_{\mathfrak{H}}$.

## 3. Magnetic Laplacians $\Delta_\nu$ on $\mathbf{P}^n(\mathbb{C})$

**3.1. Geometrical construction.** We here recall [26] the construction of magnetic Laplacian $\Delta_\nu$ in $\mathbf{P}^n(\mathbb{C}), n \geq 1$. Let $\mathbb{S}^{2n+1} = \left\{ \zeta \in \mathbb{C}^{n+1}, \langle \zeta, \zeta \rangle = 1 \right\}$ be the $(2n+1)-$dimensional unit sphere of $\mathbb{C}^{n+1}$. Then the unit circle $U(1) \equiv \mathbb{S}^1$ acts freely on $\mathbb{S}^{2n+1}$ and one can define the complex projective space by $\mathbf{P}^n(\mathbb{C}) = \mathbb{S}^1 \backslash\ \mathbb{S}^{2n+1}$ to be the set of all complex lines of $\mathbb{C}^{n+1}$. Indeed, the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbf{P}^n(\mathbb{C})$ defines a principal $U(1)-$bundle on $\mathbf{P}^n(\mathbb{C})$ whose associated complex line is $\mathfrak{L} = \left\{ (l, z) \in \mathbf{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}, z \in l \right\}$. It is endowed with the Fubini–Study metric $ds^2_{FS}$ which reads in a standard charte $\mathbb{C}^n \equiv \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}; z_{n+1} = 1 \right\}$ or local coordinates as

$$ds^2_{FS} := \sum_{i,j=1}^{n} \left( (1 + \langle z, z \rangle) \, \delta_{ij} - z_i \overline{z}_j \right) dz_i \otimes d\overline{z}_j, \tag{3.1}$$

where $g_{ij}(z) = (1 + \langle z, z \rangle)^{-2} \left( (1 + \langle z, z \rangle) \, \delta_{ij} - z_i \overline{z}_j \right)$. The projective space $\mathbf{P}^n(\mathbb{C})$ equipped with this metric is a compact Kählerian manifold of complex dimension $n$. The associated Laplace–Beltrami operator is given by

$$\sum_{i,j=1}^{n} g^{ij}(z) \frac{\partial^2}{\partial z_i \partial \overline{z}_j}, \tag{3.2}$$

where $\left( g^{ij}(z) \right)$ is the inverse of the matrix and reads

$$\Delta_{FS} = 4(1 + \langle z, z \rangle) \sum_{i,j=1}^{n} (\delta_{ij} + z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j}. \tag{3.3}$$

The Fubini–Study distance is defined by

$$\cos^2 d_{FS}(z, w) = \frac{|1 + \langle z, w \rangle|^2}{(1 + \langle z, z \rangle)(1 + \langle w, w \rangle)}. \tag{3.4}$$

Let $\nabla = d + \partial \log (1 + \langle z, z \rangle)$ be the unique hermitian connection associated with $ds^2_{FS}$ on $\mathfrak{L}$. Now, for a positive integer $\nu$, let $\mathfrak{L}^\nu := (\overline{\mathfrak{L}^*})^{\otimes \nu} \otimes (\mathfrak{L}^*)^{\otimes \nu}$ be the complex line bundle over $\mathbf{P}^n(\mathbb{C})$, where $\overline{\mathfrak{L}^*}$ denotes the conjugate dual of $\mathfrak{L}$. Then the corresponding hermitian connection on $\mathbf{P}^n(\mathbb{C})$ reads

$$\nabla_\nu = d + \nu \left( \partial - \overline{\partial} \right) \partial \log (1 + \langle z, z \rangle). \tag{3.5}$$

Next, consider the operator

$$\Delta_\nu := -\left( \nabla_\nu \right)^* \nabla_\nu \tag{3.6}$$

acting on the space of smooth sections $\Gamma^\infty_{n,\nu} := C^\infty \left( \mathbf{P}^n(\mathbb{C}), \mathfrak{L}^\nu \right)$, which is also known as the Bochner Laplacian on Hermitian line bundles parametrized by the magnetic field strength $\nu$. Precisely, in the local coordinates, the operator $\Delta_\nu$ takes the form (we omit the dependence on $n$):

$$\Delta_\nu = 4(1+\langle z,z\rangle)\left(\sum_{i,j=1}^{n}(\delta_{ij}+z_i\bar{z}_j)\frac{\partial^2}{\partial z_i\partial\bar{z}_j} - \nu\sum_{j=1}^{n}\left(z_j\frac{\partial}{\partial z_j}-\bar{z}_j\frac{\partial}{\partial\bar{z}_j}\right)-\nu^2\right)+4\nu^2 \tag{3.7}$$

and will be called, according to [26], a magnetic Laplacians on $\mathbf{P}^n(\mathbb{C})$. The dependence of the operator $\Delta_\nu$ on $n$ is omitted.

