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A Complete Study of the Lack of Compactness and Existence Results of a Fractional Nirenberg Equation via a Flatness Hypothesis. Part II

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This is a sequel to [2] where the prescribed σ-curvature problem on the standard sphere was studied under the hypothesis that the flatness order at critical points of the prescribed function lies in $(1, n-2\sigma]$. We provide a complete description of the lack of compactness of the problem when the flatness order varies in $(1, n)$ and we establish an existence theorem based on an Euler–Hopf type formula. As a product, we extend the existence results of [2, 17, 18] and deliver a new one.

Key words: conformal geometry, fractional curvature, variational calculus, critical points at infinity

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1. Introduction and main results

In this paper, we are interested in a critical fractional problem arising in conformal differential geometry. Namely, we consider the problem of existence of conformal metrics with prescribed fractional order curvatures on the unit n-dimensional sphere. Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}, n \geq 2$, endowed with the standard metric $g_{S^n} = \sum_{i=1}^{n+1} dx_i^2$. Let $g \in [g_{S^n}]$, the conformal class of metrics be associated to g_{S^n} and write $g = u^{\frac{4}{n-2\sigma}} g_{S^n}$, where $\sigma \in (0,1)$ and u is a smooth positive function on S^n. The fractional curvature R_g^σ of order σ for (S^n, g) called also σ-curvature is given by

$$R_g^\sigma = u^{-\frac{n+2\sigma}{n-2\sigma}} P_\sigma(u), \tag{1.1}$$

where

$$P_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_{S^n}} + \left(\frac{n-1}{2}\right)^2},$$

Γ is the Gamma function and $\Delta_{g_{S^n}}$ is the Laplace–Beltrami operator of (S^n, g_{S^n}). The conformal fractional operator P_σ can be considered as the pull back operator of the fractional Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection.

The prescribed σ-curvature problem on the sphere S^n can be described by the following question: *For which function $K : \mathbb{S}^n \to \mathbb{R}$, there exists a conformal metric $g \in [g_{S^n}]$ such that the associated σ-curvature function R_g^σ is equal to K?* According to (1.1), the problem is equivalent to solving the following fractional Nirenberg equation

$$P_\sigma u = c(n,\sigma) K u^{\frac{n+2\sigma}{n-2\sigma}}, \quad u > 0 \quad \text{on } \mathbb{S}^n, \tag{1.2}$$

where $c(n,\sigma) = \frac{\Gamma(\frac{n}{2}+\sigma)}{\Gamma(\frac{n}{2}-\sigma)}$.

Equation (1.2) and related fractional problems have been the target of investigations during the last several years because they have proved to be of great importance in geometry, analysis and physics (see [1, 2, 9, 10, 15, 16, 19, 22–25, 28] and the references therein).

In general, equation (1.2) may have no solution. Besides the necessary condition that the function K has to be positive somewhere, there is a Kazdan–Warner type obstruction found in [17]: if u solves (1.2), then

$$\int_{S^n} \langle \nabla_{g_{S^n}} K, \nabla_{g_{S^n}} \xi \rangle \, u^{\frac{2n}{n-2\sigma}} d\xi = 0.$$

This identity gives rise to many examples of function K for which (1.2) has no solution, see [17].

There have been many studies devoted to the existence results trying to understand under what conditions equation (1.2) is solvable. See, for example, the works of [1, 13, 26] under a “non-degeneracy” condition on K, [2, 17, 18] under a suitable “β-flatness” condition on K and [12, 20] for $K = 1$.

This paper is a continuation of the work of [2]. Therefore, throughout this paper, we assume that K satisfies the following β-flatness hypothesis:

$(f)_\beta$: $K : \mathbb{S}^n \to \mathbb{R}, n \geq 2$ is a C^1-positive function such that near each of its critical point y there exists a real number $\beta = \beta(y) \in (1, n-2\sigma]$ such that system centered at y in some geodesic normal coordinates and the following expansion holds

$$K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + R(x-y),$$

where $b_k = b_k(y) \neq 0$ for all $k = 1, \ldots, n$, $\sum_{k=1}^n b_k(y) \neq 0$, and

$$\sum_{s=0}^{[\beta]} |\nabla^s R(x-y)||x-y|^{s-\beta} = o(1) \quad \text{as } x \text{ tends to } y.$$

We point out that problem (1.2) was first addressed in [17] and [18] under the above $(f)_\beta$ condition. In these two seminal papers, Jin, Li, and Xiong where able to obtain an a priori restriction for solutions and derive an index-counting criteria for existence of solutions when the flatness order $\beta = \beta(y)$ lies in $(n-2\sigma, n)$ at any critical point y of K. Their approach is based on tricky variational tools and blow up subcritical approximations. Later in [2], the authors studied problem (1.2) under $(f)_\beta$ condition when the flatness order $\beta = \beta(y)$ lies in $(1, n-2\sigma]$ at any critical point y of K. The situation is different in this case. More precisely, when $\beta(y) \in (n-2\sigma, n)$ at any critical point y of K, a sequence of subcritical solutions cannot blow up at more than one point (see [17,18]). However, if $\beta(y) \in (1, n-2\sigma]$ at any critical point y of K, there could be blow up at many points. The complete description of the critical points at infinity (blow up points) in the later case with the related index-counting criteria of the existence of solutions were given in [2]. The method of [2] is based on the critical points at infinity of A. Bahri [4,5].

The purpose of the present paper is to study equation (1.2) when the prescribed function K is flat near its critical points for an order $\beta \in (1, n)$. We are then mainly interested in the statement when the function K admits critical points of flatness order in $(1, n-2\sigma]$ and others of flatness order in $(n-2\sigma, n)$. This leads to a new interesting phenomenon drastically different from the previous ones. Indeed, when studying the lack of compactness of the problem and trying to identify the location of the critical points at infinity of the associated variational structure, it turn out that the mutual interaction among two different bubbles (solutions of (1.2) when $K = 1$ on S^n), dominates the self interaction of the bubbles if $\beta(y) \in (n-2\sigma, n)$ for any critical points y of K. If $\beta(y) \in (1, n-2\sigma)$ for any critical points y of K, the reverse phenomenon happens. While if $\beta(y) = n-2\sigma$ for any critical points y of K, we have a phenomenon of balance.

Now if β varies in $(1, n)$, particularly if we single out two bubbles at two critical points y_i and y_j of K with $\beta(y_i) \in (1, n-2\sigma]$ and $\beta(y_j) \in (n-2\sigma, n)$, the above three phenomena may occur, and each phenomenon (as we will see in Section 3 of this paper) will be related to the sign of

$$\beta(y_i) + \beta(y_j) - 2\frac{\beta(y_i)\beta(y_j)}{n-2\sigma}.$$

We first recall the existence results of [2,17,18]. Let

$$\begin{aligned}
\mathcal{K} &= \{y \in \mathbb{S}^n \mid \nabla_{g_{S^n}} K(y) = 0\}, \\
\mathcal{K}^+ &= \left\{ y \in \mathcal{K} \mid -\sum_{k=1}^{n} b_k(y) > 0 \right\}, \\
\mathcal{K}_{\leq n-2\sigma} &= \{y \in \mathcal{K} \mid \beta(y) \in (1, n-2\sigma]\} \\
\mathcal{K}_{> n-2\sigma} &= \{y \in \mathcal{K} \mid \beta(y) \in (n-2\sigma, n)\}.
\end{aligned}$$

