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Membership Deformation of Commutativity and Obscure n-ary Algebras

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A general mechanism for "breaking" commutativity in algebras is proposed: if the underlying set is taken to be not a crisp set, but rather an obscure/fuzzy set, the membership function, reflecting the degree of truth that an element belongs to the set, can be incorporated into the commutation relations. The special "deformations" of commutativity and ε-commutativity are introduced in such a way that equal degrees of truth result in the "non-deformed" case. We also sketch how to "deform" ε-Lie algebras and Weyl algebras. Further, the above constructions are extended to n-ary algebras for which the projective representations and ε-commutativity are studied.

Key words: almost commutative algebra, obscure algebra, membership deformation, fuzzy set, membership function, n-ary algebra, Lie algebra, projective representation

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1. Introduction

Noncommutativity is the main mathematical idea of modern physics which, on the one hand, is grounded in quantum mechanics, in which generators of the algebra of observables do not commute, and, on the other hand, in supersymmetry and its variations based on the graded commutativity concept. Informally, "deformation" of commutativity in algebras is mostly a special way to place a "scalar" multiplier from the algebra field before the permuted product of two arbitrary elements. The general approach is based on the projective representation theory and realized using almost commutative (ε-commutative) graded algebras, where the role of the multipliers is played by bicharacters of the grading group (as suitable "scalar" objects taking values in the algebra field $\Bbbk$).

Here we propose a principally new mechanism for "deformation" of commutativity which comes from incorporating the ideas of vagueness in logic [22] to algebra. First, take the underlying set of the algebra not as a crisp set, but as an obscure/fuzzy set [23]. Second, consider an algebra (over $\Bbbk = \mathbb{C}$), such that each element can be endowed with a membership function (representing the degree of truth), a scalar function that takes values in the unit interval and describes the containment of a given element in the underlying set [5]. Third, introduce a special "membership deformation" of the commutation relations, so to speak the difference of the degree of truth, which determines a "measure of noncommutativity", whereby the elements having equal membership functions commute. Such procedure can be also interpreted as the "continuous noncommutativity", because the membership function is usually continuous. Likewise we "deform" ε-commutativity relations and ε-Lie algebras [20, 21]. Then we universalize and apply the above constructions to n-ary algebras [17] for which we also study projective representations generalizing the binary ones [27].

2. Preliminaries

First recall the main features of the standard gradation concept and of generalized (almost) commutativity (or ε-commutativity) [20, 21].

Let $\Bbbk$ be a unital field (with unit $1 \in \Bbbk$ and zero $0 \in \Bbbk$) and $\mathcal{A} = \langle A \mid \cdot, + \rangle$ be an associative algebra over $\Bbbk$ having zero $z \in A$ and unit $e \in A$ (for unital algebras). A *graded algebra* $\mathcal{A}_{\mathcal{G}}$ (G-graded $\Bbbk$-algebra) is a direct sum of subalgebras $\mathcal{A}_{\mathcal{G}} = \bigoplus_{g \in G} \mathcal{A}_g$, where $\mathcal{G} = \langle G \mid +' \rangle$ is a *grading group* (an abelian (finite) group with "unit" $\mathtt{0} \in G$) and the set multiplication is ("respecting gradation")

$$A_g \cdot A_h \subseteq A_{g+'h}, \quad g, h \in G. \tag{2.1}$$

The elements of subsets $a = a_{(g)} \in A_g$ with "full membership"are G -*homogeneous of degree* g

$$g \equiv \deg\left(a_{(g)}\right) = \deg\left(a\right) = a'_{(g)} \equiv a' \in G, \quad a = a_{(g)} \in A_g. \tag{2.2}$$

The graded algebra $\mathcal{A}_{\mathcal{G}}$ is called a *cross product* if in each subalgebra $\mathcal{A}_g$ there exists at least one invertible element. If all nonzero homogeneous elements are invertible, then $\mathcal{A}_{\mathcal{G}}$ is called a *graded division algebra* [8]. The morphisms $\varphi : \mathcal{A}_{\mathcal{G}} \to \mathcal{B}_{\mathcal{G}}$ acting on homogeneous elements from A_i should be compatible with the grading $\varphi\left(A_g\right) \subset B_g, g \in G$, while $\ker \varphi$ is a homogeneous ideal. The category of binary G-graded algebras G-$\mathtt{Alg}$ consists of the corresponding class of algebras and the homogeneous morphisms (see, e.g., [4, 8]).

In binary graded algebras there exists a way to generalize noncommutativity such that it can be dependent on the gradings ("coloring"). Indeed, some (two-place) function on grading degrees (bicharacter), a (*binary*) *commutation factor* $\varepsilon^{(2)} : G \times G \to \Bbbk^{\times}$ (where $\Bbbk^{\times} = \Bbbk \setminus 0$) can be introduced [20, 21] as

$$a \cdot b = \varepsilon^{(2)}\left(a', b'\right) b \cdot a, \quad a, b \in A,\ a', b' \in G. \tag{2.3}$$

The properties of the commutation factor $\varepsilon^{(2)}$ under double permutation and associativity

$$\varepsilon^{(2)}\left(a', b'\right) \varepsilon^{(2)}\left(b', a'\right) = 1, \tag{2.4}$$

$$\varepsilon^{(2)}\left(a', b' + c'\right) = \varepsilon^{(2)}\left(a', b'\right) \varepsilon^{(2)}\left(a', c'\right), \tag{2.5}$$

$$\varepsilon^{(2)}\left(a' + b', c'\right) = \varepsilon^{(2)}\left(a', c'\right) \varepsilon^{(2)}\left(b', c'\right), \qquad a', b', c' \in G, \tag{2.6}$$

make $\varepsilon^{(2)}$ a special 2-cocycle on the group $\mathcal{G}$ [4]. The conditions (2.4)–(2.6) imply that $\varepsilon^{(2)}(a',b') \neq 0$, $\left(\varepsilon^{(2)}(a',a')\right)^2 = 1$, $\varepsilon^{(2)}(a',\mathtt{0}) = \varepsilon^{(2)}(\mathtt{0},a') = 1$, and $\mathtt{0} \in G$. A graded algebra $\mathcal{A}_{\mathcal{G}}$ endowed with the commutation factor $\varepsilon^{(2)}$ satisfying (2.3)–(2.6) is called an *almost commutative* ($\varepsilon^{(2)}$*-commutative*, *color*) *algebra* [20, 21] (for a review, see [10]).

The simplest example of a commutation factor is a *sign rule*

$$\varepsilon^{(2)}\left(a',b'\right) = (-1)^{\langle a',b'\rangle}, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle : G \times G \to \mathbb{Z}_2$ is a bilinear form ("scalar product"), and for $G = \mathbb{Z}_2$ the form is a product, i.e. $\langle a',b'\rangle = a'b' \equiv gh \in \mathbb{Z}_2$. This gives the standard supercommutative algebra [3, 15, 19].

In the case $G = \mathbb{Z}_2^n$, the "scalar product" $\langle\cdot,\cdot\rangle : G \times G \to \mathbb{Z}$ is defined by (see (2.2))

$$\left\langle \left(a'_1 \ldots a'_n\right), \left(b'_1 \ldots b'_n\right) \right\rangle = a'_1 b'_1 + \ldots + a'_n b'_n \equiv g_1 h_1 + \ldots + g_n h_n \in \mathbb{Z}, \tag{2.8}$$

which leads to $\mathbb{Z}_2^n$-commutative associative algebras [7].

A classification of the commutation factors $\varepsilon^{(2)}$ can be made in terms of the *factors*, binary *Schur multipliers*, $\pi^{(2)} : G \times G \to \Bbbk^\times$ such that [21, 26]

$$\varepsilon_\pi^{(2)}\left(a',b'\right) = \frac{\pi^{(2)}\left(a',b'\right)}{\pi^{(2)}\left(b',a'\right)}, \quad a',b' \in G. \tag{2.9}$$

The (Schur) factors $\pi^{(2)}$ naturally appear in the projective representation theory [27] and satisfy the relation

$$\pi^{(2)}\left(a',b'+c'\right)\pi^{(2)}\left(b',c'\right) = \pi^{(2)}\left(a',b'\right)\pi^{(2)}\left(a'+b',c'\right), \quad a',b' \in G, \tag{2.10}$$

which follows from associativity. Using (2.9), we can rewrite the $\varepsilon^{(2)}$-commutation relation (2.3) of the graded algebra $\mathcal{A}_{\mathcal{G}}$ in the form

$$\pi^{(2)}\left(b',a'\right) a \cdot b = \pi^{(2)}\left(a',b'\right) b \cdot a, \quad a,b \in A, \; a',b' \in G. \tag{2.11}$$

We call (2.11) a π*-commutativity*. A factor set $\left\{\pi_{\mathrm{sym}}^{(2)}\right\}$ is *symmetric* if

$$\pi_{\mathrm{sym}}^{(2)}\left(b',a'\right) = \pi_{\mathrm{sym}}^{(2)}\left(a',b'\right), \tag{2.12}$$

$$\varepsilon_{\pi_{\mathrm{sym}}}^{(2)}\left(a',b'\right) = 1, \quad a',b' \in G, \tag{2.13}$$

and thus the graded algebra $\mathcal{A}_{\mathcal{G}}$ becomes commutative.

For further details, see Section 5 and [20, 21].