**3.2. An example.** Take $n=1$ and consider the elements of $\Gamma^\infty_{1,\nu}$. These are smooth sections of the $U(1)$-bundle with the first Chern class $\nu$ [25] and are acted on by the Hamiltonian $H_\nu = -\Delta_\nu + 4\nu^2$ of the Dirac (point) monopole in $\mathbb{R}^3$ with magnetic charge $\nu$ (in suitable units, see (3.8) below). Indeed, eigenfunctions of this monopole were identified as *sections* by Wu and Yang [44] and are known as *monopole harmonics*. Their explicit expression in the coordinate $z$ are given below by (3.9). For more information on Dirac monopoles, see [39]. The restriction $\nu\in\mathbb{Z}_+$ results from Dirac's quantization condition for monopole charges, which requires that the total flux of the magnetic field across a closed surface be an integer multiple of a universal constant. It can also be understood in the context of cohomology groups for hermitian line bundles [28] or as the Weil–Souriau–Kostant quantization condition [40]. In the stereographic coordinate $z\in\mathbb{C}\cup\{\infty\}\equiv\mathbb{S}^2\equiv\mathbf{P}^1(\mathbb{C})$ (and suitable units) this Hamiltonian reads [21, p. 598]:

$$H_\nu = -(1+|z|^2)\left((1+|z|^2)\frac{\partial^2}{\partial z\partial\bar{z}}+\nu\left(z_j\frac{\partial}{\partial z_j}-\bar{z}_j\frac{\partial}{\partial\bar{z}_j}\right)-\nu^2\right)-\nu^2. \tag{3.8}$$

Note that $\frac{-1}{4}\Delta_\nu = H_\nu$ for $n=1$. The stereorgraphic projection bridges the monopole system and the Landau system which describes spinless charged particles in perpendicular homogeneous magnetic fields [17, p. 240]. Precisely, to determine eigenstates of the monopole Hamiltonian (3.8) one proceeds exactly as for the Landau Hamiltonian of the Euclidean setting. This leads, for each fixed integer $m\in\mathbb{Z}_+$, to a finite dimensional $L^2$ eigenspace whose orthonormal basis vectors $\left\{\Phi_k^{\nu,m}\right\}_k, -m\le k\le 2\nu+m$, are given by [34]:

$$\Phi_k^{\nu,m}(z) := \sqrt{\frac{(2\nu+2m+1)\,(2\nu+m)!m!}{(m+k)!(2\nu+m-k)!}}\,(1+z\bar{z})^{-\nu}\,z^k P_m^{(k,2\nu-k)}\left(\frac{1-z\bar{z}}{1+z\bar{z}}\right), \tag{3.9}$$

where $P_m^{(\alpha,\beta)}(\cdot)$ is the Jacobi polynomial [3]. The eigenstates (3.9) are associated with the eigenvalue

$$\tau_m := (2m+1)\,\nu + m\,(m+1) \tag{3.10}$$

called a spherical Landau level.

## 4. Spaces of bounded eigenfunctions of $\Delta_\nu$

In order to summarize some needed results [26] about eigenspaces of the operator $\Delta_\nu$, we first need to fix some notations [37].

**4.1. Spherical harmonics.** Let $\mathcal{P}(\mathbb{C}^n)$ denote the space of polynomials in the independent variables $z$ and $\bar{z}$ of $\mathbb{C}^n$. The elements of this space can be written in the form

$$u(z,\bar{z}) = \sum_{|\alpha|\le k}\sum_{|\beta|\le l} c_{\alpha,\beta} z^\alpha \bar{z}^\beta, \qquad c_{\alpha,\beta}\in\mathbb{C}, \quad \alpha,\beta\in\mathbb{Z}_+^n, \tag{4.1}$$

for nonnegative integers $k$ and $l$, where the standard multi-index is used. The subspace of $\mathcal{P}(\mathbb{C}^n)$ composed of polynomials that are homogeneous of degree $p$ in $z$ and of degree $q$ in $\bar{z}$ is denoted by $\mathcal{P}_{p,q}(\mathbb{C}^n)$. The dimension of $\mathcal{P}_{p,q}(\mathbb{C}^n)$ is given by

$$\kappa(n,p,q) = \binom{p+n-1}{p-1}\binom{q+n-1}{q-1} \tag{4.2}$$

in terms of binomial coefficients

$$\binom{\alpha}{s} := \alpha(\alpha-1)\cdots(\alpha-s+1)/s! \ \text{ if } \ s\in\mathbb{Z}_+\setminus\{0\} \text{ and } \binom{\alpha}{0} := 1, \text{ for all } \alpha\in\mathbb{R}.$$