For each p-tuple $\tau_p = (y_1, \dots, y_p)$, $p \geq 1$, of distinct critical points of K such that $\beta(y_i) = n-2\sigma$ for all $i = 1, \dots, p$, we associate a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})_{1 \leq i \neq j \leq p}$ by denoting

$$m_{ii} = m(y_i, y_i) = \frac{n-2\sigma}{2} c_i \frac{-\sum_{k=1}^{n} b_k(y_i)}{K(y_i)^{\frac{n}{2\sigma}}}, \quad 1 \leq i \leq p,$$

$$m_{ij} = m(y_i, y_j) = \tilde{c}2^{\frac{n-2\sigma}{2}} \frac{G(y_i, y_j)}{\Big(K(y_i)K(y_j)\Big)^{\frac{n-2\sigma}{4\sigma}}}, \quad 1 \leq i \neq j \leq p,$$

where

$$G(y_i, y_j) = \frac{1}{(1 - \cos d(y_i, y_j))^{\frac{n-2\sigma}{2}}},$$

$$c_i = c_0^{\frac{2n}{n-2\sigma}} \int_{\mathbb{R}^n} \frac{|x_1|^{\beta(y_i)}}{(1+|x|^2)^n} dx, \quad \tilde{c} = c_0^{\frac{2n}{n-2\sigma}} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2\sigma}{2}}}.$$

Here x_1 is the first component of x in the geodesic normal coordinate system, and c_0 is a fixed positive constant defined in Section 2.

($\mathbf{H_1}$) Assume that the first eigenvalue $\rho(\tau_p)$ of $M(\tau_p)$ is different to zero for any $M(\tau_p)$.

Let $\mathcal{C}^\infty_{\leq n-2\sigma}$ be the set of $\tau_p = (y_1, \dots, y_p) \in \big(\mathcal{K}_{\leq n-2\sigma} \cap \mathcal{K}^+\big)^p$ such that $y_i \neq y_j$ for all $1 \leq i \neq j \leq p$ and $\rho(y_{i_1}, \dots, y_{i_q}) > 0$ for $y_{i_1}, \dots, y_{i_q}$ satisfying $\beta(y_{i_j}) = n - 2\sigma$, $p \geq 1$. For any $y \in \mathcal{K}$, let

$$\widetilde{i}(y) = \sharp\{b_k(y), 1 \leq k \leq n, b_k(y) < 0\}.$$

Theorem 1.1 ([17, 18]). *Assume that K satisfies $(f)_\beta$-condition, $\beta \in (n - 2\sigma, n)$. If*

$$\sum_{y \in \mathcal{K}_{>n-2\sigma} \cap \mathcal{K}^+} (-1)^{n - \widetilde{i}(y)} \neq 1,$$

then (1.2) has a solution.

For any $\tau_p = (y_1, \dots, y_p) \in \mathcal{K}^p, p \geq 1$, we set

$$\widetilde{i}(\tau_p) = p - 1 + \sum_{j=1}^{p} n - \widetilde{i}(y_j).$$

Theorem 1.2 ([2]). *Assume that K satisfies* **($\mathbf{H_1}$)** *and $(f)_\beta$-condition, $\beta \in (1, n - 2\sigma]$. If*

$$\sum_{\tau_p \in \mathcal{C}^\infty_{\leq n-2\sigma}} (-1)^{\widetilde{i}(\tau_p)} \neq 1,$$

then (1.2) has a solution.

We are now in position to state our main theorem. Let $y \in \mathcal{K}_{>n-2\sigma}$. We denote

$$B_y = \left\{ q \in \mathcal{K}_{\leq n-2\sigma} \mid \beta(y) + \beta(q) - 2\frac{\beta(y)\beta(q)}{n - 2\sigma} = 0 \right\}.$$

($\mathbf{H_2}$) We assume that for any $\tau_p = (q_1, \dots, q_s) \in (B_y)^s, 1 \leq s \leq \sharp B_y$ such that $q_i \neq q_j$ for all $1 \leq i \neq j \leq s$, we have

$$\sum_{i=1}^{s}\frac{c_i\beta(q_i)}{n}\frac{|\sum_{k=1}^{n}b_k(q_i)|}{K(q_i)^{1+\frac{n-2\sigma}{n}}}\left(\frac{n\tilde{c}_1 2^{\frac{n-2\sigma}{2}}K(q_i)^{1+\frac{n-2\sigma}{2}}G(q_i,y)}{\beta(q_i)c_i\big(K(q_i)K(y)\big)^{\frac{n-2\sigma}{4}}|\sum_{k=1}^{n}b_k(q_i)|}\right)^{\frac{2\beta(y)}{n-2\sigma}} + \frac{c(y)\beta(y)\sum_{k=1}^{n}b_k(y)}{nK(y)^{1+\frac{n-2\sigma}{2}}}\neq 0,$$

where $c(y)=\int_{\mathbb{R}^n}\frac{|x_1|^{\beta(y)}}{(1+|x|^2)^n}dx$.

We point out that an assumption like **$(\mathbf{H_2})$** was used for the scalar curvature problem on S^n as a standard assumption to guarantee the existence of solution (see [11, Theorem 10.3] and [14]).

Set

$$\mathcal{C}^\infty=\Big\{(y,\tau_p)\mid y\in(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+),\ \tau_p=(z_1,\dots,z_p)\in\mathcal{C}^\infty_{\leq n-2\sigma},\ \beta(y)+\beta(z_i)-2\frac{\beta(y)\beta(z_i)}{n-2\sigma}>0,\ 1\leq i\leq p\Big\}.$$

We shall prove the following theorem.

Theorem 1.3. *Assume that K satisfies* **$(\mathbf{H_1})$**, **$(\mathbf{H_2})$**, *and $(f)_\beta$-condition, $\beta\in(1,n)$. If*

$$\sum_{y\in(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+)}(-1)^{n-\tilde{i}(y)}+\sum_{\tau_p\in\mathcal{C}^\infty_{\leq n-2\sigma}}(-1)^{\tilde{i}(\tau_p)}+\sum_{(y,\tau_p)\in\mathcal{C}^\infty}(-1)^{\tilde{i}(y,\tau_p)}\neq 1,$$

then (1.2) has a solution.

In the next section, we state some preliminary results. In Section 3 we describe the lack of compactness of the problem by identifying the location of all the critical points at infinity of the associated variational functional. Finally, in Section 4 we prove our existence theorem.

2. Preliminaries

Problem (1.2) has a variational structure. The variational space is $H^\sigma(S^n)$. It is the Sobolev space defined by the closer of $C^\infty(S^n)$ by the norm

$$\|u\|=\left(\int_{S^n}P_\sigma u\,u\,dv_{g_{S^n}}\right)^{\frac{1}{2}}.$$

The variational functional associated to (1.2) is

$$J(u)=\frac{\|u\|^2}{\left(\int_{S^n}Ku^{\frac{2n}{n-2\sigma}}dv_{g_{S^n}}\right)^{\frac{n-2\sigma}{n}}},\quad u\in H^\sigma(S^n).$$

Let

$$\Sigma=\{u\in H^\sigma(S^n)\mid \|u\|=1\}\quad\text{and}\quad\Sigma^+=\{u\in\Sigma,u\geq 0\}.$$

Up to a multiplicative constant, a solution of (1.2) is a critical point of J subjected to the constraint $u \in \Sigma^+$. Since the Sobolev embedding $H^(S^n) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(S^n)$ is not compact, the functional J does not satisfy the Palais–Smale condition. In order to describe the sequences which violate the Palais–Smale condition, we first recall the classification result of the solution of (1.2) when $K = 1$ on S^n.