3. Membership function and obscure algebras

Now let us consider a generalization of associative algebras and graded algebras to the case where the degree of truth (containment of an element in the underlying set) is not full or distinct (for a review, see, e.g., [23, 25]). In this case, an element $a \in A$ of the algebra $\mathcal{A} = \langle A \mid \cdot, + \rangle$, over $\Bbbk$, can be endowed with an additional function, the *membership function* $\mu : A \to [0,1]$, $0, 1 \in \Bbbk$, which "measures" the degree of truth as a "grade of membership" [5]. If A is a *crisp set*, then $\mu \in \{0,1\}$ and $\mu^{\mathrm{crisp}}(a) = 1$ if $a \in A$ and $\mu^{\mathrm{crisp}}(a) = 0$ if $a \notin A$. Also, we assume that the zero has full membership $\mu(z) = 1$, $z \in A$ (for details, see, e.g., [23, 25]). If the membership function μ is positive, we can identify an *obscure* (*fuzzy*) *set* $\mathfrak{A}^{(\mu)}$ with support the universal set A consisting of pairs

$$\mathfrak{A}^{(\mu)} = \{(a|\mu(a))\}, \quad a \in A, \mu(a) > 0. \tag{3.1}$$

Sometimes, instead of the operations with obscure sets it is convenient to consider the corresponding operations only in terms of the membership function μ itself. Denote $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$. Then, for inclusion $\subseteq$, union $\cup$, intersection $\cap$ and negation of obscure sets, one can write $\mu(a) \leq \mu(b)$, $\mu(a) \vee \mu(b)$, $\mu(a) \wedge \mu(b)$, $1 - \mu(a)$, $a, b \in \mathfrak{A}^{(\mu)}$, respectively.

Definition 3.1. An *obscure algebra* $\mathcal{A}(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid \cdot, + \right\rangle$ is an algebra over $\Bbbk$ having an obscure set $\mathfrak{A}^{(\mu)}$ as its underlying set, where the following conditions hold:

$$\mu(a+b) \geq \mu(a) \wedge \mu(b), \tag{3.2}$$

$$\mu(a \cdot b) \geq \mu(a) \wedge \mu(b), \tag{3.3}$$

$$\mu(ka) \geq \mu(a), \quad a, b \in A, \ k \in \Bbbk. \tag{3.4}$$

Definition 3.2. An *obscure unity* η is given by $\eta(a) = 1$ if $a = 0$, and $\eta(a) = 0$ if $a \neq 0$.

For two obscure sets their direct sum $\mathfrak{A}^{(\mu)} = \mathfrak{A}^{(\mu_g)} \oplus \mathfrak{A}^{(\mu_h)}$ can be defined if $\mu_g \wedge \mu_h = \eta$ and the joint membership function has two "nonintersecting" components

$$\mu(a) = \mu_g\left(a_{(g)}\right) \vee \mu_h\left(a_{(h)}\right), \quad \mu_g \wedge \mu_h = \eta, \tag{3.5}$$

$$a = a_{(g)} + a_{(h)}, \quad a \in \mathfrak{A}^{(\mu)}, \ a_{(g)} \in \mathfrak{A}^{(\mu_g)}, \ a_{(h)} \in \mathfrak{A}^{(\mu_h)}, \ g, h \in G, \tag{3.6}$$

satisfying

$$\mu\left(a_{(g)} \cdot a_{(h)}\right) \geq \mu_g\left(a_{(g)}\right) \wedge \mu_h\left(a_{(h)}\right). \tag{3.7}$$

Definition 3.3. An *obscure G-graded algebra* is a direct sum decomposition

$$\mathcal{A}_{\mathcal{G}}(\mu) = \bigoplus\nolimits_{g \in G} \mathcal{A}(\mu_g) \tag{3.8}$$

such that the relation (2.1) holds and the joint membership function is

$$\mu(a) = \vee_{g \in G} \mu_g\left(a_{(g)}\right), \quad a = \sum_{g \in G} a_{(g)}, \qquad a \in \mathfrak{A}^{(\mu)}, \ a_{(g)} \in \mathfrak{A}^{(\mu_g)}, \ g \in G. \tag{3.9}$$

4. Membership deformation of commutativity

The membership concept leads to the question whether it is possible to generalize commutativity and $\varepsilon^{(2)}$-commutativity (2.3) for the obscure algebras. A positive answer to this question can be based on a consistent usage of the membership μ as ordinary functions which are pre-defined for each element and satisfy some conditions (see, e.g., [23, 25]). In this case, the commutation factor (and the Schur factors) may depend not only on the element grading, but also on the element membership function μ, and therefore it becomes "individual" for each pair of elements, and moreover it can be continuous. We call this procedure a *membership deformation* of commutativity.

4.1. Deformation of commutative algebras Let us consider an obscure commutative algebra $\mathcal{A}(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid \cdot, + \right\rangle$ (see Definition 3.1), in which $a \cdot b = b \cdot a,\ a, b \in \mathfrak{A}^{(\mu)}$. Now we "deform" this commutativity by the membership function μ and introduce a new algebra product $(*)$ for the elements in A to get a noncommutative algebra.

Definition 4.1. An *obscure membership deformed algebra* is $\mathcal{A}_*(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid *, + \right\rangle$ in which the noncommutativity relation is

$$\mu(b)\, a * b = \mu(a)\, b * a, \quad a, b \in \mathfrak{A}^{(\mu)}. \tag{4.1}$$

Remark 4.2. The relation (4.1) is a reminiscent of (2.11), where the role of the Schur factors is played by the membership function μ which depends on the element itself, but not on the element grading. Therefore, the membership deformation of commutativity (4.1) is highly nonlinear as opposed to the gradation.

As in our consideration the membership function μ cannot be zero, it follows from (4.1) that

$$a * b = \epsilon_\mu^{(2)}(a, b)\, b * a, \tag{4.2}$$

$$\epsilon_\mu^{(2)}(a, b) = \frac{\mu(a)}{\mu(b)}, \quad a, b \in \mathfrak{A}^{(\mu)}, \tag{4.3}$$

where $\epsilon_\mu^{(2)}$ is the *membership commutation factor*.

Corollary 4.3. *An obscure membership deformed algebra $\mathcal{A}_*(\mu)$ is a (kind of) ϵ_μ-commutative algebra with a membership commutation factor* (4.3) *which now depends not on the element gradings as in* (2.9)*, but on the membership function μ.*

Remark 4.4. As can be seen from (4.3), the noncommutativity "measures" the difference between the element degree of truth and $\mathfrak{A}^{(\mu)}$. So, if two elements have the same membership function, they commute.

Proposition 4.5. *In $\mathcal{A}_*(\mu)$, the membership function satisfies the equality* (*cf.* (3.4))

$$\mu(ka) = \mu(a), \quad a \in \mathfrak{A}^{(\mu)},\ k \in \Bbbk. \tag{4.4}$$

Proof. From the distributivity of the scalar multiplication $k(a*b) = ka*b = a*kb$ and the membership noncommutativity (4.1), we get

$$\mu(b)\, ka*b = \mu(ka)\, b*ka, \tag{4.5}$$

$$\mu(kb)\, a*kb = \mu(a)\, kb*a. \tag{4.6}$$

So, $\mu(ka)\,\mu(kb) = \mu(a)\,\mu(b)$, $a \in \mathfrak{A}^{(\mu)}$, $\quad k \in \Bbbk$, and thus (4.4) follows. □

In general, the algebra $\mathcal{A}_*(\mu)$ is not associative without further conditions (for instance, similar to (2.10)) on the membership function μ which is assumed predefined and satisfies the properties (3.2)–(3.3) and (4.4) only.

Proposition 4.6. *The obscure membership deformed algebra $\mathcal{A}_*(\mu)$ cannot be associative with any additional conditions on the membership function μ.*

Proof. Since the ε-commutativity (2.3) and the ϵ-commutativity (4.2) have the same form, the derivations of associativity coincide and give a cocycle relation similar to (2.5)–(2.6) also for $\epsilon_\mu^{(2)}$, e.g.,

$$\epsilon_\mu^{(2)}(a, b*c) = \epsilon_\mu^{(2)}(a, b)\,\epsilon_\mu^{(2)}(a, c). \tag{4.7}$$

This becomes $\mu(a) = \mu(b)\,\mu(c) \diagup \mu(b*c)$ in terms of the membership function (4.3), but it is impossible for all $a, b, c \in \mathfrak{A}^{(\mu)}$. On the other side, after double commutation in $(a*b)*c \to c*(b*a)$ and $a*(b*c) \to (c*b)*a$ we obtain (if the associativity of $(*)$ is implied)

$$\epsilon_\mu^{(2)}(a, b*c)\,\epsilon_\mu^{(2)}(b, c) = \epsilon_\mu^{(2)}(a*b, c)\,\epsilon_\mu^{(2)}(a, b), \tag{4.8}$$

which in terms of the membership function becomes $\mu(b)^2 = \mu(a*b)\,\mu(b*c)$, and this is also impossible for arbitrary $a, b, c \in \mathfrak{A}^{(\mu)}$. □

Nevertheless, distributivity of the algebra multiplication and algebra addition in $\mathcal{A}_*(\mu)$ is possible, but can only be one-sided.

Proposition 4.7. *The algebra $\mathcal{A}_*(\mu)$ is right-distributive, but has the membership deformed left distributivity*

$$\mu(b+c)\, a*(b+c) = \mu(b)\, a*b + \mu(c)\, a*c, \tag{4.9}$$

$$(b+c)*a = b*a + c*a, \quad a, b, c \in \mathfrak{A}^{(\mu)}. \tag{4.10}$$

Proof. Applying the membership noncommutativity (4.1) to (4.9), we obtain $\mu(a)\,(b+c)*a = \mu(a)\, b*a + \mu(a)\, c*a$, and then (4.10), because $\mu > 0$. □

Theorem 4.8. *The binary obscure algebra $\mathcal{A}_*(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid *, + \right\rangle$ is necessarily nonassociative and $\epsilon_\mu^{(2)}$-commutative (4.2), right-distributive (4.10) and membership deformed left distributive (4.9).*

Remark 4.9. If membership noncommutativity is valid for generators of the algebra only, then the form of (4.2) coincides with that of the quantum polynomial algebra [1], but the latter is two-sided distributive, in distinction to the obscure algebra $\mathcal{A}_* (\mu)$.