The subspace of $\mathcal{P}_{p,q}(\mathbb{C}^n)$ composed of harmonic elements, that is, elements in the kernel of the complex Laplacian

$$\Delta_{\mathbb{C}^n} := 4\sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j} \tag{4.3}$$

is denoted by $\mathfrak{H}_{p,q}(\mathbb{C}^n)$. The set of restrictions of elements of $\mathfrak{H}_{p,q}(\mathbb{C}^n)$ to the unit sphere $\mathbb{S}^{2n-1} = \{\zeta\in\mathbb{C}^n, \langle\zeta,\zeta\rangle = 1\}$, denoted by $\mathcal{H}(p,q)$, is called the space of complex spherical harmonics of degree $p$ in $z$ and degree $q$ in $\bar{z}$. Note that $\mathcal{H}(p,0)$ consists of holomorphic polynomials, and $\mathcal{H}(0,q)$ consists of polynomials whose complex conjugates are holomorphic. The dimensions of spaces $\mathcal{H}(p,q)$, denoted by $d(n,p,q)$, are given by

$$d(n,p,q) = \kappa(n,p,q) - \kappa(n,p-1,q-1), \quad p,q\neq 0, \tag{4.4}$$

$$d(n,p,0) = \kappa(n,p,0) \qquad \text{and} \qquad d(n,0,q) = \kappa(n,0,q). \tag{4.5}$$

For $n=1$, $d(n,p,0) = d(n,0,q) = 1$, but $\mathcal{H}(p,q) = \{0\}$ if both $p>0$ and $q>0$. It is a standard fact that the spaces $\mathcal{H}(p,q)$ are pairwise orthogonal in $L^2(\mathbb{S}^{2n-1}, d\omega)$, where $d\omega$ is the uniform measure on the sphere.

**4.2. Bounded eigenfunctions of $\Delta_\nu$.** For $\lambda\in\mathbb{C}$, we set $\Lambda_{n,\nu}(\lambda) := n^2 - \lambda^2 + 4\nu^2$ and consider the equation [26, p. 149]:

$$\Delta_\nu F(z) = \Lambda_{n,\nu}(\lambda)\, F(z), \tag{4.6}$$

where $F$ is a bounded function on $\mathbb{C}^n$. Define the eigenspace

$$\mathcal{A}_m^\nu := \{F\in L^\infty(\mathbb{C}^n),\ \Delta_\nu F = \Lambda_{n,\nu}(\lambda)\, F\}. \tag{4.7}$$

From [26], the eigenspace $\mathcal{A}_m^\nu = \{0\}$ if $\lambda \notin D_\nu$, where

$$D_\nu := \{\lambda \in \mathbb{C},\ \frac{1}{2}(n \pm \lambda) + \nu \in \mathbb{Z}_-\} \cup \{\lambda \in \mathbb{C},\ \frac{1}{2}(n \pm \lambda) - \nu \in \mathbb{Z}_-\}. \tag{4.8}$$

Otherwise it is not trivial if and only if $\lambda$ has the form $\lambda = \pm(2(m+\nu)+n)$ for some $m \in \mathbb{Z}_+$. Note that when $n = 1$ and $\lambda = \pm(2(m+\nu)+1)$ the above parametrization of the eigenvalue of $\Delta_\nu$ gives that $\frac{-1}{4}\Lambda_{1,\nu}(\lambda) = \tau_m$ (where $\tau_m$ was given by (3.10)) as an expected form of the example in Subsection 3.2. For $n \geq 1$ and under the condition $\lambda = \pm(2(m+\nu)+n)$, any function $F$ in $\mathcal{A}_m^\nu$ admits the expansion

$$\begin{aligned} F(r\omega) = \frac{1}{(1+r^2)^{(m+\nu)}} & \sum_{\substack{0 \leq p \leq m \\ 0 \leq q \leq m+2\nu}} r^{p+q} {}_2F_1\left( \begin{matrix} p-m, q-m-2\nu \\ n+p+q \end{matrix} \middle| -r^2 \right) \\ & \times \sum_{j=1}^{d(n,p,q)} a_j^{\nu,p,q} h_{p,q}^j(\omega, \bar{\omega}), \end{aligned} \tag{4.9}$$

where $r > 0$, $\omega \in \mathbb{S}^{2n-1}$, $a_j^{\nu,p,q}$ are constant complex numbers, ${}_2F_1$ is the Gauss hypergeometric function [3] and $\left\{h_{p,q}^j\right\}_{j=0}^{d(n,p,q)}$ is an orthonormal basis of $\mathcal{H}(p,q)$. Note that $F$ satisfies the growth condition

$$\lim_{r\to\infty} F(r\omega) = \sum_{0 \leq p \leq m} \frac{\Gamma(m-p+1)\Gamma(n+2p+2\nu)}{(-1)^{p-m}\Gamma(m+n+p+2\nu)} \sum_{j=1}^{d(n,p,p+2\nu)} a_j^{\nu,p}\, h_{p,p+2\nu}^j(\omega, \bar{\omega}) \tag{4.10}$$

where we wrote $a_j^{\nu,p} = a_j^{\nu,p,p+2\nu}$. In particular, $\mathcal{A}_m^\nu$ has a finite dimension given by

$$\dim_{\mathbb{C}} \mathcal{A}_m^\nu = \frac{(2m+n+2\nu)\Gamma(m+n)\Gamma(m+n+2\nu)}{n\left(\Gamma(n)\right)^2 \Gamma(m+1)\Gamma(m+2\nu+1)} \tag{4.11}$$

and admits a reproducing kernel given by [26]:

$$K_{\nu,m}(z,w) := c_m^{\nu,n} \left( \frac{(1+\langle z,w \rangle)^2}{(1+|z|^2)(1+|w|^2)} \right)^\nu P_m^{(n-1,2\nu)}(\cos 2d_{FS}(z,w)) \tag{4.12}$$

with

$$c_m^{\nu,n} = \frac{(2m+2\nu+n)\Gamma(m+n+2\nu)}{\pi^n \Gamma(m+2\nu+1)}. \tag{4.13}$$