For any $a \in \mathbb{S}^n$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = c_0 \left(\frac{\lambda}{\lambda^2 + 1 + (1 - \lambda^2)\cos d(a,x)} \right)^{\frac{n-2\sigma}{2}}, \quad x \in \mathbb{S}^n,$$

where d is the distance induced by the standard metric g_{S^n} and c_0 is a fixed constant chosen so that $\delta_{(a,\lambda)}$ solves

$$P_\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}}, \quad u > 0 \ \text{ on } \mathbb{S}^n$$

(see [12, 20]). Let $p \in \mathbb{N}^*$ and $u = \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v \in H^\sigma(S^n)$, $\alpha_i > 0$, $\lambda_i > 0$, $a_i \in S^n$ for all $i = 1, \dots, p$. We say that $v \in (V_0)$ if v satisfies the following assumption

$$(V_0) \quad \langle v, \varphi \rangle = 0 \quad \text{for } \varphi \in \left\{ \delta_{(a_i,\lambda_i)}, \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial \lambda_i}, \frac{\partial \delta_{(a_i,\lambda_i)}}{\partial a_i}, i = 1, \dots, p \right\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product of $H^\sigma(\mathbb{S}^n)$ defined by

$$\langle u, w \rangle = \int_{\mathbb{S}^n} P_\sigma u \, w \, dv_{g_{S^n}}.$$

Let $p \in \mathbb{N}^*$ and $\varepsilon > 0$. Let $V(p, \varepsilon)$ be the set of $u \in \Sigma$ such that there exist $\lambda_1, \dots, \lambda_p > \varepsilon^{-1}$ and $\alpha_1, \dots, \alpha_p > 0$ that

$$u = \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v,$$

where $\|v\| < \varepsilon$, $v \in (V_0)$, $|J(u)^{\frac{n}{n-2\sigma}} \alpha_i^{\frac{4}{n-2\sigma}} K(a_i) - 1| < \varepsilon$ for all $i = 1, \dots, p$ and

$$\varepsilon_{ij} := \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d(a_i, a_j)^2 \right)^{\frac{\sigma 2-n}{2}} < \varepsilon \quad \text{for all } 1 \le i \ne j \le p.$$

Proposition 2.1 ([21, 27]). *Let $(u_k)_k$ be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ tends to zero. If $(u_k)_k$ has no convergent subsequence in Σ^+, then there exist an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$ as $\varepsilon_k \to 0$ and a subsequence of $(u_k)_k$ denoted again $(u_k)_k$ such that $u_k \in V(p, \varepsilon_k)$ for all $k \in \mathbb{N}$.*

The following Morse Lemma completely gets rid of the v-contributions.

Proposition 2.2 ([5, 6]). *There exists a C^1-map associating $\bar{v} = \bar{v}(\alpha, a, \lambda)$ to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} \in V(p, \varepsilon)$, where $\bar{v}$ is unique and satisfies*

$$J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + \bar{v} \right) = \min_{v \in (V_0)} J\left(\sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)} + v \right).$$

Moreover, there exists a change of variables $v - \bar{v} \to V$ *such that*

$$J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\right) = J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \bar{v}\right) + \|V\|^2.$$

Furthermore, under $(f)_\beta$*-condition,* $1 < \beta < n$*, there exists* $c > 0$ *such that the following estimate holds:*

$$\|\bar{v}\| \le c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+2\sigma}{2n}}}{\lambda_i^{\frac{n+2\sigma}{2}}} \right) + c \begin{cases} \sum_{k\neq r} \varepsilon_{kr}^{\frac{n+2\sigma}{2(n-2\sigma)}} (\log \varepsilon_{kr}^{-1})^{\frac{n+2\sigma}{2n}} & \text{if } n \ge 6\sigma, \\ \sum_{k\neq r} \varepsilon_{kr} (\log \varepsilon_{kr}^{-1})^{\frac{n-2\sigma}{n}} & \text{if } n < 6\sigma. \end{cases}$$

We now state the definition of a critical point at infinity.

Definition 2.3 ([4]). A critical point at infinity of J is a limit of a non-compact flow line $u(s)$ of the equation

$$\dot{u}(s) = -\partial J(u(s)), \quad u(0) = u_0 \in \Sigma^+.$$

If we assume that (1.2) has no solution, according to Propositions 2.1 there exist $p \in \mathbb{N}^*$ and a positive function $\varepsilon(s)$ which tends to zero as s tends to $+\infty$ such that

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) \delta_{(a_i(s),\lambda_i(s))} + v(s) \in V(p, \varepsilon(s)) \quad \text{if } s \text{ is large}.$$

Setting $\tilde{\alpha_i} = \lim \alpha_i(s)$ and $\tilde{a_i} = \lim a_i(s)$, we denote a critical point at infinity by

$$\sum_{i=1}^{p} \tilde{\alpha_i} \delta_{(\tilde{a_i},\infty)} \quad \text{or} \quad (\tilde{a_1}, \dots, \tilde{a_p})_\infty.$$

The following two results rule out the existence of critical points at infinity of J in $V(p,\varepsilon) \setminus V_\delta(p,\varepsilon)$, where δ is a small positive constant which depends only on K and

$$V_\delta(p,\varepsilon) = \Bigg\{ u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v \in V(p,\varepsilon) \mid \forall i = 1, \dots, p \ \exists y_i \in \mathcal{K} \ \forall 1 \le i \neq j \le p \quad \left(\lambda_i |a_i - y_i| < \delta \text{ and } y_i \neq y_j\right) \Bigg\}.$$

Proposition 2.4. *Assume that* K *satisfies* $(f)_\beta$*-condition,* $\beta \in (1, n)$*. Let*

$$\beta = \max\{\beta(y), y \in \mathcal{K}\}.$$

There exists a bounded pseudo-gradient W_0 *in* $V(p,\varepsilon) \setminus V_\delta(p,\varepsilon)$*,* $p \ge 1$*, and a positive constant* c *independent of*

$$u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V(p,\varepsilon) \setminus V_\delta(p,\varepsilon)$$

such that

(i) $\langle \partial J(u), W_0(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{ij}\right).$

(ii) $$\left\langle \partial J(u+\bar v), W_0(u)+\frac{\partial \bar v}{\partial(\alpha_i,a_i,\lambda_i)}(W_0(u))\right\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\beta}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big).$$

(iii) *Along any flow line*

$$u(s)=\sum_{i=1}^{p}\alpha_i(s)\delta_{(a_i(s),\lambda_i(s))} \quad \textit{of } W_0,$$

$\max_{1\leq i\leq p}\lambda_i(s)$ *is bounded as long as* $u(s)$ *remains in* $V(p,\varepsilon)\setminus V_\delta(p,\varepsilon)$.

Proof. The construction of W_0 proceeds exactly as the one of [2]. More precisely, see the construction in the regions $V_1^3(p,\varepsilon)$ and $V_1^4(p,\varepsilon)$ in [2, pp. 1300–1304]. □

The above-mentioned result shows that no concentration phenomenon happens in $V(p,\varepsilon)\setminus V_\delta(p,\varepsilon)$, since any flow line of W_0 remains in a compact set. This allows us to derive the following corollary.

Corollary 2.5. *There is no critical points at infinity of* J *in* $V(p,\varepsilon)\setminus V_\delta(p,\varepsilon)$, $p\geq 1$.

The aim of the next section is the characterization of the critical points at infinity of J when the flatness order $\beta(y)$ varies in $(1,n)$ for any $y\in\mathcal{K}$. According to the corollary mentioned above, we are only interested to characterize these critical points in $V_\delta(p,\varepsilon)$, $p\geq 1$.