Example 4.10 (Deformed Weyl algebra). Consider an obscure algebra

$$\mathcal{A}_{\odot}^{\mathrm{Weyl}} (\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid \odot, + \right\rangle$$

generated by two generators $x, y \in \mathfrak{A}^{(\mu)}$ satisfying the Weyl-like relation (cf. (4.1))

$$\mu (y) \, x \odot y = \mu (x) \, y \odot x + \mu (x \odot y) \, e, \quad x, y, e \in \mathfrak{A}^{(\mu)}. \tag{4.11}$$

We call $\mathcal{A}_{\odot}^{\mathrm{Weyl}} (\mu)$ a *membership deformed Weyl algebra*. Because the membership function is predefined and $\mu (x \odot y)$ is not a symplectic form, the algebra $\mathcal{A}_{\odot}^{\mathrm{Weyl}} (\mu)$ is not isomorphic to the ordinary Weyl algebra (see, e.g., [16]). In the same way, the graded Weyl algebra [24] can be membership deformed in a similar way.

Remark 4.11. The special property of the membership noncommutativity is the fact that each pair of elements has their own "individual" commutation factor ϵ which depends on the membership function (4.3) that can also be continuous.

4.2. Deformation of ε-commutative algebras. Here we apply the membership deformation procedure (4.1) to the obscure G-graded algebras (3.8) which are $\varepsilon^{(2)}$-commutative (2.3). We now "deform" (2.11) by analogy with (4.1).

Let $\mathcal{A}_{\mathcal{G}} (\mu)$ be a binary obscure G-graded algebra (3.8) which is $\varepsilon^{(2)}$-commutative with the Schur factor $\pi^{(2)}$ (2.9).

Definition 4.12. An *obscure membership deformed $\varepsilon^{(2)}$-commutative G-graded algebra* is $\mathcal{A}_{\mathcal{G}\star} (\mu) = \langle \mathfrak{A}^{(\mu)} \mid \star, + \rangle$ in which the noncommutativity relation is given by

$$\mu (b) \, \pi^{(2)} \left(b', a'\right) a \star b = \mu (a) \, \pi^{(2)} \left(a', b'\right) b \star a, \quad a, b \in \mathfrak{A}^{(\mu)}, \;\; a', b' \in G. \tag{4.12}$$

Both functions $\pi^{(2)}$ and μ being nonvanishing, we can combine (2.9) and (4.3).

Definition 4.13. An algebra $\mathcal{A}_{\mathcal{G}\star} (\mu)$ is called a *double $\varepsilon_\pi^{(2)} / \epsilon_\mu^{(2)}$-commutative algebra* when

$$a \star b = \varepsilon_\pi^{(2)} \left(a', b'\right) \epsilon_\mu^{(2)} (a, b) \, b \star a, \quad a, b \in \mathfrak{A}^{(\mu)}, \;\; a', b' \in G \tag{4.13}$$

$$\epsilon_\mu^{(2)} (a, b) = \frac{\mu (a)}{\mu (b)}, \tag{4.14}$$

$$\varepsilon_\pi^{(2)} \left(a', b'\right) = \frac{\pi^{(2)} (a', b')}{\pi^{(2)} (b', a')}, \tag{4.15}$$

where $\varepsilon_\pi^{(2)}$ is the grading commutation factor and $\epsilon_\mu^{(2)}$ is the membership commutation factor.

In the *first version*, we assume that $\varepsilon_{\pi}^{(2)}$ is still a cocycle and satisfies (2.4)–(2.6). Then in $\mathcal{A}_{\mathcal{G}\star}(\mu)$ the relation (4.4) is satisfied as well, because the Schur factors cancel in the derivation from (4.5)–(4.6). For the same reason Assertion 4.6 holds, and therefore the double $\varepsilon_{\pi}^{(2)}/\epsilon_{\mu}^{(2)}$-commutative algebra $\mathcal{A}_{\mathcal{G}\star}(\mu)$ with the fixed grading commutation factor $\varepsilon_{\pi}^{(2)}$ is necessarily nonassociative.

As the *second version*, we consider the case when the grading commutation factor does not satisfy (2.4)–(2.6), but the double commutation factor does satisfy them (the membership function is fixed, being predefined for each element), which can lead to an associative algebra.

Proposition 4.14. *A double $\varepsilon_{\pi\mu}^{(2)}/\epsilon_{\mu}^{(2)}$-commutative algebra*

$$\mathcal{A}_{\mathcal{G}\circledast}(\mu)=\left\langle \mathfrak{A}^{(\mu)} \mid \circledast, +\right\rangle$$

with

$$a\circledast b=\varepsilon_{\pi\mu}^{(2)}\left(a',b'\right)\epsilon_{\mu}^{(2)}(a,b)\, b\circledast a,\quad \epsilon_{\mu}^{(2)}(a,b)=\frac{\mu(a)}{\mu(b)} \tag{4.16}$$

is associative if the noncocycle commutation factor $\varepsilon_{\pi\mu}^{(2)}$ satisfies the "membership deformed cocycle-like" conditions

$$\varepsilon_{\pi\mu}^{(2)}\left(a',b'\right)\varepsilon_{\pi\mu}^{(2)}\left(b',a'\right)=\frac{1}{\epsilon_{\mu}^{(2)}(a,b)\,\epsilon_{\mu}^{(2)}(b,a)}, \tag{4.17}$$

$$\varepsilon_{\pi\mu}^{(2)}\left(a',b'+c'\right)=\varepsilon_{\pi\mu}^{(2)}\left(a',b'\right)\varepsilon_{\pi\mu}^{(2)}\left(a',c'\right)\frac{\epsilon_{\mu}^{(2)}(a,b)\,\epsilon_{\mu}^{(2)}(a,c)}{\epsilon_{\mu}^{(2)}(a,b\circledast c)}, \tag{4.18}$$

$$\varepsilon_{\pi\mu}^{(2)}\left(a'+b',c'\right)=\varepsilon_{\pi\mu}^{(2)}\left(a',c'\right)\varepsilon_{\pi\mu}^{(2)}\left(b',c'\right)\frac{\epsilon_{\mu}^{(2)}(a,c)\,\epsilon_{\mu}^{(2)}(b,c)}{\epsilon_{\mu}^{(2)}(a\circledast b,c)}, \tag{4.19}$$

for all $a,b\in\mathfrak{A}^{(\mu)}$ and $a',b'\in G$.

Proof. Now indeed the double commutation factor (the product of the grading and membership factors) $\varepsilon_{\pi\mu}^{(2)}\epsilon_{\mu}^{(2)}$ satisfies (2.4)–(2.6). Then (4.17)–(4.19) immediately follow. □

We can find the deformed equation for the Schur-like factors (similar to (4.17)–(4.19))

$$\varepsilon_{\pi\mu}^{(2)}\left(a',b'\right)=\frac{\pi_{\mu}^{(2)}\left(a',b'\right)}{\pi_{\mu}^{(2)}\left(b',a'\right)}, \tag{4.20}$$

such that the following "membership deformed" commutation takes place (see (4.16)):

$$\pi_{\mu}^{(2)}\left(b',a'\right)\mu(b)\,a\circledast b=\pi_{\mu}^{(2)}\left(a',b'\right)\mu(a)\,b\circledast a,\quad a,b\in\mathfrak{A}^{(\mu)},\ a',b'\in G, \tag{4.21}$$

and the algebra $\mathcal{A}_{\mathcal{G}\circledast}(\mu)$ becomes associative in distinct with (4.12), where the grading commutation factor $\varepsilon_{\pi}^{(2)}$ satisfies (2.4)–(2.6) and the algebra multiplications $(\star)$ are different.

Thus the deformed binary Schur-like factors $\pi_\mu^{(2)}$ of the obscure membership deformed associative double commutative algebra $\mathcal{A}_{\mathcal{G}\circledast}(\mu)$ satisfy

$$\pi_\mu^{(2)}(a',b'+c')\,\pi_\mu^{(2)}(b',c') = \pi_\mu^{(2)}(a',b')\,\pi_\mu^{(2)}(a'+b',c')\,\frac{\mu(a\circledast b)}{\mu(b)}, \tag{4.22}$$

for all $a,b\in\mathfrak{A}^{(\mu)}$ and $a',b'\in G$, which should be compared with the corresponding nondeformed relation (2.10).

4.3. Double $\varepsilon\epsilon$-Lie algebras. Consider the second version of an obscure double $\varepsilon_{\pi\mu}^{(2)}/\epsilon_\mu^{(2)}$-commutative algebra $\mathcal{A}_{\mathcal{G}\circledast}(\mu)$ defined in (4.16)–(4.19) and construct a corresponding analog of the Lie algebra by following the same procedure as for associative ε-commutative algebras [18, 20, 21].