**4.3. The free-magnetic case $\nu = 0$.** In this case, the operator in (3.7) reduces to the Fubini-Study Laplacian (3.3), i.e., $\Delta_0 = \Delta_{FS}$ which has a discrete spectral decomposition with eigenvalues $(-4k(k+n))_{k\geq 0}$. Besides, each eigenspace is finite-dimensional and has an orthonormal basis given by homogeneous spherical harmonics of degree zero. Let

$$d\mu_n(w) = (1+\langle w,w \rangle)^{-(n+1)} d\mu(w), \quad w \in \mathbb{C}^n, \tag{4.14}$$

where $d\mu(w)$ is the Lebesgue measure on $\mathbb{C}^n$. Letting $\nu = 0$ in (4.12), the kernel of the orthogonal projection from the space $L^2(\mathbb{C}^n, d\mu_n)$ onto the $k$-th eigenspace of $\Delta_{FS}$ reduces to

$$K_{0,k}(z,w) = \pi^{-n}(2k+n)\frac{\Gamma(n+k)}{k!}P_k^{(n-1,0)}(\cos 2d_{FS}(z,w)). \tag{4.15}$$

This is in agreement with the formula derived by Koornwinder in [30, Theorem 3.8, p. 19]. Consequently, the spectral theorem implies that for any function $L$ continuous on an open set containing the spectrum of $-\Delta_{FS}$, the operator $L(-\Delta_{FS})$ is an integral operator whose kernel is given by

$$\sum_{k\geq 0} L(4k(k+n))K_{0,k}(z,w). \tag{4.16}$$

## 5. Coherent states quantization

**5.1. Coherent states attached to $\mathcal{A}_m^\nu$.** Now we specialize the definition (2.5) of coherent states to the eigenspace (4.7) by taking $X \equiv \mathbb{C}^n$ endowed with $d\mu_n$, $x \equiv z \in \mathbb{C}^n$, $\mathcal{A} \equiv \mathcal{A}_m^\nu$, and an orthonormal basis $(\Phi_{p,q,j}^{\nu,m})_{p,q,j}$ of $\mathcal{A}_m^\nu$, where $1 \leq j \leq d(n,p,q)$ and $0 \leq q \leq m+2\nu, 0 \leq p \leq m$. The Hilbert space $\mathfrak{H}$ carrying the quantum states of some physical system and its basis $(\phi_{p,q,j})_{p,q,j}$ will be specified when needed. With these data, we define for any $n \geq 1, \nu, m \in \mathbb{Z}_+$, the following coherent states:

$$\Psi_z^{\nu,m} := (\mathcal{N}^{\nu,m}(z))^{-1/2} \sum_{\substack{0\leq p\leq m \\ 0\leq q\leq m+2\nu \\ 1\leq j\leq d(n,p,q)}} \overline{\Phi_{p,q,j}^{\nu,m}(z)}.\phi_{p,q,j}, \tag{5.1}$$

where

$$\mathcal{N}^{\nu,m}(z) = \mathcal{N}^{\nu,m} = \frac{(2m+2\nu+n)\Gamma(m+n+2\nu)}{\pi^n\Gamma(m+2\nu+1)}\frac{(n)_m}{m!}. \tag{5.2}$$

To check (5.2), observe that the normalizing factor can be expressed as the diagonal of the reproducing kernel (4.12) and use the special value ( [20], p.169),

$$P_m^{(n-1,2\nu)}(1) = \frac{(n)_m}{m!}. \tag{5.3}$$

The states defined above satisfy the resolution of the identity

$$\mathbf{1}_{\mathfrak{H}} = \int_{\mathbb{C}^n} T_w^{\nu,m}\mathcal{N}^{\nu,m}\, d\mu_n(w), \tag{5.4}$$

where $T_w^{\nu,m}$ is the operator which projects a vector $\phi \in \mathfrak{H}$ onto a coherent state $\Psi_w^{\nu,m}$ as $T_w^{\nu,m}[\phi] = \langle \Psi_w^{\nu,m}, \phi\rangle_{\mathfrak{H}}\Psi_w^{\nu,m}$ and allows the quantization scheme described in Section 2.