3. Critical points at infinity

In this section we construct a decreasing pseudo-gradient of J in $V_\delta(p,\varepsilon)$, $p\geq 1$, for which the Palais–Smale condition is satisfied along its flow lines as long as these flow lines do not enter in neighborhood of critical points y_i, $i=1,\dots,p$, such that $(y_1,\dots,y_p)\in(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+)\cup\mathcal{C}^\infty_{\leq n-2\sigma}\cup\mathcal{C}^\infty$. Namely, we shall prove the following main result.

Theorem 3.1. *Let* $\beta=\max\{\beta(y), y\in\mathcal{K}\}$. *Under assumptions* **($\mathbf{H_1}$)**, **($\mathbf{H_2}$)**, *and* $(f)_\beta$, $\beta\in(1,n)$, *there exist a bounded pseudo-gradient* W *in* $V_\delta(p,\varepsilon)$, $p\geq 1$, *and a fixed positive constant* c *such that for any*

$$u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)}\in V_\delta(p,\varepsilon),$$

the following assertions hold:

(i) $\langle \partial J(u), W(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{ij}\right).$

(ii) $$\left\langle \partial J(u+\bar{v}), W(u)+\frac{\partial \bar{v}}{\partial(\alpha_i,a_i,\lambda_i)}(W(u))\right\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{ij}\right).$$

(iii) *All the component $\lambda_i(s)$, $i=1,\dots,p$, of the flow line*

$$u(s)=\sum_{i=1}^{p}\alpha_i(s)\delta_{(a_i(s),\lambda_i(s))}$$

remain bounded as long as $u(s)$ is out side a small neighborhood $\mathcal{N}(p,\varepsilon)$ of $\sum_{i=1}^{p}\delta_{(y_i,\infty)}/K(y_i)^{\frac{n-2\sigma}{2}}$, where $(y_1,\dots,y_p)\in(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+)\cup\mathcal{C}^{\infty}_{\leq n-2\sigma}\cup\mathcal{C}^{\infty}$. However, if $u(s)$ enter $\mathcal{N}(p,\varepsilon)$, all concentration $\lambda_i(s)$, $i=1,\dots,p$ tend to ∞.

Before giving the proof of Theorem 3.1, we state the following result which is an immediate corollary of Theorem 3.1.

Corollary 3.2. *Let K be a positive function satisfying* **($\mathbf{H_1}$)**, **($\mathbf{H_2}$)**, *and $(f)_\beta$, $\beta\in(1,n)$. If (1.2) has no solution then the only critical points at infinity of J are*

$$\sum_{i=1}^{p}\frac{\delta_{(y_i,\infty)}}{K(y_i)^{\frac{n-2\sigma}{2}}},\quad \text{where } (y_1,\dots,y_p)\in(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+)\cup\mathcal{C}^{\infty}_{\leq n-2\sigma}\cup\mathcal{C}^{\infty}.$$

We recall, in [2, Theorem 3.1] and [3, Proposition 2.5], the authors studied the subject of Theorem 3.1 and provided the description of the critical points at infinity of J when the flatness order $\beta(y)\in(1,n-2\sigma]$ for any $y\in\mathcal{K}$ and $\beta(y)\in(n-2\sigma,n)$ for any $y\in\mathcal{K}$ respectively. In these two paper it is proved that

Proposition 3.3 ([2]). *Assume that $\beta(y)\in(1,n-2\sigma]$ for any $y\in\mathcal{K}$. Under assumption* **($\mathbf{H_1}$)**, *there exists a bounded pseudo-gradient W_1 in $V(p,\varepsilon)$, $p\geq 1$, satisfying (i), (ii), and (iii) of Theorem 3.1, where $(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+)\cup\mathcal{C}^{\infty}_{\leq n-2\sigma}\cup\mathcal{C}^{\infty}$ is replaced by $\mathcal{C}^{\infty}_{\leq n-2\sigma}$ in (iii).*

Proposition 3.4 ([3]). *Assume that $\beta(y)\in(n-2\sigma,n)$ for any $y\in\mathcal{K}$. There exists a bounded pseudo-gradient W_2 in $V(p,\varepsilon)$, $p\geq 1$, satisfying (i), (ii), and (iii) of Theorem 3.1, where $(\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+)\cup\mathcal{C}^{\infty}_{\leq n-2\sigma}\cup\mathcal{C}^{\infty}$ is replaced by $\mathcal{K}_{>n-2\sigma}\cap\mathcal{K}^+$ in (iii).*

We now state the proof of Theorem 3.1.

Proof of Theorem 3.1. Let

$$u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)}\in V_\delta(p,\varepsilon),\quad p\geq 1.$$

For any $i=1,\dots,p$ there exists $y_i\in\mathcal{K}$ such that $\lambda_i|a_i-y_i|<\delta$ and $y_i\neq y_j$ for all $1\leq i\neq j\leq p$. According to two above-mentioned results, we only consider here the situation where u is expressed as follows

$$u=\sum_{i=1}^{q}\alpha_i\delta_{(a_i,\lambda_i)}+\sum_{i=q+1}^{p}\alpha_i\delta_{(a_i,\lambda_i)}=u_1+u_2,\quad 1\leq q<p,$$

where $\beta(y_i)\in(1,n-2\sigma]$ for all $i=1,\dots,q$ and $\beta(y_i)\in(n-2\sigma,n)$, for all $i=q+1,\dots,p$.

The construction of the required pseudo-gradient $W(u)$ will depend on the following three statements. In each statement, we construct an appropriate pseudo-gradient V_i satisfying the requirements of Theorem 3.1 and the global pseudo-gradient W in $V_\delta(p,\varepsilon)$ will be a convex combination of V_i, $i=1,2,3$.

In the next reasoning we denote β_i instead of $\beta(y_i)$ for each i.

Statement 1: $u_1=\sum_{i=1}^{p-1}\alpha_i\delta_{(a_i,\lambda_i)}$ and $u_2=\alpha_p\delta_{(a_p,\lambda_p)}$ with $\beta_i+\beta_p-2\dfrac{\beta_i\beta_p}{n-2\sigma}>0$, $1\leq i\leq p-1$. For any $i=1,\dots,p-1$, we claim the following

$$\varepsilon_{ip}=o\left(\frac{1}{\lambda_i^{\beta_i}}\right)+o\left(\frac{1}{\lambda_p^{\beta_p}}\right)\quad\text{if } \varepsilon \quad\text{is small.}\tag{3.1}$$

Indeed, let γ be a small positive constant. We have

$$\varepsilon_{ip}\sim\frac{1}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}},\quad i=1,\dots,p-1.$$

If $\lambda_i^{\frac{n-2\sigma}{2}}\geq\frac{1}{\gamma}\lambda_p^{\beta_p-\frac{n-2\sigma}{2}}$, then $\varepsilon_{ip}\leq\gamma\frac{1}{\lambda_p^{\beta_p}}=o\left(\frac{1}{\lambda_p^{\beta_p}}\right)$ if γ is small. If $\lambda_i^{\frac{n-2\sigma}{2}}\leq\frac{1}{\gamma}\lambda_p^{\beta_p-\frac{n-2\sigma}{2}}$, then β_p will be strictly larger than $\frac{n-2\sigma}{2}$ since $\lambda_i>\varepsilon^{-1}$ and ε is arbitrary small. Thus,

$$\lambda_i^{\frac{n-2\sigma}{2\beta_p-(n-2\sigma)}}\leq\frac{1}{\gamma^{\frac{2}{2\beta_p-(n-2\sigma)}}}\lambda_p.$$

Therefore,

$$\frac{1}{\lambda_p^{\frac{n-2\sigma}{2}}}\leq\frac{1}{\gamma^{\frac{n-2\sigma}{2\beta_p-(n-2\sigma)}}}\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}\frac{n-2\sigma}{2\beta_p-(n-2\sigma)}}}.$$

Consequently,

$$\varepsilon_{ip}\leq\frac{1}{\gamma^{\frac{n-2\sigma}{2\beta_p-(n-2\sigma)}}}\frac{1}{\lambda_i^{\frac{n-2\sigma}{2}(1+\frac{n-2\sigma}{2\beta_p-(n-2\sigma)})}}.$$

Using the fact that $\beta_i + \beta_p - 2\frac{\beta_i\beta_p}{n-2\sigma} > 0$, we derive that

$$\frac{n-2\sigma}{2}(1+\frac{n-2\sigma}{2\beta_p-(n-2\sigma)}) > \beta_i$$

and therefore $\varepsilon_{ip} = o\left(\frac{1}{\lambda_i^{\beta_i}}\right)$ for ε small. Claim (3.1) is then valid.