Take $\mathcal{A}_{\mathcal{G}\circledast}(\mu)$ and define a *double $\varepsilon\epsilon$-Lie bracket* $L_{\varepsilon\epsilon}:\mathfrak{A}^{(\mu)}\otimes\mathfrak{A}^{(\mu)}\to\mathfrak{A}^{(\mu)}$ by

$$L_{\varepsilon\epsilon}[a,b] = a\circledast b - \varepsilon_{\pi\mu}^{(2)}(a',b')\,\epsilon_\mu^{(2)}(a,b)\,b\circledast a,\quad a,b\in\mathfrak{A}^{(\mu)},\ a',b'\in G, \tag{4.23}$$

where $\varepsilon_{\pi\mu}^{(2)}$ and $\epsilon_\mu^{(2)}$ are given in (4.16) and (4.20), respectively.

Proposition 4.15. *The double $\varepsilon\epsilon$-Lie bracket is $\varepsilon\epsilon$-skew commutative, i.e. it satisfies double commutativity with the commutation factor* $\left(-\varepsilon_{\pi\mu}^{(2)}\epsilon_\mu^{(2)}\right)$.

Proof. Multiply (4.23) by $\varepsilon_{\pi\mu}^{(2)}(b',a')\,\epsilon_\mu^{(2)}(b,a)$ and use (4.17) to obtain

$$\varepsilon_{\pi\mu}^{(2)}(b',a')\,\epsilon_\mu^{(2)}(b,a)\,L_{\varepsilon\epsilon}[a,b] = \varepsilon_{\pi\mu}^{(2)}(b',a')\,\epsilon_\mu^{(2)}(b,a)\,a\circledast b - b\circledast a = -L_{\varepsilon\epsilon}[b,a].$$

Therefore,

$$L_{\varepsilon\epsilon}[a,b] = -\varepsilon_{\pi\mu}^{(2)}(a',b')\,\epsilon_\mu^{(2)}(a,b)\,L_{\varepsilon\epsilon}[b,a], \tag{4.24}$$

which should be compared with (4.16). □

Proposition 4.16. *The double $\varepsilon\epsilon$-Lie bracket satisfies the membership deformed $\varepsilon\epsilon$-Jacobi identity*

$$\varepsilon_{\pi\mu}^{(2)}(a',b')\,\epsilon_\mu^{(2)}(a,b)\,L_{\varepsilon\epsilon}[a,L_{\varepsilon\epsilon}[b,c]] + \textit{cyclic permutations} = 0,$$
$$a,b,c\in\mathfrak{A}^{(\mu)},\quad a',b',c'\in G. \tag{4.25}$$

Definition 4.17. A *double $\varepsilon\epsilon$-Lie algebra* is an obscure G-graded algebra with the double $\varepsilon\epsilon$-Lie bracket (satisfying the $\varepsilon\epsilon$-skew commutativity (4.24) and the membership deformed Jacobi identity (4.25)) as a multiplication, i.e. $\mathcal{A}_{\mathcal{G}L}(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid L_{\varepsilon\epsilon}, +\right\rangle$.

5. Projective representations

To generalize the ε-commutative algebras to the n-ary case, we need to introduce n-ary projective representations and study them in brief.

5.1. Binary projective representations. First, recall some general properties of the Schur factors and corresponding commutation factors in the projective representation theory of Abelian groups [26,27] (see also [14] and the unitary ray representations in [2]). We show some known details in our notation which can be useful in further extensions of the well-known binary constructions to the n-ary case.

Let $\mathcal{H}^{(2)} = \langle H \mid \dot{+} \rangle$ be a binary Abelian group and $f : \mathcal{H}^{(2)} \to \mathcal{E}^{(2)}$, where $\mathcal{E}^{(2)} = \langle \operatorname{End} V \mid \circ \rangle$, and V is a vector space over a field $\Bbbk$. A map f is a (binary) *projective representation* (*σ-representation* [26]) if $f(x_1) \circ f(x_2) = \pi_0^{(2)}(x_1, x_2) f(x_1 \dot{+} x_2)$, $x_1, x_2 \in H$, and $\pi_0^{(2)} : \mathcal{H}^{(2)} \times \mathcal{H}^{(2)} \to \Bbbk^\times$ is a (Schur) factor, while $(\circ)$ is a (noncommutative binary) product in $\operatorname{End} V$). The "associativity relation" of factors follows immediately from the associativity of $(\circ)$ such that (cf. (2.10))

$$\pi_0^{(2)}(x_1, x_2 \dot{+} x_3)\, \pi_0^{(2)}(x_2, x_3) = \pi_0^{(2)}(x_1, x_2)\, \pi_0^{(2)}(x_1 \dot{+} x_2, x_3), \qquad x_1, x_2, x_3 \in H. \tag{5.1}$$

Two factor systems are equivalent $\left\{\pi_0^{(2)}\right\} \overset{\lambda}{\sim} \left\{\tilde{\pi}_0^{(2)}\right\}$ (or associated [27]) if there exists $\lambda : \mathcal{H}^{(2)} \to \Bbbk^\times$ such that

$$\tilde{\pi}_0^{(2)}(x_1, x_2) = \frac{\lambda(x_1)\,\lambda(x_2)}{\lambda(x_1 \dot{+} x_2)} \pi_0^{(2)}(x_1, x_2), \quad x_1, x_2 \in H, \tag{5.2}$$

and the $\tilde{\pi}_0^{(2)}$-representation is given by $\tilde{f}(x) = \lambda(x) f(x)$, $x \in H$. The cocycle condition (5.1) means that $\pi_0^{(2)}$ belongs to the group $Z^2\left(\mathcal{H}^{(2)}, \Bbbk^\times\right)$ of 2-cocycles of $\mathcal{H}^{(2)}$ over $\Bbbk^\times$, the quotient $\left\{\pi_0^{(2)}\right\} / \overset{\lambda}{\sim}$ gives the corresponding multiplier group, if $\Bbbk = \mathbb{C}$, and, in the general case, coincides with the exponents of the 2-cohomology classes $H^2\left(\mathcal{H}^{(2)}, \Bbbk\right)$ (for details, see, e.g., [10,26,27]).

The group $\mathcal{H}^{(2)}$ is Abelian, and therefore we have (cf. (2.11))

$$\begin{aligned} \pi_0^{(2)}(x_2, x_1) f(x_1) \circ f(x_2) &= \pi_0^{(2)}(x_2, x_1)\, \pi_0^{(2)}(x_1, x_2) f(x_1 \dot{+} x_2) \\ &= \pi_0^{(2)}(x_1, x_2) f(x_2) \circ f(x_1), \end{aligned} \tag{5.3}$$

which allows us to introduce a (binary) *commutation factor* $\varepsilon_{\pi_0}^{(2)} : \mathcal{H}^{(2)} \times \mathcal{H}^{(2)} \to \Bbbk^\times$ by (see (2.3), (2.9))

$$\varepsilon_{\pi_0}^{(2)}(x_1, x_2) = \frac{\pi_0^{(2)}(x_1, x_2)}{\pi_0^{(2)}(x_2, x_1)}, \tag{5.4}$$

$$f(x_1) \circ f(x_2) = \varepsilon_{\pi_0}^{(2)}(x_1, x_2) f(x_2) \circ f(x_1), \quad x_1, x_2 \in H, \tag{5.5}$$

with the obvious "normalization" (cf. (2.4))

$$\varepsilon_{\pi_0}^{(2)}(x_1, x_2)\, \varepsilon_{\pi_0}^{(2)}(x_2, x_1) = 1, \quad \varepsilon_{\pi_0}^{(2)}(x, x) = 1, \quad x_1, x_2, x \in H, \tag{5.6}$$

which will be important in the n-ary case below.

The multiplication of the commutation factors follows from different permutations of the three terms:

$$\begin{aligned} f(y)\circ f(x_1)\circ f(x_2) &= \varepsilon^{(2)}_{\pi_0}(y,x_1)\, f(x_1)\circ f(y)\circ f(x_2) \\ &= \varepsilon^{(2)}_{\pi_0}(y,x_1)\,\varepsilon^{(2)}_{\pi_0}(y,x_2)\, f(x_1)\circ f(x_2)\circ f(y) \\ &= \pi^{(2)}_0(x_1,x_2)\, f(y)\circ f(x_1 \dotplus x_2) \\ &= \varepsilon^{(2)}_{\pi_0}(y,x_1 \dotplus x_2)\left(\pi^{(2)}_0(x_1,x_2)\, f(x_1 \dotplus x_2)\right)\circ f(y) \\ &= \varepsilon^{(2)}_{\pi_0}(y,x_1 \dotplus x_2)\, f(x_1)\circ f(x_2)\circ f(y), \quad x_1,x_2,y\in H. \end{aligned} \tag{5.7}$$

Thus, it follows that the commutation factor multiplication is (and similarly for the second place, cf. (2.5)–(2.6))

$$\varepsilon^{(2)}_{\pi_0}(y,x_1 \dotplus x_2) = \varepsilon^{(2)}_{\pi_0}(y,x_1)\,\varepsilon^{(2)}_{\pi_0}(y,x_2), \quad x_1,x_2,y\in H, \tag{5.8}$$

which means that $\varepsilon^{(2)}_{\pi_0}$ is a (binary) bicharacter on $\mathcal{H}^{(2)}$, because for $\chi^{(2)}_y(x) \equiv \varepsilon^{(2)}_{\pi_0}(y,x)$ we have

$$\chi^{(2)}_y(x_1)\,\chi^{(2)}_y(x_2) = \chi^{(2)}_y(x_1 \dotplus x_2), \quad x_1,x_2\in H. \tag{5.9}$$

Denote the group of bicharacters $\chi^{(2)}_y$ on $\mathcal{H}^{(2)}$ with the multiplication (5.9) by $\mathcal{B}^{(2)}(\mathcal{H}^{(2)},\Bbbk)$. We observe that the mapping $\pi^{(2)}_0 \to \varepsilon^{(2)}_{\pi_0}$ is a homomorphism of $Z^2\left(\mathcal{H}^{(2)},\Bbbk^\times\right)$ to $\mathcal{B}^{(2)}(\mathcal{H}^{(2)},\Bbbk)$ whose kernel is a subgroup of the 2-coboundaries of $\mathcal{H}^{(2)}$ over $\Bbbk^\times$ (for more details, see [27]).