*Example* 5.1. For $n = 1$, coherent states (5.1) can be defined via the superposition

$$\Psi_z^{\nu,m}(x) = (2(\nu+m)+1)^{-\frac{1}{2}} \sum_{k=0}^{2(\nu+m)} \overline{\Phi_k^{(\nu,m)}(z)}\phi_k(x), \tag{5.5}$$

where $\Phi_k^{(\nu,m)}(z)$ are given by (3.9) and

$$\phi_k(x) \equiv \phi_k^{(r)}(x,N) := K_k^{(r)}(x+Nr,N)\sqrt{\frac{k!(N-k)!r^{Nr+x}s^{Ns-x}}{r^k s^k \Gamma(Nr+x+1)\Gamma(Ns-x+1)}} \tag{5.6}$$

with $-Nr \le x \le (1-r)N$, $s = 1-r$, $N \in \mathbb{Z}_+$ and $K_k^{(r)}(.,N)$ being the Kravchuk polynomial [29]. The functions $x \mapsto \phi_k^{(r)}(x,N)$ satisfy the orthogonality relations $\sum_{j=0}^{N} \phi_k^{(r)}(x_j,N)\phi_\ell^{(r)}(x_j,N) = \delta_{\ell,k}$ at points $x_j$ of the discrete set $\Omega_{N+1}^{(r)} = \{x_j = j - rN, j = 0,1,\dots,N\}$. Here, the Hilbert space $\mathfrak{H}$ in the above formalism is specified to be $\ell^2\left(\Omega_{N+1}^{(r)}\right)$, the space of square summable functions on $\Omega_{N+1}^{(r)}$, and it stands for the states space of the Kravchuk oscillator, see [32,34] for more details on such coherent states.

**5.2. Berezin transform $\mathcal{B}_m^\nu$.** Now, to an arbitrary function $\varphi \in L^\infty(\mathbb{C}^n)$, viewed as a classical observable, we associate (*quantization*) the operator-valued integral

$$P_\varphi = \int_{\mathbb{C}^n} T_w^{\nu,m}[\varphi]\mathcal{N}^{\nu,m}\, d\mu_n(w), \tag{5.7}$$

viewed as a quantum observable, where $T_w^{\nu,m}$ is the rank-one operator

$$T_w^{\nu,m}[\varphi] = \langle \varphi, \Psi_w^{\nu,m}\rangle \Psi_w^{\nu,m}. \tag{5.8}$$

Next, we define a new classical observable by taking the expectation (*dequantization*) of $P_\varphi$ with respect to the set of coherent states $\{\Psi_z^{\nu,m}\}$ as follows:

$$z \mapsto \langle \Psi_z^{\nu,m}, P_\varphi[\Psi_z^{\nu,m}]\rangle_{\mathfrak{H}}. \tag{5.9}$$

Then, according to the formalism in Subsection 2.2, the map defined by

$$\varphi \mapsto \mathcal{B}_m^\nu[\varphi](z) := \langle \Psi_z^{\nu,m}, P_\varphi[\Psi_z^{\nu,m}]\rangle_{\mathfrak{H}} \tag{5.10}$$

is the Berezin transform of $\varphi$. Starting from the scalar product (5.10) and replacing there $P_\varphi$ and $T_w^{\nu,m}[\varphi]$ by their expressions (5.7) and (5.8) successively, straightforward computations readily give

$$\mathcal{B}_m^\nu[\varphi](z) = \frac{m!c_m^{\nu,n}}{(n)_m}\int_{\mathbb{C}^n}\left(\cos^2 d_{FS}(z,w)\right)^{2\nu}\left(P_m^{(n-1,2\nu)}(\cos 2d(z,w))\right)^2 \varphi(w)\, d\mu_n(w), \tag{5.11}$$

where $c_m^{\nu,n}$ is the constant given by (4.13).

**Proposition 5.2.** *For $\nu,\ m \in \mathbb{Z}_+$ there exists a measurable function $W$ such that $\mathcal{B}_m^\nu = W(\Delta_{FS})$.*

*Proof.* Notice that the kernel of the Berezin transform $B_m^\nu$ depends only on the geodesic distance $d_{FS}$ defined by (3.4) which is an $SU(n+1)$-biinvariant function. It follows that $B_m^\nu$ commutes with the translation operators $\tau_g$ defined by

$$\tau_g\left[f\right](z) = f\left(g.z\right), \quad g \in SU(n+1),\ z \in \mathbb{C}^n, \tag{5.12}$$

where for the group element written into bloc matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A, B, C, D$ there are $n \times n$, $n \times 1$, $1 \times n$, $1 \times 1$ matrices, respectively, and the group action is defined by $g.z = (Az + B)\left(Cz + D\right)^{-1}$. According to [11] (see Section 4 there), for every operator $\delta$ in the space $L^2$ on a symmetric domain which commutes with operators (5.12) there is a function of the Laplace operators $\Delta^{(1)}, \ldots, \Delta^{(r)}$, where $r$ is the rank of the domain. That is, $\delta = W\left(\Delta^{(1)}, \ldots, \Delta^{(r)}\right)$. But since here $\mathbf{P}^n(\mathbb{C})$ is a rank-one symmetric domain, then $B_m^\nu = W\left(\Delta^{(1)}\right)$, where $\Delta^{(1)} = \Delta_{FS}$ is the Fubini-Study operator. $\square$

In the subsequent section, we shall determine explicitly $W$ relying on the spectral theory of $\Delta_{FS}$.