The construction of the required pseudo-gradient V_1 in this statement depends to the two following three cases.

Case 1.1: $(y_1,\ldots,y_{p-1}) \in \mathcal{C}^{\infty}_{\leq n-2\sigma}$ and $y_p \in (\mathcal{K}_{>n-2\sigma} \cap \mathcal{K}^+)$. In this case, we set $V_1^1(u) = W_1(u_1) + W_2(u_2)$ where W_1 and W_2 are the pseudo-gradients defined in Propositions 3.3 and 3.4 respectively. From the estimate (i) of Proposition 3.3 we have

$$\langle \partial J(u), W_1(u_1)\rangle \leq -c\left(\sum_{i=1}^{p-1}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{1\leq i\neq j\leq p-1}\varepsilon_{ij}\right) + \sum_{i=1}^{p-1} O\left(\varepsilon_{ip}\right).$$

In addition, from the estimate (i) of Proposition 3.4, we have

$$\langle \partial J(u), W_2(u_2)\rangle \leq -c\left(\frac{1}{\lambda_p^{\beta_p}} + \frac{|\nabla K(a_p)|}{\lambda_p}\right) + \sum_{i=1}^{p-1} O\left(\varepsilon_{ip}\right).$$

Using claim (3.1), we obtain

$$\langle \partial J(u), V_1^1(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{j\neq i}\varepsilon_{ij}\right).$$

The properties of W_1 and W_2 in Propositions 3.3 and 3.4 show that a concentration phenomenon happens in this case in the sense that all components $\lambda_i(s)$, $i = 1,\ldots,p$, tend to $+\infty$ as s tends to $+\infty$ along the flow lines of V_1^1.

Case 1.2: $y_p \notin (\mathcal{K}_{>n-2\sigma} \cap \mathcal{K}^+)$. We use $W_2(u_2)$ which is the pseudo-gradient defined in Proposition 3.4. In this case $W_2(u_2)$ satisfies the Palais–Smale condition on its flow lines in the sense that $\lambda_p(s)$ remains bounded along $u_2(s) = \alpha_p(s)\delta_{(a_p(s),\lambda_p(s))}$. Moreover, by estimate (i) of Proposition 3.4 we have

$$\langle \partial J(u), W_2(u_2)\rangle \leq -c\left(\frac{1}{\lambda_p^{\beta_p}} + + \frac{|\nabla K(a_p)|}{\lambda_p}\right) + \sum_{i=1}^{p-1} O(\varepsilon_{ip}).$$

Using claim (3.1), we obtain

$$\langle \partial J(u), W_2(u_2)\rangle \leq -c\left(\frac{1}{\lambda_p^{\beta_p}} + + \frac{|\nabla K(a_p)|}{\lambda_p}\right) + \sum_{i=1}^{p-1} o\left(\frac{1}{\lambda_i^{\beta_i}}\right).$$

Let

$$I = \left\{i \mid 1 \leq i \leq p \text{ and } \lambda_i^{\beta_i} \geq \frac{1}{2}\lambda_p^{\beta_p}\right\}.$$

From the above inequality, we get

$$\langle \partial J(u), W_2(u_2)\rangle \leq -c\left(\sum_{i\in I}\frac{1}{\lambda_i^{\beta_i}}++\frac{|\nabla K(a_p)|}{\lambda_p}\right)+\sum_{i\notin I} o\left(\frac{1}{\lambda_i^{\beta_i}}\right).$$

Observe that under $(f)_\beta$-condition, we have

$$|\nabla K(a_i)| \sim |a_i - y_i|^{\beta_i - 1}, \quad i = 1, \dots, p.$$

Using the fact that $\lambda_i|a_i - y_i| \leq \delta$, we get

$$\frac{|\nabla K(a_i)|}{\lambda_i} \leq \frac{c_\delta}{\lambda_i^{\beta_i}}, \quad i = 1, \dots, p.$$

We then have

$$\langle \partial J(u), W_2(u_2)\rangle \leq -c\sum_{i\in I}\left(\frac{1}{\lambda_i^{\beta_i}}++\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{i\notin I} o\left(\frac{1}{\lambda_i^{\beta_i}}\right).$$

For any $i = 1, \dots, p$, we set

$$Z_i(u) = \alpha_i\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}.$$

Using the estimate (iii) of Proposition A_1 of [2], we have

$$\langle \partial J(u), Z_i(u)\rangle = 2J(u)\left(\frac{n-2\sigma}{2n}\beta_i c_i\frac{\alpha_i^2}{K(a_i)}\frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta_i}} - \tilde{c}\sum_{j\neq i}\alpha_i\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}\right) + \sum_{j=1}^p o\left(\frac{1}{\lambda_j^{\beta_j}}\right) + \sum_{j\neq i} o(\varepsilon_{ij}), \quad (3.2)$$

where

$$c_i = c_0^{\frac{2n}{n-2\sigma}}\int_{\mathbb{R}^n}\frac{|x_1|^{\beta_i}}{(1+|x|^2)^n}\,dx$$
$$\tilde{c} = c_0^{\frac{2n}{n-2\sigma}}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^{\frac{n+2\sigma}{2}}}.$$

Using the fact that

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \leq -c\varepsilon_{ij}, \quad 1 \leq i \neq j \leq p,$$

we obtain from (3.2).

$$\left\langle \partial J(u), -\sum_{i\in I} Z_i(u)\right\rangle \leq -c\sum_{i\in I, j\neq i}\varepsilon_{ij} + \sum_{i\in I} O\left(\frac{1}{\lambda_i^{\beta_i}}\right) + \sum_{i\notin I} o\left(\frac{1}{\lambda_i^{\beta_i}}\right) \quad (3.3)$$

Therefore, for $m > 0$ being small enough, we get

$$\left\langle \partial J(u), W_2(u_2) - m\sum_{i\in I} Z_i(u)\right\rangle \leq -c\left(\sum_{i\in I}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{j\neq i}\varepsilon_{ij}\right) + \sum_{i\notin I} o\left(\frac{1}{\lambda_i^{\beta_i}}\right).$$

Let $J_1 = \{1,\dots,p\}\setminus I$ and let $\hat{u}_1 = \sum_{i\in J_1}\alpha_i\delta_{(a_i,\lambda_i)}$. For any $i\in J_1$, $\beta_i \leq n-2\sigma$. Thus, by using the pseudo-gradient W_1 of Proposition 3.3, we have

$$\langle \partial J(u), W_1(\hat{u}_1)\rangle \leq -c\left(\sum_{i\in J_1}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j\in J_1}\varepsilon_{ij}\right) + \sum_{i\in J_1, j\in I} O\left(\varepsilon_{ij}\right).$$

Let $m' > 0$ be small enough and let

$$V_1^2(u) = W_2(u_2) - m\sum_{i\in I} Z_i(u) + m' W_1(\hat{u}_1).$$

These above two inequalities yield

$$\langle \partial J(u), V_1^2(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right).$$

By construction, V_1^2 satisfies the Palais–Smale condition on its flow lines, since under the action $W_1(\hat{u}_1)$, all the concentration $\lambda_i, i\in J_1$ satisfies $\lambda_i^{\beta_i} < \frac{1}{2}\lambda_p^{\beta_p}$ and λ_p does not move.