5.2. n-ary projective representations. Here we consider some features of n-ary projective representations and corresponding particular generalizations of binary ε-commutativity.

Let $\mathcal{H}^{(n)} = \left\langle H \mid [\ ]^{(n)}_+ \right\rangle$ be an n-ary Abelian group with the totally commutative multiplication $[\ ]^{(n)}_+$, and the mapping $f:\mathcal{H}^{(n)} \to \mathcal{E}^{(n)}$, where $\mathcal{E}^{(n)} = \left\langle \operatorname{End} V \mid [\ ]^{(n)}_\circ \right\rangle$ (for general polyadic representations, see [12] and references therein). Here we suppose that V is a vector space over a field $\Bbbk$, and $[\ ]^{(n)}_\circ$ is an n-ary associative product in $\operatorname{End} V$, which means that

$$\begin{aligned} &f\left[[f(x_1),\dots,f(x_n)]^{(n)}_\circ, f(x_{n+1}),\dots,f(x_{2n-1})\right]^{(n)}_\circ \\ &\quad = f\left[f(x_1),[f(x_2),\dots,f(x_{n+1})]^{(n)}_\circ, f(x_{n+2}),\dots,f(x_{2n-1})\right]^{(n)}_\circ \\ &\quad \dots \\ &\quad = f\left[f(x_1),f(x_2),\dots,f(x_{n-1}),[f(x_n),\dots,f(x_{2n-1})]^{(n)}_\circ\right]^{(n)}_\circ. \end{aligned} \tag{5.10}$$

Definition 5.1. A map f is an *n-ary projective representation* if

$$[f(x_1),\dots,f(x_n)]^{(n)}_\circ = \pi^{(n)}_0(x_1,\dots,x_n)\, f\left([x_1,\dots,x_n]^{(n)}_+\right), \; x_1,\dots,x_n\in H, \tag{5.11}$$

and $\pi_0^{(n)}:\overbrace{\mathcal{H}^{(n)}\times\ldots\times\mathcal{H}^{(n)}}^{n}\to\Bbbk^{\times}$ is an n-ary *(Schur-like) factor*.

Proposition 5.2. *The factors $\pi_0^{(n)}$ satisfy the n-ary 2-cocycle conditions (cf. (5.1))*

$$\begin{aligned}\pi_0^{(n)}(x_1,\ldots,x_n)\,\pi_0^{(n)}\left([x_1,\ldots,x_n]_+^{(n)},x_{n+1},\ldots,x_{2n-1}\right)\\ =\pi_0^{(n)}(x_2,\ldots,x_{n+1})\,\pi_0^{(n)}\left(x_1,[x_2,\ldots,x_{n+1}]_+^{(n)},x_{n+2},\ldots,x_{2n-1}\right)\\ \ldots\\ =\pi_0^{(n)}(x_n,\ldots,x_{n+1})\,\pi_0^{(n)}\left(x_1,\ldots,x_{n-1},[x_n,\ldots,x_{2n-1}]_+^{(n)}\right),\\ x_1,\ldots,x_{2n-1}\in H.\quad(5.12)\end{aligned}$$

Proof. These immediately follow from the n-ary associativity in $\operatorname{End}V$ (5.10) and (5.11). □

Two n-ary factor systems are equivalent $\left\{\pi_0^{(n)}\right\}\overset{\lambda}{\sim}\left\{\tilde{\pi}_0^{(n)}\right\}$ if there exists $\lambda:\mathcal{H}^{(n)}\to\Bbbk^{\times}$ such that

$$\tilde{\pi}_0^{(n)}(x_1,\ldots,x_n)=\frac{\lambda(x_1)\,\lambda(x_2)\ldots\lambda(x_n)}{\lambda\left([x_1,\ldots,x_n]_+^{(n)}\right)}\pi_0^{(n)}(x_1,\ldots,x_n),\quad x_1,x_2\in H,\quad(5.13)$$

and the $\tilde{\pi}_0^{(n)}$-representation is given by $\tilde{f}(x)=\lambda(x)\,f(x)$, $x\in H$.

To understand how properly and uniquely to introduce the commutation factors for n-ary projective representations, we need to consider an n-ary analog of (5.3) (see also (2.11)).

Proposition 5.3. *The commutativity of a $\pi_0^{(n)}$-representation is given by $(n!-1)$ relations of the form*

$$\begin{aligned}\pi_0^{(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)[f(x_1),\ldots,f(x_n)]_\circ^{(n)}\\ =\pi_0^{(n)}(x_1,\ldots,x_n)\left[f\left(x_{\sigma(1)}\right),\ldots,f\left(x_{\sigma(n)}\right)\right]_\circ^{(n)},\quad(5.14)\end{aligned}$$

where $\sigma\in S_n$, $\sigma\neq I$, S_n is the symmetry permutation group on n elements.

Proof. Using the definition of the n-ary projective representation (5.11), we obtain for the left-hand side and right-hand side of (5.14)

$$\pi_0^{(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)\pi_0^{(n)}(x_1,\ldots,x_n)\,f\left([x_1,\ldots,x_n]_+^{(n)}\right)\quad(5.15)$$

and

$$\pi_0^{(n)}(x_1,\ldots,x_n)\,\pi_0^{(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)f\left(\left[x_{\sigma(1)},\ldots,x_{\sigma(n)}\right]_+^{(n)}\right),\quad(5.16)$$

respectively.

Since $\mathcal{H}^{(n)}$ is totally commutative

$$[x_1,\ldots,x_n]_+^{(n)} = \left[x_{\sigma(1)},\ldots,x_{\sigma(n)}\right]_+^{(n)}, \quad x_1,\ldots,x_n \in H, \sigma \in S_n, \tag{5.17}$$

then $f\left([x_1,\ldots,x_n]_+^{(n)}\right) = f\left(\left[x_{\sigma(1)},\ldots,x_{\sigma(n)}\right]_+^{(n)}\right)$. Taking into account all non-identical permutations, we get $(n!-1)$ relations in (5.14). □

Corollary 5.4. *To describe the noncommutativity of an n-ary projective representation, we need to have not one relation between Schur factors (as in the binary case* (5.3) *and* (2.11)*), but $(n!-1)$ relations* (5.14)*. This leads to the concept of the set of $(n!-1)$ commutation factors.*

Definition 5.5. The commutativity of the n-ary projective representation with the Schur-like factor $\pi_0^{(n)}(x_1,\ldots,x_n)$ is governed by the *set of* $(n!-1)$ *commutation factors*

$$\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)} \equiv \varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}(x_1,\ldots,x_n) = \frac{\pi_0^{(n)}(x_1,\ldots,x_n)}{\pi_0^{(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right)}, \tag{5.18}$$

$$[f(x_1),\ldots,f(x_n)]_\circ^{(n)} = \varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left[f\left(x_{\sigma(1)}\right),\ldots,f\left(x_{\sigma(n)}\right)\right]_\circ^{(n)}, \tag{5.19}$$

where $\sigma \in S_n$. We call all $\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}$ as a *commutation factor "vector"* and denote it by $\vec{\varepsilon}$, where its components will be enumerated in lexicographic order.

Thus, each component of $\vec{\varepsilon}$ is responsible for the commutation of any two n-ary monomials with different permutations $\sigma, \sigma' \in S_n$ since from (5.19) it follows that

$$\left[f\left(x_{\sigma'(1)}\right),\ldots,f\left(x_{\sigma'(n)}\right)\right]_\circ^{(n)} = \varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left(x_{\sigma'(1)},\ldots,x_{\sigma'(n)}\right)\left[f\left(x_{\sigma(1)}\right),\ldots,f\left(x_{\sigma(n)}\right)\right]_\circ^{(n)}, \tag{5.20}$$

It follows from (5.18) that an n-ary analog of the normalization property (5.6) is

$$\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left(x_{\sigma'(1)},\ldots,x_{\sigma'(n)}\right)\varepsilon^{(n)}_{\sigma'(1),\ldots,\sigma'(n)}\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right) = 1, \tag{5.21}$$

$$\varepsilon^{(n)}_{1,\ldots,n}(x,\ldots,x) = 1, \tag{5.22}$$

where σ,$\sigma' \in S_n$, $x_1,\ldots,x_n, x \in H$.

In this notation, the binary commutation factor (5.4) is $\varepsilon^{(2)}_{\pi_0}(x_1,x_2) = \varepsilon^{(2)}_{21}(x_1,x_2)$.

Example 5.6 (Ternary projective representation). Consider the minimal non-binary case $n=3$. The ternary projective representation of the Abelian ternary group $\mathcal{H}^{(3)}$ is given by

$$[f(x_1), f(x_2), f(x_3)] = \pi_0(x_1,x_2,x_3) f(x_1 \dotplus x_2 \dotplus x_3), \quad x_1,x_2,x_3 \in H, \tag{5.23}$$

where we denote $\pi_0^{(3)} \equiv \pi_0$, $[\ \ ]_{\circ}^{(3)} \equiv [\ \ ]$ and $[x_1, x_2, x_3]_{\dot{+}}^{(3)} \equiv x_1 \dot{+} x_2 \dot{+} x_3$.