## 6. A formula for the Berezin transform $\mathcal{B}_m^\nu$

Here is our main result.

**Theorem 6.1.** *For any $n \geq 1$, $\nu, m \in \mathbb{Z}_+$, the transform $B_m^\nu$ can be expressed as a function of the Fubini–Study Laplacian $\Delta_{FS}$,*

$$\begin{aligned}\mathcal{B}_m^\nu &= \frac{\beta_{\nu,n,m}}{\Gamma\left(2\nu + 1 + \frac{n}{2} - \frac{\sqrt{n^2 - \Delta_{FS}}}{2}\right)\Gamma\left(2\nu + 1 + \frac{n}{2} + \frac{\sqrt{n^2 - \Delta_{FS}}}{2}\right)} \\ &\times F_{2:1,1}^{2:2,2}\left[\left.\begin{matrix} 2\nu+1,\ 2\nu+1: \quad -m\ , \quad 2\nu+m+n, \quad -m\ , \quad 2\nu+m+n \\ 2\nu+1+\frac{n}{2} - \frac{\sqrt{n^2-\Delta_{FS}}}{2}, 2\nu+1+\frac{n}{2} + \frac{\sqrt{n^2-\Delta_{FS}}}{2} : 2\nu+1, 2\nu+1 \end{matrix}\right| 1, 1\right],\end{aligned} \tag{6.1}$$

*where*

$$\beta_{\nu,n,m} = (2m + 2\nu + n)\,\frac{\Gamma\left(m + 2\nu + n\right)\Gamma\left(m + 2\nu + 1\right)}{\Gamma\left(m + n\right)\Gamma\left(m + 1\right)}\Gamma(n), \tag{6.2}$$

*and $F_{2:1,1}^{2:2,2}$ denotes the Kampé de Fériet function. In particular, for $n = 1$ and $m = 0$ corresponding to the lowest Landau level on the Riemann sphere, we retrieve Berezin's formula*

$$\mathcal{B}_0^\nu = \prod_{\ell \geq 1}\left(1 + \frac{\Delta_{FS}}{(\ell + 2\nu)(\ell + 2\nu + 1)}\right). \tag{6.3}$$

*Proof.* Appealing to (4.16) and the kernel of $\mathcal{B}_m^\nu$ in (5.11), the requirement $B_m^\nu = W(-\Delta_{FS})$ is equivalent to the equation

$$\frac{m!}{(n)_m} c_m^{\nu,n} \left(\cos^2 d_{FS}(z,w)\right)^{2\nu} \left(P_m^{(n-1,2\nu)}(\cos 2d_{FS}(z,w))\right)^2 = \sum_{k=0}^{\infty} W(\lambda_k) K_{0,k}(z,w), \quad (6.4)$$

where $\lambda_k := 4k(k+n)$, and $K_{0,k}(z,w)$ was defined in (4.15). Setting $t = \cos^2 d_{FS}(z,w)$ and using the symmetry relation $P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\beta,\alpha)}(x)$ satisfied by Jacobi polynomials, (6.4) takes the form

$$t^{2\nu}\left(P_m^{(2\nu,n-1)}(1-2t)\right)^2 = \sum_{k=0}^{\infty}\left[\frac{W(\lambda_k)(m+2\nu)!\,(n)_m\,(2k+n)\Gamma(n+k)}{(2m+2\nu+n)(n+m+2\nu-1)!m!\,(-1)^k\,(n)_k}\right] P_k^{(0,n-1)}(1-2t)\,. \quad (6.5)$$

To get a solution for the unknown quantity $W(\lambda_k)$, we need to expand the left-hand side of (6.5) as a series of Jacobi polynomials $(P_k^{(0,n-1)})_{k\geq 0}$. To proceed, we make use of the following formula [42, p. 4467] satisfied by any $\alpha_1, \alpha, \beta > -1$, any $m, \mu \in \mathbb{Z}_+$ and any $t \in [0,1]$:

$$\begin{aligned} t^\mu P_m^{(\alpha_1,\beta)}(1-2t) P_m^{(\alpha_1,\beta)}(1-2t) &= (\alpha+1)_\mu \binom{\alpha_1+m}{m}^2 \sum_{k=0}^{+\infty} \frac{(\alpha+\beta+2k+1)(-\mu)_k}{(\alpha+1)_k(\alpha+\beta+k+1)_{\mu+1}} P_k^{(\alpha,\beta)}(1-2t) \\ &\times F^{2:2,2}_{2:1,1}\left[\left.\begin{matrix}\mu+1,\ \alpha+\mu+1 & : -m\ ,\alpha_1+\beta+m+1, & -m,\ \alpha_1+\beta+m+1 \\ \mu-k+1\ , & \alpha+\beta+\mu+2+k\ : & \alpha_1+1\ , \quad \alpha_1+1\end{matrix}\right| 1,1\right], \end{aligned} \quad (6.6)$$