Case 1.3: $(y_1,\dots,y_{p-1}) \notin \mathcal{C}^{\infty}_{\leq n-2\sigma}$. In this case $W_1(u_1)$ defined in Proposition 3.3 satisfies the Palais–Smale condition on its flow lines in the sense that the $\max_{1\leq i\leq p-1}\lambda_i(s)$ remains bounded along $u_1(s) = \sum_{i=1}^{p-1}\alpha_i(s)\delta_{(a_i(s),\lambda_i(s))}$. Moreover, it satisfies

$$\begin{aligned}
\langle \partial J(u), W_1(u_1)\rangle &\leq -c\left(\sum_{i=1}^{p-1}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{1\leq i\neq j\leq p-1}\varepsilon_{ij}\right) + \sum_{i=1}^{p-1} O(\varepsilon_{ip}) \\
&\leq -c\left(\sum_{i=1}^{p-1}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{1\leq i\neq j\leq p-1}\varepsilon_{ij}\right) + o\left(\frac{1}{\lambda_p^{\beta_p}}\right). \qquad (3.4)
\end{aligned}$$

If $\lambda_p^{\beta_p} > \frac{1}{2}\lambda_1^{\beta_1}$, we set $V_1^3(u) = W_1(u_1)$. From the above inequality and claim (3.1), we have

$$\langle \partial J(u), V_1^3(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right).$$

If $\lambda_p^{\beta_p} < \frac{1}{2}\lambda_1^{\beta_1}$, we set $Z(u) = (-\sum_{k=1}^{n} b_k(y_p))Z_p(u)$. Observe that $Z(u)$ satisfies the Palais–Smale condition, since λ_1 does not move under the action of Z. Moreover, by the expansion (3.2), we have

$$\langle \partial J(u), Z(u)\rangle \leq -\frac{c}{\lambda_p^{\beta_p}} + \sum_{i=1}^{p-1} O(\varepsilon_{ip}) \leq -\frac{c}{\lambda_p^{\beta_p}} + \sum_{i=1}^{p-1} o\left(\frac{1}{\lambda_i^{\beta_i}}\right). \tag{3.5}$$

In this situation, we set $V_1^3(u) = Z(u) + W_1(u_1)$. From (3.4) (3.5), we have

$$\langle \partial J(u), V_1^3(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right).$$

This conclude the construction of the required pseudo-gradient V_1 in this first statement. It is defined by a convex combination of V_1^1, V_1^2 and V_1^3.

Statement 2:

$u_1 = \sum_{i=1}^{p-1}\alpha_i\delta_{(a_i,\lambda_i)}$ and $u_2 = \alpha_p\delta_{(a_p,\lambda_p)}$ with $\beta_i + \beta_p - 2\frac{\beta_i\beta_p}{n-2\sigma} \geq 0,\ 1 \leq i \leq p-1$

and there exists at least $i_0 \in \{1,\dots,p-1\}$ such that $\beta_{i_0} + \beta_p - 2\frac{\beta_{i_0}\beta_p}{n-2\sigma} = 0$. Setting

$$A_p = \left\{i \mid 1 \leq i \leq p-1 \text{ and } \beta_i + \beta_p - 2\frac{\beta_i\beta_p}{n-2\sigma} = 0\right\}.$$

It is easy to verify that $\frac{n-2\sigma}{2} < \beta_i$ for all $i \in A_p$. We introduce the following Lemma.

Lemma 3.5. *Under assumption* **($\mathbf{H_2}$)**, *there exists a bounded pseudo-gradient $Y_1(u)$ satisfying*

$$\langle \partial J(u), Y_1(u)\rangle \leq -c\sum_{i\in A_p}\left(\frac{1}{\lambda_i^{\beta_i}} + \varepsilon_{ip}\right) + \sum_{i\notin A_p} o\left(\frac{1}{\lambda_i^{\beta_i}}\right).$$

Moreover, $\max_{1\leq i\leq p}\lambda_i(s)$ *remains bounded along the associated flow-lines*

$$u(s) = \sum_{i=1}^{p}\alpha_i(s)\,\delta_{(a_i(s),\lambda_i(s))}.$$

Proof. For any $1 \leq i \neq j \leq p$, we have

$$\varepsilon_{ij} = \left(\frac{2}{(1-\cos d(a_i,a_j))\lambda_i\lambda_j}\right)^{\frac{n-2\sigma}{2}}(1+o(1)) = 2^{\frac{n-2\sigma}{2}}\frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\big(1+o(1)\big)$$

Thus,

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = -\frac{n-2\sigma}{2}2^{\frac{n-2\sigma}{2}}\frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\big(1+o(1)\big).$$

Estimate (3.2) is then reduced to

$$\langle \partial J(u), Z_i(u)\rangle = (n-2\sigma)J(u)\left(2^{\frac{n-2\sigma}{2}}\tilde{c}\sum_{j\neq i}\alpha_i\alpha_j 2^{\frac{n-2}{2}}\frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\right.$$
$$\left. + \alpha_i^2\frac{c_i\beta_i}{nK(y_i)}\frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta_i}}\right) + \sum_{j=1}^p o\left(\frac{1}{\lambda_j^{\beta_j}}\right) + o\left(\sum_{j\neq i}\varepsilon_{ij}\right), \quad i=1,\dots,p.$$

Using the fact that $\alpha_i^{\frac{4}{n-2\sigma}}J(u)^{\frac{n}{n-2\sigma}}K(a_i) = 1+o(1)$, we get

$$\alpha_i\alpha_j = \frac{1}{\Big(K(a_i)K(a_j)\Big)^{\frac{n-2\sigma}{4}}}J(u)^{\frac{-n}{2}}(1+o(1))$$
$$\alpha_i^2 = \frac{1}{K(a_i)^{\frac{n-2\sigma}{2}}}J(u)^{\frac{-n}{2}}(1+o(1)).$$

Therefore,

$$\langle \partial J(u), Z_i(u)\rangle = (n-2\sigma)J(u)^{1-\frac{n}{2}}\left(2^{\frac{n-2\sigma}{2}}\tilde{c}\sum_{j\neq i}\frac{G(y_i,y_j)}{(K(y_i)K(y_j))^{\frac{n-2\sigma}{4}}}\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\right.$$
$$\left. + \frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta_i}}\right) + \sum_{j=1}^p o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\sum_{j\neq i}\varepsilon_{ij}\right). \tag{3.6}$$

Let γ_0 be a fixed positive constant small enough and let

$$I_{\gamma_0} = \left\{ i\in A_p \mid \frac{2^{\frac{n-2\sigma}{2}}\tilde{c}}{\big(K(y_i)K(y_j)\big)^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}} < (1-\gamma_0)\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^n b_k(y_i)|}{\lambda_i^{\beta_i}}\right\}.$$

The proof of Lemma 3.5 depends on the following two cases.