The ternary 2-cocycle conditions for the ternary Schur-like factor π_0 now become

$$\pi_0 (x_1, x_2, x_3) \pi_0 (x_1 \dot{+} x_2 \dot{+} x_3, x_4, x_5) \\ = \pi_0 (x_2, x_3, x_4) \pi_0 (x_1, x_2 \dot{+} x_3 \dot{+} x_4, x_5) \\ = \pi_0 (x_3, x_4, x_5) \pi_0 (x_1, x_2, x_3 \dot{+} x_4 \dot{+} x_5) . \tag{5.24}$$

Thus, we obtain $(3! - 1) = 5$ different ternary commutation relations and the corresponding 5-dimensional "vector" $\vec{\varepsilon}$

$$[f (x_1), f (x_2), f (x_3)] = \varepsilon_{\sigma(1)\sigma(2)\sigma(3)} \left[f \left(x_{\sigma(1)} \right), f \left(x_{\sigma(2)} \right), f \left(x_{\sigma(3)} \right) \right], \tag{5.25}$$

$$\varepsilon_{132} = \frac{\pi_0 (x_1, x_2, x_3)}{\pi_0 (x_1, x_3, x_2)}, \quad \varepsilon_{231} = \frac{\pi_0 (x_1, x_2, x_3)}{\pi_0 (x_2, x_3, x_1)}, \quad \varepsilon_{213} = \frac{\pi_0 (x_1, x_2, x_3)}{\pi_0 (x_2, x_1, x_3)}, \\ \varepsilon_{312} = \frac{\pi_0 (x_1, x_2, x_3)}{\pi_0 (x_3, x_1, x_2)}, \quad \varepsilon_{321} = \frac{\pi_0 (x_1, x_2, x_3)}{\pi_0 (x_3, x_2, x_1)}, \quad x_1, x_2, x_3 \in H, \ \sigma \in S_3. \tag{5.26}$$

6. n-ary double commutative algebras

6.1. n-ary ε-commutative algebras. Here we introduce grading noncommutativity for n-ary algebras ("n-ary coloring"), which is closest to the binary "coloring" case (2.3). We are exploiting an n-ary analog of the (Schur) factor (2.9) and its relation (2.11) by means of the n-ary projective representation theory from Section 5.

Let $\mathcal{A}^{(n)} = \left\langle A \mid [\ \ ]^{(n)}, + \right\rangle$ be an associative n-ary algebra [6, 11, 17] over a binary field $\Bbbk$ (with the binary addition) having zero $z \in A$ and unit $e \in A$ if $\mathcal{A}^{(n)}$ is unital (for polyadic algebras with all nonbinary operations, see [13]). An n-ary *graded algebra* $\mathcal{A}_{\mathcal{G}}^{(n)}$ (an n-ary G-graded $\Bbbk$-algebra) is a direct sum of subalgebras $\mathcal{A}_{\mathcal{G}}^{(n)} = \bigoplus_{g \in G} \mathcal{A}_g$, where $\mathcal{G} = \langle G \mid +' \rangle$ is a (binary Abelian) *grading group* and the set n-ary multiplication "respects the gradation"

$$[A_{g_1}, \ldots, A_{g_n}]^{(n)} \subseteq A_{g_1 +' \ldots +' g_n}, \quad g_1, \ldots, g_n \in G. \tag{6.1}$$

As in the binary case (2.2), the elements from $A_g \subset A$ are homogeneous of degree $a' = g \in G$.

It is natural to start our n-ary consideration from the Schur-like factors (5.12) which generalize (2.10) and (5.1).

Definition 6.1. In an n-ary graded algebra $\mathcal{A}_{\mathcal{G}}^{(n)}$, the *$n$-ary Schur-like factor* is an n-place function on gradings $\pi^{(n)} : \overbrace{G \times \ldots \times G}^{n} \to A$ satisfying the n-ary cocycle condition (cf. (2.10) and (5.12)):

$$\pi^{(n)} \left(a'_1, \ldots, a'_n \right) \pi^{(n)} \left(\left[a'_1, \ldots, a'_n \right]^{(n)}, a'_{n+1}, \ldots, a'_{2n-1} \right) \\ = \pi^{(n)} \left(a'_2, \ldots, a'_{n+1} \right) \pi^{(n)} \left(a'_1, \left[a'_2, \ldots, a'_{n+1} \right]^{(n)}, a'_{n+2}, \ldots, a'_{2n-1} \right)$$

$$\begin{aligned} &\dots \\ &= \pi^{(n)}\left(a'_n,\dots,a'_{n+1}\right)\pi^{(n)}\left(a'_1,\dots,a'_{n-1},\left[a'_n,\dots,a'_{2n-1}\right]^{(n)}\right), \\ &\qquad\qquad a'_1,\dots,a'_{2n-1}\in G. \end{aligned} \tag{6.2}$$

There are many possible ways to introduce noncommutativity for n-ary algebras [9, 17]. We propose a "projective version" of noncommutativity in $\mathcal{A}_{\mathcal{G}}^{(n)}$ which naturally follows from the n-ary projective representations (5.18) and can be formulated in terms of the Schur-like factors as in (2.11) and (5.14).

Definition 6.2. An n-ary graded algebra $\mathcal{A}_{\mathcal{G}}^{(n)}$ is called *π-commutative* if the $(n!-1)$ relations (cf. (5.14))

$$\begin{aligned} &\pi^{(n)}\left(a'_{\sigma(1)},\dots,a'_{\sigma(n)}\right)\left[a_1,\dots,a_n\right]^{(n)} \\ &\qquad = \pi^{(n)}\left(a'_1,\dots,a'_n\right)\left[a_{\sigma(1)},\dots,a_{\sigma(n)}\right]^{(n)} \end{aligned} \tag{6.3}$$

hold for all $a_1,\dots,a_n\in A$, and $\sigma\in S_n$, $\sigma\neq I$, where $\pi^{(n)}$ are the n-ary Schur-like factors satisfying (6.2).

Two n-ary Schur-like factor systems are equivalent $\left\{\pi^{(n)}\right\}\overset{\tilde{\lambda}}{\sim}\left\{\tilde{\pi}^{(n)}\right\}$ if there exists $\tilde{\lambda}:\mathcal{A}_{\mathcal{G}}^{(n)}\to\Bbbk^{\times}$ such that (cf. (5.13))

$$\tilde{\pi}^{(n)}\left(a'_1,\dots,a'_n\right)=\frac{\tilde{\lambda}\left(a'_1\right)\dots\tilde{\lambda}\left(a'_n\right)}{\tilde{\lambda}\left(\left[a'_1,\dots,a'_n\right]^{(n)}\right)}\pi^{(n)}\left(a'_1,\dots,a'_n\right),\quad a'_1,\dots,a'_2\in G. \tag{6.4}$$

The quotient, by this equivalence relation $\left\{\pi^{(n)}\right\}/\overset{\tilde{\lambda}}{\sim}$, is the corresponding multiplier group as in the binary case [21].

Let $\varphi\in\operatorname{Aut}G$ and $\left\{\pi^{(n)}\right\}$ be a factor system, then its pullback

$$\pi_*^{(n)}\left(a'_1,\dots,a'_n\right)=\pi^{(n)}\left(\varphi\left(a'_1\right),\dots,\varphi\left(a'_n\right)\right) \tag{6.5}$$

is also a factor system $\left\{\pi_*^{(n)}\right\}$. In the n-ary case, the "homotopic" analog of (6.5) is possible

$$\pi_{**}^{(n)}\left(a'_1,\dots,a'_n\right)=\pi^{(n)}\left(\varphi_1\left(a'_1\right),\dots,\varphi_n\left(a'_n\right)\right), \tag{6.6}$$

where $\varphi_1,\dots,\varphi_n\in\operatorname{Aut}G$ such that $\left\{\pi_{**}^{(n)}\right\}$ is also a factor system.

Comparing (6.3) with the binary π-commutativity (2.11), we observe that the most general description of n-ary graded algebras $\mathcal{A}_{\mathcal{G}}^{(n)}$ can be achieved by using at least $(n!-1)$ commutation factors.

Definition 6.3. An n-ary graded algebra $\mathcal{A}_{\mathcal{G}}^{(n)}$ is *$\varepsilon^{(n)}$-commutative* if there are $(n!-1)$ commutation factors

$$\varepsilon_{\sigma(1),\dots,\sigma(n)}^{(n)}\equiv\varepsilon_{\sigma(1),\dots,\sigma(n)}^{(n)}\left(a'_1,\dots,a'_n\right)=\frac{\pi^{(n)}\left(a'_1,\dots,a'_n\right)}{\pi^{(n)}\left(a'_{\sigma(1)},\dots,a'_{\sigma(n)}\right)}, \tag{6.7}$$

$$[a_1,\ldots,a_n]^{(n)}=\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left[a_{\sigma(1)},\ldots,a_{\sigma(n)}\right]^{(n)}, \tag{6.8}$$

where $\sigma\in S_n$, $\sigma\neq I$, and the factors $\pi^{(n)}$ satisfy (6.2). The set of $\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}$ is a *commutation factor* "*vector*" $\overrightarrow{\varepsilon}$ of the algebra $\mathcal{A}^{(n)}_{\mathcal{G}}$ having $(n!-1)$ components.