where

$$F^{2:2,2}_{2:1,1}\left[\left.\begin{matrix}a_1,a_2 : b_1,b_2,b_3,b_4 \\ c_1,c_2 : \quad d_1,d_2\end{matrix}\right| x,y\right] := \sum_{s,l=0}^{\infty} \frac{(a_1)_{l+s}(a_2)_{l+s}}{(c_1)_{l+s}(c_2)_{l+s}} \frac{(b_1)_l(b_2)_l(b_3)_s(b_4)_s}{(d_1)_l(d_2)_s} \frac{x^l y^s}{l!s!} \quad (6.7)$$

is the Kampé de Fériet function [31]. Specializing (6.6) with $\mu = \alpha_1 = 2\nu$, $\beta = n-1$, $\alpha = 0$, we get, after identification, that

$$\begin{aligned} W(\lambda_k) &= (2m+2\nu+m)\frac{\Gamma(m+2\nu+n)\Gamma(m+2\nu+1)}{\Gamma(m+n)\Gamma(m+1)}\Gamma(n) \\ &\times \frac{1}{\Gamma(n+2\nu+1+k)\Gamma(2\nu-k+1)} \\ &\times F^{2:2,2}_{2:1,1}\left[\left.\begin{matrix}2\nu+1,\ 2\nu+1 : -m\ ,2\nu+m+n, -m,\ 2\nu+m+n \\ 2\nu-k+1,\ n+2\nu+k+1 : \quad 2\nu+1,\ 2\nu+1\end{matrix}\right| 1,1\right]. \end{aligned} \quad (6.8)$$

Observe that in (6.6) the infinite sum over $k$ terminates at $2\nu$ and that

$$(-2\nu)_k = (-1)^k \frac{(2\nu)!}{(2\nu-k)!}, \quad k \leq 2\nu. \tag{6.9}$$

Similarly, the Kampé de Fériet series in (6.8) terminates at $m$. Solve the equation $\lambda_k = 4k(k+n) = \lambda$ in the variable $k \geq 0$ so that we can extend the function $W$ for $\lambda \in \mathbb{R}_+$. Precisely, we may write $W(\lambda)$ by replacing $k$ in (6.8) by $\frac{-n+\sqrt{n^2+\lambda}}{2}$. Finally, by replacing $\lambda$ by $-\Delta_{FS}$, we have ended with this point. In the case $n = 1$, using the definition (6.7), we see that $F^{2:2,2}_{2:1,1}$ reduces to 1 for $m = 0$ and that

$$W(4k(k+1)) = \frac{\Gamma(2\nu+1)\Gamma(2\nu+2)}{\Gamma(2\nu+1-k)\Gamma(2\nu+2+k)}. \tag{6.10}$$

Now recall the Weierstrass product for the Gamma function [3],

$$\frac{1}{\Gamma(s+1)} = e^{\gamma s} \prod_{\ell \geq 1} \left(1 + \frac{s}{\ell}\right) e^{-s/\ell}, \tag{6.11}$$

where $\gamma$ is Euler's constant given by $\gamma = \lim_{q \to +\infty} \left(\sum_{j=1}^{q} \frac{1}{j} - \log q\right)$. It follows that

$$W(4k(k+1)) = \beta_{\nu,1,0} e^{\gamma(4\nu+3)} \prod_{\ell \geq 1} \left(1 + \frac{2\nu-k}{\ell}\right)\left(1 + \frac{2\nu+1+k}{\ell}\right) e^{-(4\nu+3)/\ell}. \tag{6.12}$$

Writing

$$\begin{aligned} &\left(1 + \frac{2\nu-k}{\ell}\right)\left(1 + \frac{2\nu+1+k}{\ell}\right) \\ &\quad = \left(1 + \frac{2\nu}{\ell}\right)\left(1 + \frac{2\nu+1}{\ell}\right)\left(1 - \frac{k}{\ell+2\nu}\right)\left(1 + \frac{k}{\ell+2\nu+1}\right) \end{aligned} \tag{6.13}$$

and using again the Weierstrass product

$$e^{\gamma(4\nu+3)} \prod_{\ell \geq 1} \left(1 + \frac{2\nu}{\ell}\right)\left(1 + \frac{2\nu+1}{\ell}\right) e^{-(4\nu+3)/\ell} = \frac{1}{\Gamma(2\nu+1)\Gamma(2\nu+2)}, \tag{6.14}$$

we get

$$W(4k(k+1)) = \frac{\beta_{\nu,1,0}}{\Gamma(2\nu+1)\Gamma(2\nu+2)} \prod_{\ell \geq 1} \left(1 - \frac{k(k+1)}{(\ell+2\nu)(\ell+2\nu+1)}\right). \tag{6.15}$$

As a matter of the fact, $W$ can be chosen as

$$W(\lambda) = \frac{((2\nu)!)^2(2\nu+1)}{\Gamma(2\nu+1)\Gamma(2\nu+2)} \prod_{\ell \geq 1} \left(1 - \frac{\lambda}{(\ell+2\nu)(\ell+2\nu+1)}\right), \quad \lambda \geq 0, \tag{6.16}$$

so that

$$\mathcal{B}_0^\nu = W(-\Delta_{FS}) = \prod_{\ell\geq 1}\left(1+\frac{\Delta_{FS}}{(\ell+2\nu)(\ell+2\nu+1)}\right). \tag{6.17}$$