Case 1: $I_{\gamma_0}\neq\emptyset$. In this case, we move each λ_i, $i\in I_{\gamma_0}$, according to the following differential equation

$$\dot{\lambda}_i = \left(-\sum_{k=1}^n b_k(y_i)\right)\lambda_i.$$

The corresponding vector field is

$$X_i(u) = \left(-\sum_{k=1}^n b_k(y_i)\right)Z_i(u).$$

Observe that $X_i, i \in I_{\gamma_0}$ satisfies the Palais–Smale condition along its flow lines, since for any $i \in I_{\gamma_0}$ we have $\lambda_i^{\beta_i - \frac{n-2\sigma}{2}} \le M \lambda_p^{\frac{n-2\sigma}{2}}$, where M is a fixed positive constant and λ_p does not move under the action of X_i. Using estimate (3.6), we have

$$\langle \partial J(u), X_i(u)\rangle = (n-2\sigma)J(u)^{1-\frac{n}{2}} \Bigg(-\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \frac{(\sum_{k=1}^n b_k(y_i))^2}{\lambda_i^{\beta_i}} - 2^{\frac{n-2\sigma}{2}} \tilde{c} \sum_{j\neq i} \frac{(\sum_{k=1}^n b_k(y_i))}{(K(y_i)K(y_j))^{\frac{n-2\sigma}{4}}} \frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}} \Bigg) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta_j}}\right) + o\Bigg(\sum_{j\neq i} \varepsilon_{ij}\Bigg).$$

In our statement, we have

$$\beta_i + \beta_p - 2\frac{\beta_i\beta_p}{n-2\sigma} \ge 0, \quad i = 1, \dots, p-1.$$

This implies that $\beta_i < n - 2\sigma$ for all $i = 1, \dots, p-1$ and therefore

$$\varepsilon_{ij} = o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\frac{1}{\lambda_j^{\beta_j}}\right), \text{ as } \varepsilon \to 0, \quad i,j = 1,\dots,p-1, \ i \neq j. \tag{3.7}$$

Thus for any $i \in I_{\gamma_0}$, the following assertion holds

$$\langle \partial J(u), X_i(u)\rangle = (n-2\sigma)J(u)^{1-\frac{n}{2}} \Bigg(-\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \frac{(\sum_{k=1}^n b_k(y_i))^2}{\lambda_i^{\beta_i}} - 2^{\frac{n-2\sigma}{2}} \tilde{c} \sum_{i=1}^{n} \frac{|\sum_{k=1}^n b_k(y_i)|}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}} \frac{G(y_i,y_p)}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}} \Bigg) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_j^{\beta_j}}\right).$$

If $(\sum_{k=1}^n b_k(y_i)) < 0$, then from the above inequality, we obtain

$$\langle \partial J(u), X_i(u)\rangle \le -c\left(\frac{1}{\lambda_i^{\beta_i}} + \varepsilon_{ip}\right) + \sum_{j\neq i} o\left(\frac{1}{\lambda_j^{\beta_j}}\right) \le \frac{-c}{\lambda_i^{\beta_i}} + \sum_{j\neq i} o\left(\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3.8}$$

If $(\sum_{k=1}^n b_k(y_i)) > 0$, using the fact that $i \in I_{\gamma_0}$, we obtain

$$\langle \partial J(u), X_i(u)\rangle \le (n-2\sigma)J(u)^{1-\frac{n}{2}} \Bigg(-\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \frac{(\sum_{k=1}^n b_k(y_i))^2}{\lambda_i^{\beta_i}} - (1-\gamma_0)\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \sum_{j\neq i} \frac{(\sum_{k=1}^n b_k(y_i))^2}{\lambda_i^{\beta_i}} \Bigg) + \sum_{j\neq i} o\left(\frac{1}{\lambda_j^{\beta_j}}\right) \le \frac{-c}{\lambda_i^{\beta_i}} + \sum_{j\neq i} o\left(\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3.9}$$

Then (3.8) and (3.9) imply that

$$\left\langle \partial J(u), \sum_{i\in I_{\gamma_0}} X_i(u)\right\rangle \le -c\sum_{i\in I_{\gamma_0}} \frac{1}{\lambda_i^{\beta_i}} + \sum_{j\notin I_{\gamma_0}} o\left(\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3.10}$$

For the indices $i\in A_p\setminus I_{\gamma_0}$, we have

$$\frac{2^{\frac{n-2\sigma}{2}}\tilde{c}}{(K(y_i)K(y_j))^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_j)}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}} \ge (1-\gamma_0)\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta_i}}.$$

This implies that

$$\lambda_i^{\beta_i-\frac{n-2\sigma}{2}} \ge m_1\lambda_p^{\frac{n-2\sigma}{2}}, \quad i\in A_p\setminus I_{\gamma_0},$$

where m_1 is a fixed positive constant. Now let $i_0\in I_{\gamma_0}$. We have

$$\lambda_p^{\frac{n-2\sigma}{2}} \ge \frac{1}{M}\lambda_{i_0}^{\beta_{i_0}-\frac{n-2\sigma}{2}}.$$

It follows that

$$\lambda_i \ge m'\lambda_{i_0}, \quad i\in A_p\setminus I_{\gamma_0}, \tag{3.11}$$

since $\beta_i=\beta_{i_0}$ for all $i\in A_p$. Here m' is a fixed positive constant. Using (3.10) and (3.11), we obtain

$$\left\langle \partial J(u), \sum_{i\in I_{\gamma_0}} X_i(u)\right\rangle \le -c\sum_{i\in A_p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{j\notin A_p} o\left(\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3.12}$$

Moreover, as in estimate (3.3), we have

$$\left\langle \partial J(u), -\sum_{i\in A_p} Z_i(u)\right\rangle \le -c\sum_{i\in A_p, j\neq i}\varepsilon_{ij} + \sum_{i\in A_p} O\left(\frac{1}{\lambda_j^{\beta_j}}\right) + \sum_{i\notin A_p} o\left(\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3.13}$$

Let in this case

$$Y_1(u) = \sum_{i\in I_{\gamma_0}} X_i(u) - m\sum_{i\in A_p} Z_i(u),$$

where m is a small positive constant. From (3.12) and (3.13), we have

$$\begin{aligned}\langle \partial J(u), Y_1(u)\rangle &\le -c\sum_{i\in A_p}\left(\frac{1}{\lambda_i^{\beta_i}} + \sum_{j\neq i}\varepsilon_{ij}\right) + \sum_{j\notin A_p} o\left(\frac{1}{\lambda_j^{\beta_j}}\right)\\ &\le -c\sum_{i\in A_p}\left(\frac{1}{\lambda_i^{\beta_i}} + \varepsilon_{ip}\right) + \sum_{j\notin A_p} o\left(\frac{1}{\lambda_j^{\beta_j}}\right).\end{aligned}$$

This conclude the proof of Lemma 3.5 in this case.

Case 2: $I_{\gamma_0}=\emptyset$. It follows that for any $i\in A_p$, we have

$$\frac{2^{\frac{n-2\sigma}{2}}\tilde{c}}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_p)}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}} \ge (1-\gamma_0)\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^n b_k(y_i)|}{\lambda_i^{\beta_i}}.$$

Let

$$I = \left\{ i \in A_p \mid (1-2\gamma_0)\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^n b_k(y_i)|}{\lambda_i^{\beta_i}} < \frac{2^{\frac{n-2\sigma}{2}}\tilde{c}}{\big(K(y_i)K(y_p)\big)^{\frac{n-2\sigma}{4}}} \right.$$
$$\left. \text{and } \frac{G(y_i,y_p)}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}} < (1+2\gamma_0)\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^n b_k(y_i)|}{\lambda_i^{\beta_i}} \right\}.$$