Each component of the "vector" $\overrightarrow{\varepsilon}$ is responsible for the commutation of two n-ary monomials in $\mathcal{A}^{(n)}_{\mathcal{G}}$ such that from (6.8) we have

$$\left[a_{\sigma'(1)},\ldots,a_{\sigma'(n)}\right]^{(n)}=\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left(a'_{\sigma'(1)},\ldots,a'_{\sigma'(n)}\right)\left[a_{\sigma(1)},\ldots,a_{\sigma(n)}\right]^{(n)} \tag{6.9}$$

with permutations $\sigma,\sigma'\in S_n$.

It follows from (5.18) that an n-ary analog of the normalization property (5.6) is

$$\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left(a'_{\sigma'(1)},\ldots,a'_{\sigma'(n)}\right)\varepsilon^{(n)}_{\sigma'(1),\ldots,\sigma'(n)}\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)=1, \tag{6.10}$$

$$\varepsilon^{(n)}_{1,\ldots,n}\left(a',\ldots,a'\right)=1, \tag{6.11}$$

where σ,$\sigma'\in S_n$, $a'_1,\ldots,a'_n,a'\in G$.

If $\varphi\in\operatorname{Aut}G$ and $\left\{\pi^{(n)}_*\right\}$ is a pullback of $\left\{\pi^{(n)}\right\}$ (6.5), then the corresponding

$$\varepsilon^{(n)}_{*\sigma(1),\ldots,\sigma(n)}\left(a'_1,\ldots,a'_n\right)=\frac{\pi^{(n)}_*\left(a'_1,\ldots,a'_n\right)}{\pi^{(n)}_*\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)} \tag{6.12}$$

are commutation factors as well (if $\Bbbk$ is algebraically closed by analogy with [21]). The same is true for their "homotopic" analog $\left\{\pi^{(n)}_{**}\right\}$ (6.6).

Definition 6.4. A factor set $\left\{\pi^{(n)}\right\}$ is called *totally symmetric* if $\overrightarrow{\varepsilon}$ is the unit commutation factor "vector"such that all of its $(n!-1)$ components are identities (in $\Bbbk$)

$$\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left(a'_1,\ldots,a'_n\right)=1,\quad \sigma\in S_n,\sigma\neq I,\ a'_1,\ldots,a'_n\in G. \tag{6.13}$$

Suppose we have two factor sets $\left\{\pi^{(n)}_1\right\}$ and $\left\{\pi^{(n)}_2\right\}$ which correspond to the same commutation factor $\varepsilon^{(n)}$ in $\mathcal{A}^{(n)}_{\mathcal{G}}$ such that

$$ex\varepsilon^{(n)}_{\sigma(1),\ldots,\sigma(n)}\left(a'_1,\ldots,a'_n\right)=\frac{\pi^{(n)}_1\left(a'_1,\ldots,a'_n\right)}{\pi^{(n)}_1\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)}=\frac{\pi^{(n)}_2\left(a'_1,\ldots,a'_n\right)}{\pi^{(n)}_2\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)}. \tag{6.14}$$

We can always choose the same order for the commutation factor "vector" components σ. Define a new factor set $\left\{\pi_{12}^{(n)}\right\}$ by

$$\pi_{12}^{(n)}\left(a'_1,\ldots,a'_n\right)=\frac{\pi_1^{(n)}\left(a'_1,\ldots,a'_n\right)}{\pi_2^{(n)}\left(a'_1,\ldots,a'_n\right)},\quad a'_1,\ldots,a'_n\in G. \tag{6.15}$$

Then $\left\{\pi_{12}^{(n)}\right\}$ becomes totally symmetric, because the corresponding communication factor is

$$\varepsilon_{12,\sigma(1),\ldots,\sigma(n)}^{(n)}\left(a'_1,\ldots,a'_n\right)=\frac{\pi_{12}^{(n)}\left(a'_1,\ldots,a'_n\right)}{\pi_{12}^{(n)}\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)}=1 \tag{6.16}$$

as follows from (6.14)–(6.15). Therefore, as in the binary case, if the grading group $\mathcal{G}$ is finitely generated and $\Bbbk$ is algebraically closed, then the communication factor $\varepsilon^{(n)}$ is constructed from the unique multiplier $\left\{\pi^{(n)}\right\}$ [21].

There are two possible differences from the simple binary case (2.13), where for identity commutation factor the symmetry condition (2.12) is sufficient:

1) Not all $\varepsilon^{(n)}$ need to be equal to the identity.

2) Some arguments of the Schur-like factors $\pi^{(n)}$ can be intact.

Definition 6.5. An $\varepsilon^{(n)}$-commutative n-ary graded algebra $\mathcal{A}_{\mathcal{G}}^{(n)}$ is called *m-partially* (or *partially*) *commutative* if exactly m commutation factors (with permutations $\tilde{\sigma}$) from the $(n!-1)$ total are equal to 1,

$$\varepsilon_{\tilde{\sigma}(1),\ldots,\tilde{\sigma}(n)}^{(n)}\left(a'_1,\ldots,a'_n\right)=1,\quad \#\tilde{\sigma}\leq\#\sigma,\ \sigma\in S_n,\quad a'_1,\ldots,a'_n\in G, \tag{6.17}$$

where we denote $\#\tilde{\sigma}=m$, $\#\sigma=n!-1$. If $\#\tilde{\sigma}=\#\sigma$, then $\mathcal{A}_{\mathcal{G}}^{(n)}$ is totally commutative, see (6.13).

In an m-partially commutative n-ary graded algebra $\mathcal{A}_{\mathcal{G}}^{(n)}$, the Schur-like factors $\pi^{(n)}$ satisfy m additional symmetry conditions (cf. (2.12))

$$\pi^{(n)}\left(a'_1,\ldots,a'_n\right)=\pi^{(n)}\left(a'_{\tilde{\sigma}(1)},\ldots,a'_{\tilde{\sigma}(n)}\right),\quad \#\tilde{\sigma}\leq\#\sigma,\quad \sigma\in S_n. \tag{6.18}$$

Example 6.6 (Ternary ε-commutative algebra). Let $\mathcal{A}_{\mathcal{G}}^{(3)}=\langle A\mid[\ \ ],+\rangle$ be a ternary associative G-graded algebra over $\Bbbk$. The ternary Schur-like factor $\pi_0\left(a'_1,a'_2,a'_3\right)$ satisfies the ternary 2-cocycle conditions (cf. (5.24))

$$\begin{aligned}\pi^{(3)}&\left(a'_1,a'_2,a'_3\right)\pi^{(3)}\left(a'_1+a'_2+a'_3,a'_4,a'_5\right)\\&=\pi^{(3)}\left(a'_2,a'_3,a'_4\right)\pi^{(3)}\left(a'_1,a'_2+a'_3+a'_4,a'_5\right)\\&=\pi^{(3)}\left(a'_3,a'_4,a'_5\right)\pi^{(3)}\left(a'_1,a'_2,a'_3+a'_4+a'_5\right),\quad a'_1,\ldots,a'_5\in G.\end{aligned} \tag{6.19}$$

We can introduce $(3!-1)=5$ different ternary commutation relations explicitly,

$$[a_1,a_2,a_3]=\varepsilon^{(3)}_{\sigma(1)\sigma(2)\sigma(3)}\left[a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}\right],\quad a_1,a_2,a_3\in A,\ \sigma\in S_3, \tag{6.20}$$

$$\varepsilon^{(3)}_{132}=\frac{\pi^{(3)}\left(a_1',a_2',a_3'\right)}{\pi^{(3)}\left(a_1',a_3',a_2'\right)},\quad \varepsilon^{(3)}_{231}=\frac{\pi^{(3)}\left(a_1',a_2',a_3'\right)}{\pi^{(3)}\left(a_2',a_3',a_1'\right)},\quad \varepsilon^{(3)}_{213}=\frac{\pi^{(3)}\left(a_1',a_2',a_3'\right)}{\pi^{(3)}\left(a_2',a_1',a_3'\right)},$$
$$\varepsilon^{(3)}_{312}=\frac{\pi^{(3)}\left(a_1',a_2',a_3'\right)}{\pi^{(3)}\left(a_3',a_1',a_2'\right)},\quad \varepsilon^{(3)}_{321}=\frac{\pi^{(3)}\left(a_1',a_2',a_3'\right)}{\pi^{(3)}\left(a_3',a_2',a_1'\right)},\quad a_1',a_2',a_3'\in G. \tag{6.21}$$

The 1-partial commutativity in $\mathcal{A}^{(3)}_{\mathcal{G}}$ can be realized if, for instance,

$$\pi^{(3)}\left(a_1',a_2',a_3'\right)=\pi^{(3)}\left(a_1',a_3',a_2'\right),\quad a_1',a_2',a_3'\in G, \tag{6.22}$$

and in this case, $\varepsilon^{(3)}_{132}=1$ so that we obtain one commutativity relation

$$[a_1,a_2,a_3]=[a_1,a_3,a_2],\quad a_1,a_2,a_3\in A, \tag{6.23}$$

while the other 4 relations in (6.20) will be ε-commutative.

6.2. Membership deformed n-ary algebras. Now we consider n-ary algebras over obscure sets as their underlying sets, where each element of them is endowed with the membership function μ as a degree of truth (see Section 3).