Identifying $2\nu$ with $1/h$ in the notation of ( [9], p.171), we get Berezin's formula

$$\mathcal{B}_0^\nu = \prod_{\ell\geq 1}\left(1+h^2\frac{\Delta_{FS}}{(1+\ell h)(1+(\ell+1)h)}\right). \tag{6.18}$$

This ends the proof. □

To decompose the Kampé de Fériet series as a finite sum in terms of the hypergeometric function ${}_4F_3$, recall the expansion ( [20], p.101),

$${}_4F_3\left(\begin{matrix} a,b,c,d \\ e,f,g\end{matrix}\middle| u\right) = \sum_{l=0}^{\infty}\frac{(a)_l(b)_l(c)_l(d)_l}{(e)_l(f)_l(g)_l}\frac{u^l}{l!},$$

provided the series converges. If, moreover, the quantity $s=(e+f+g)-(a+b+c+d)$ is an integer, we say that ${}_4F_3$ is $s$-balanced. In the case $s=1$, this series is called Saalschützian.

**Corollary 6.2.** *For any $n\geq 1$, $\nu,m\in\mathbb{Z}_+$ and $\lambda_k=4k\,(k+n)\,,\ k\in\mathbb{Z}_+$, we can decompose $W(\lambda_k)$ as*

$$\begin{aligned} W(\lambda_k) &= (2m+2\nu+n)(m+2\nu+1)_{n-1}\frac{((2\nu+m)!)^2}{m!} \\ &\times\frac{k!}{(n)_k(n)_m\Gamma(2\nu-k+1)\Gamma(n+k+2\nu+1)}\sum_{s=0}^{m}\frac{(-m)_s(2\nu+1)_s(2\nu+m+n)_s}{s!(2\nu-k+1)_s(n+2\nu+k+1)_s} \\ &\times{}_4F_3\left[\begin{matrix}-m,\ 2\nu+1+s,\ l2\nu+1+s, 2\nu+m+n, \\ 2\nu-k+1+s, n+2\nu+1+k+s, 2\nu+1\end{matrix}\middle| 1\right]. \end{aligned}$$

## 7. Concluding remarks

In this paper, we dealt with the complex projective $n$-space $\mathbf{P}^n(\mathbb{C})$, $n\geq 1$, endowed with its Fubini–Study metric and the Schrödinger operator $\Delta_\nu$ with a uniform magnetic field whose strength is proportional to $\nu\in\mathbb{Z}_+$. This operator is nothing else but the Bochner Laplacian on powers of the Hopf line bundle and of its conjugate. When acting on the space of smooth bounded functions, $\Delta_\nu$ admits a discrete spectrum consisting of eigenvalues called spherical Landau levels, and to each eigenspace $\mathcal{A}_m^\nu$, we attached a set of coherent states used afterwards to define the Berezin transform via a quantization-dequantization process. Our main result is a formula expressing this transform as a function of the Fubini-Study Laplacian $\Delta_{FS}=\Delta_0$ given by a terminating Kampé de Fériet function $F^{2:2,2}_{2:1,1}$. In particular, we recovered Berezin's original formula which corresponds in our framework to the lowest Landau level on the Riemann sphere ($n=1, m=0$). A more general formula was subsequently derived ( [8], p.370) for arbitrary $n\geq 1$ and is also an instance of ours. We also derived another formula, where the Kampé de Fériet function is further expressed as a terminating ${}_4F_3$ hypergeometric series which is Saalschützian, taken at unit argument and having integer parameters. These functions are known to satisfy parameter transformations ( [6]) and we hope to exploit them in a future project for further simplification of the formula obtained in Corollary 6.2. It would be also interesting to derive a suitable integral representation for it: doing so would lead to a sharp estimate of the norm of the Berezin transform extending the one corresponding to weighted Bergman spaces [4, 16].

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## Перетворення Березіна приєднані до рівнів Ландау в комплексному проєктивному просторі $\mathbf{P}^n(\mathbb{C})$

Nizar Demni, Zouhaïr Mouayn, and Houda Yaqine

Ми будуємо когерентні стани для кожного власного простору магнітного лапласіана в комплексному проєктивному $n$-просторі для того, щоб застосувати метод квантизації-деквантизації. Це дозволяє визначити перетворення Березіна для цих просторів. Потім ми встановлюємо формулу для цього перетворення як функцію від лапласіана Фубіні–Штуді в замкненій формі, яка містить кінцеву функцію Кампе де Феріє. Для найнижчого сферичного рівня Ландау на рімановій сфері одержана формула зводиться до формули одержаної самим Березіним.

*Ключові слова:* комплексний проєктивний простір, когерентні стани, перетворення Березіна, магнітний лапласіан, лапласіан Фубіні–Штуді, формула Корнвіндера, співвідношення Клебша–Ґордана, функція Кампе де Феріє