Definition 6.7. An *obscure n-ary algebra* $\mathcal{A}^{(n)}(\mu)=\left\langle \mathfrak{A}^{(\mu)}\mid[\ \ ]^{(n)},+\right\rangle$ is an n-ary algebra $\mathcal{A}^{(n)}$ over $\Bbbk$ having an obscure set $\mathfrak{A}^{(\mu)}=\{(a|\mu(a)),a\in A,\mu>0\}$ as its underlying set (see (3.1)), and where the membership function $\mu(a)$ satisfies

$$\mu(a_1+a_2)\geq\min\{\mu(a_1),\mu(a_2)\}, \tag{6.24}$$
$$\mu([a_1,\ldots,a_n]_n)\geq\min\{\mu(a_1),\ldots,\mu(a_n)\}, \tag{6.25}$$
$$\mu(ka)\geq\mu(a),\quad a,a_i\in A,\ k\in\Bbbk,\quad i=1,\ldots,n. \tag{6.26}$$

Definition 6.8. An *obscure G-graded n-ary algebra* $\mathcal{A}^{(n)}_{\mathcal{G}}(\mu)$ is a direct sum decomposition (cf. (3.8)),

$$\mathcal{A}^{(n)}_{\mathcal{G}}(\mu)=\bigoplus\nolimits_{g\in G}\mathcal{A}^{(n)}_g(\mu_g), \tag{6.27}$$

where $\mathfrak{A}^{(\mu)}=\bigcup_{g\in G}\mathfrak{A}^{(\mu_g)}_g$, $\mathfrak{A}^{(\mu_g)}=\{(a|\mu_g(a)),a\in A_g,\mu_g=(0,1]\}$, and the joint membership function μ is given by (3.9).

In the obscure totally commutative n-ary algebra $\mathcal{A}^{(n)}_{\mathcal{G}}(\mu)$ (for homogeneous elements) we have $(n!-1)$ commutativity relations (cf. (5.17))

$$[a_1,\ldots,a_n]^{(n)}=\left[a_{\sigma(1)},\ldots,a_{\sigma(n)}\right]^{(n)},\quad a_1,\ldots,a_n\in\mathfrak{A}^{(\mu)},\ \sigma\in S_n,\ \sigma\neq I. \tag{6.28}$$

Let us consider a "linear" (in μ) deformation of (6.28) analogous to the binary case (4.1).

Definition 6.9. An *obscure membership deformed n-ary algebra* is

$$\mathcal{A}_{*\mathcal{G}}^{(n)}(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid [\ \ ]_*^{(n)}, + \right\rangle$$

in which there are $(n! - 1)$ possible noncommutativity relations

$$\mu_{a'_n}(a_n)\,[a_1, \dots, a_n]_*^{(n)} = \mu_{a'_{\sigma(n)}}\left(a_{\sigma(n)}\right)\left[a_{\sigma(1)}, \dots, a_{\sigma(n)}\right]_*^{(n)}, \\ a_1, \dots, a_n \in \mathfrak{A}^{(\mu)}, \quad a'_n, a'_{\sigma(n)} \in G, \quad \sigma \in S_n, \ \sigma \neq I. \tag{6.29}$$

Since $\mu > 0$, we can have

Definition 6.10. The $(n! - 1)$ membership commutation factors in $\mathcal{A}_{*\mathcal{G}}^{(n)}(\mu)$ are defined by

$$\epsilon_{\sigma(n)}^{(n)}\left(a'_n, a'_{\sigma(n)}, a_n, a_{\sigma(n)}\right) = \frac{\mu_{a'_{\sigma(n)}}\left(a_{\sigma(n)}\right)}{\mu_{a'_n}(a_n)}, \ a'_n, a'_{\sigma(n)} \in G, \ \sigma \in S_n, \ \sigma \neq I, \tag{6.30}$$

and the relations (6.29) become

$$[a_1, \dots, a_n]_*^{(n)} = \epsilon_{\sigma(n)}^{(n)}\left(a'_n, a'_{\sigma(n)}, a_n, a_{\sigma(n)}\right)\left[a_{\sigma(1)}, \dots, a_{\sigma(n)}\right]_*^{(n)}. \tag{6.31}$$

These definitions are unique if we require:

1) for connection, only two monomials with different permutations;

2) membership "linearity" (in μ);

3) compatibility with the binary case (4.2)–(4.3).

Example 6.11. For the obscure membership deformed ternary algebra

$$\mathcal{A}_{*\mathcal{G}}^{(3)}(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid [\ \ ]_*^{(3)}, + \right\rangle,$$

we obtain

$$[a_1, a_2, a_3] = \epsilon_{\sigma(1)\sigma(2)\sigma(3)}^{(3)}\left[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\right]_*^{(3)}, \ a_1, a_2, a_3 \in A, \ \sigma \in S_3, \tag{6.32}$$

$$\epsilon_{132}^{(3)} = \epsilon_{312}^{(3)} = \frac{\mu_{a'_3}(a_3)}{\mu_{a'_2}(a_2)}, \quad \epsilon_{231}^{(3)} = \epsilon_{321}^{(3)} = \frac{\mu_{a'_3}(a_3)}{\mu_{a'_1}(a_1)}, \quad \epsilon_{213}^{(3)} = 1, , \quad a'_1, a'_2, a'_3 \in G,$$

which means that it is 1-partially commutative (see (6.17)) and has only 2 independent membership commutation factors (cf. the binary case (4.3)).

We now provide a sketch construction of the n-ary ε-commutative algebras (6.8) membership deformation (see Subsection 4.2 for the binary case).

Definition 6.12. An *obscure membership deformed n-ary π-commutative algebra* over $\Bbbk$ is $\mathcal{A}_{\star\mathcal{G}}^{(n)}(\mu) = \left\langle \mathfrak{A}^{(\mu)} \mid [\ ]_{\star}^{(n)}, + \right\rangle$ in which the following $(n!-1)$ noncommutativity relations are valid:

$$\pi^{(n)}\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)\mu_{a'_n}(a_n)\,[a_1,\ldots,a_n]_{\star}^{(n)}$$
$$= \pi^{(n)}\left(a'_1,\ldots,a'_n\right)\mu_{a'_{\sigma(n)}}\left(a_{\sigma(n)}\right)\left[a_{\sigma(1)},\ldots,a_{\sigma(n)}\right]_{\star}^{(n)},$$
$$a_i \in \mathfrak{A}^{(\mu)},\quad a'_i \in G,\quad \sigma \in S_n,\ \sigma \neq I, \tag{6.33}$$

where $\pi^{(n)}$ are n-ary Schur-like factors satisfying the 2-cocycle conditions (6.2).

Definition 6.13. An algebra $\mathcal{A}_{\star\mathcal{G}}^{(n)}(\mu)$ is called a *double $\varepsilon_{\pi}^{(n)}/\epsilon_{\mu}^{(n)}$-commutative algebra* if

$$[a_1,\ldots,a_n]_{\star}^{(n)}$$
$$= \varepsilon_{\sigma(1),\ldots,\sigma(n)}^{(n)}\left(a'_1,\ldots,a'_n\right)\epsilon_{\sigma(n)}^{(n)}\left(a'_n,a'_{\sigma(n)},a_n,a_{\sigma(n)}\right)\left[a_{\sigma(1)},\ldots,a_{\sigma(n)}\right]_{\star}^{(n)}, \tag{6.34}$$

$$\varepsilon_{\sigma(1),\ldots,\sigma(n)}^{(n)}\left(a'_1,\ldots,a'_n\right) = \frac{\pi^{(n)}\left(a'_1,\ldots,a'_n\right)}{\pi^{(n)}\left(a'_{\sigma(1)},\ldots,a'_{\sigma(n)}\right)}, \tag{6.35}$$

$$\epsilon_{\sigma(n)}^{(n)}\left(a'_n,a'_{\sigma(n)},a_n,a_{\sigma(n)}\right) = \frac{\mu_{a'_{\sigma(n)}}\left(a_{\sigma(n)}\right)}{\mu_{a'_n}(a_n)},\quad a_1,\ldots,a_n,a_{\sigma(1)},\ldots,a_{\sigma(n)} \in \mathfrak{A}^{(\mu)},$$
$$a'_1,\ldots,a'_n,a'_{\sigma(1)},\ldots,a'_{\sigma(n)} \in G,\quad \sigma \in S_n,\ \sigma \neq I, \tag{6.36}$$

where $\varepsilon_{\pi}^{(n)}$ is the n-ary grading commutation factor (6.7) and $\epsilon_{\mu}^{(n)}$ is the n-ary membership commutation factor (in our definition (6.30)).

This procedure can be considered to be the membership deformation of the given ε-commutative algebra (6.8), which corresponds to the first version of the binary commutative algebra deformation as in (4.12) and leads to a nonassociative algebra in general. To achieve associativity, one should consider the second version of the algebra deformation as in the binary case (4.16): to introduce a different n-ary multiplication $[\ ]_{\circledast}^{(n)}$ such that the product of commutation factors $\varepsilon^{(n)}\epsilon^{(n)}$ satisfies the n-ary 2-cocycle-like conditions, while the n-ary noncocycle commutation factor satisfies the "membership deformed cocycle-like" conditions analogously to (4.17)–(4.19).

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Деформація приналежності комутативності та нечіткі n-арні алгебри

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Запропоновано загальний механізм “порушення” комутативності в алгебрах: якщо базова множина приймається не за чітку множину, а скоріше за нечітку, функція приналежності, що відображає ступінь істинності приналежності елемента до множини, може бути включена в комутаційні співвідношення. Спеціальні “деформації” комутативності і ε-комутативності вводяться таким чином, що рівні ступені істинності призводять до “недеформованого” випадку. Ми також наводимо схеми “деформування” ε-алгебр Лі і алгебри Вейля. Далі, наведені вище конструкції, поширюються на n-арні алгебри, для яких вивчаються проективні подання та ε-комутативність.

Ключові слова: майже комутативна алгебра, неясна алгебра, деформація приналежності, нечітка множина, функція приналежності, n-арна алгебра, алгебра Лі, проективне подання