For any $i \in I$, we have

$$(1-2\gamma_0)^{\frac{2\beta_p}{n-2\sigma}}\frac{K_{ip}}{\lambda_i^{\beta_i}} < \frac{1}{\lambda_p^{\beta_p}} < (1+2\gamma_0)^{\frac{2\beta_p}{n-2\sigma}}\frac{K_{ip}}{\lambda_i^{\beta_i}},$$

where

$$K_{ip} = \left(\frac{\beta_i c_i\big(K(y_i)K(y_p)\big)^{\frac{n-2\sigma}{4}}|\sum_{k=1}^n b_k(y_i)|}{n2^{\frac{n-2\sigma}{2}}\tilde{c}K(y_i)^{1+\frac{n-2\sigma}{2}}G(y_i,y_p)}\right)^{\frac{2\beta_p}{n-2\sigma}},$$

since $(\beta_i - \frac{n-2\sigma}{2})\frac{2\beta_p}{n-2\sigma} = \beta_i$. Moreover, we have

$$(1+2\gamma_0)^{\frac{-2\beta_p}{n-2\sigma}}\frac{K_{ip}^{-1}}{\lambda_p^{\beta_p}} < \frac{1}{\lambda_i^{\beta_i}} < (1-2\gamma_0)^{\frac{-2\beta_p}{n-2\sigma}}\frac{K_{ip}^{-1}}{\lambda_p^{\beta_p}}. \tag{3.14}$$

Using estimate (3.6), we have

$$\begin{aligned}\langle \partial J(u), Z_p(u)\rangle = (n-2\sigma)J(u)^{1-\frac{n}{2}}\Bigg(&\frac{\beta_p c_p}{nK(y_p)^{1+\frac{n-2\sigma}{2}}}\frac{\sum_{k=1}^n b_k(y_p)}{\lambda_p^{\beta_p}} \\ &+ 2^{\frac{n-2\sigma}{2}}\tilde{c}\sum_{i\neq p}\frac{G(y_i,y_p)}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}}\frac{1}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}\Bigg) \\ &+\sum_{j=1}^p o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\sum_{j\neq p}\varepsilon_{jp}\right).\end{aligned} \tag{3.15}$$

Observe that for any $i \neq p$ ad $i \notin A_p$, we have

$$\beta_i + \beta_p - \frac{2\beta_i\beta_p}{n-2\sigma} > 0.$$

Therefore, by claim 3.1, we get $\varepsilon_{ip} = o\Big(\frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\frac{1}{\lambda_p^{\beta_p}}\Big)$, as ε is small. (3.15) is then reduced to

$$\begin{aligned}\langle \partial J(u), Z_p(u)\rangle = (n-2\sigma)J(u)^{1-\frac{n}{2}}\Bigg(&\frac{\beta_p c_p}{nK(y_p)^{1+\frac{n-2\sigma}{2}}}\frac{\sum_{k=1}^n b_k(y_p)}{\lambda_p^{\beta_p}} \\ &+ 2^{\frac{n-2\sigma}{2}}\tilde{c}\sum_{i\in A_p}\frac{G(y_i,y_p)}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}}\frac{1}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{p} o\left(\frac{1}{\lambda_i^{\beta_i}}\right)+o\left(\sum_{j\in A_p}\varepsilon_{jp}\right)\\
&=(n-2\sigma)J(u)^{1-\frac{n}{2}}\left(\frac{\beta_p c_p}{nK(y_p)^{1+\frac{n-2\sigma}{2}}}\frac{\sum_{k=1}^{n}b_k(y_p)}{\lambda_p^{\beta_p}}\right.\\
&\left.+2^{\frac{n-2\sigma}{2}}\tilde{c}\sum_{i\in I}\frac{G(y_i,y_p)}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}}\frac{1}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}\right)\\
&+\sum_{j\in A_p\setminus I}^{p} O\left(\varepsilon_{jp}\right)+\sum_{i=1}^{p} o\left(\frac{1}{\lambda_i^{\beta_i}}\right)+o\left(\sum_{j\in A_p}\varepsilon_{jp}\right).
\end{aligned}\tag{3.16}
$$

Using the fact that

$$
\begin{aligned}
(1-2\gamma_0)\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{\lambda_i^{\beta_i}}&<\sum_{i\in I}\frac{2^{\frac{n-2\sigma}{2}}\tilde{c}}{\left(K(y_i)K(y_p)\right)^{\frac{n-2\sigma}{4}}},\\
\frac{G(y_i,y_p)}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}&<(1+2\gamma_0)\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{\lambda_i^{\beta_i}},
\end{aligned}
$$

we get from (3.14) that

$$
L_1<2^{\frac{n-2\sigma}{2}}\tilde{c}\sum_{i\in I}\frac{1}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_p)}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}<L_2,
$$

where,

$$
\begin{aligned}
L_1&=\frac{1-2\gamma_0}{(1+2\gamma_0)^{\frac{2\beta_p}{n-2\sigma}}}\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{K_{ip}\lambda_p^{\beta_p}},\\
L_2&=\frac{1+2\gamma_0}{(1-2\gamma_0)^{\frac{2\beta_p}{n-2\sigma}}}\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{K_{ip}\lambda_p^{\beta_p}}.
\end{aligned}
$$

Thus, from (3.16), we obtain the following two inequalities

$$
\widetilde{L}_1<\langle\partial J(u),Z_p(u)\rangle<\widetilde{L}_2,\tag{3.17}
$$

where,

$$
\begin{aligned}
\widetilde{L_1}&=(n-2\sigma)J(u)^{1-\frac{n}{2}}\frac{\beta_p c_p}{n}\frac{\sum_{k=1}^{n}b_k(y_p)}{K(y_p)^{1+\frac{n-2\sigma}{2}}}\frac{1-2\gamma_0}{(1+2_0)^{\frac{2\beta_p}{n-2\sigma}}}\\
&\times\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{K_{ip}}\frac{1}{\lambda_p^{\beta_p}}+O\left(\sum_{i\in A_p\setminus I}\varepsilon_{ip}\right)+\sum_{j=1}^{p}o\left(\frac{1}{\lambda_j^{\beta_j}}\right).\\
\widetilde{L_2}&=(n-2\sigma)J(u)^{1-\frac{n}{2}}\frac{\beta_p c_p}{n}\frac{\sum_{k=1}^{n}b_k(y_p)}{K(y_p)^{1+\frac{n-2\sigma}{2}}}\frac{1+2\gamma_0}{(1-2\gamma_0)^{\frac{2\beta_p}{n-2\sigma}}}
\end{aligned}
$$

$$\times\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{K_{ip}}\frac{1}{\lambda_p^{\beta_p}}+O\left(\sum_{i\in A_p\setminus I}\varepsilon_{ip}\right)+\sum_{j=1}^{p}o\left(\frac{1}{\lambda_j^{\beta_j}}\right).$$

Let

$$S=\sum_{i\in I}\frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{K_{ip}}.$$

For γ small enough, the sign of $\frac{\beta_p c_p}{n}\frac{\sum_{k=1}^{n}b_k(y_p)}{K(y_p)^{1+\frac{n-2\sigma}{2}}}+\frac{1\pm 2\gamma_0}{(1\mp 2\gamma_0)^{\frac{2\beta_p}{n-2\sigma}}}S$ is the sign of

$$\frac{\beta_p c_p}{n}\frac{\sum_{k=1}^{n}b_k(y_p)}{K(y_p)^{1+\frac{n-2\sigma}{2}}}+S,$$

which is non zero by assumption **($\mathbf{H_2}$)**. Setting

$$X(u)=-\ \mathrm{sign}\ \Big(\frac{\beta_p c_p}{n}\frac{\sum_{k=1}^{n}b_k(y_p)}{K(y_p)^{1+\frac{n-2\sigma}{2}}}+S\Big)Z_p(u).$$

Observe that by (3.14), we have $\lambda_p^{\beta_p}\le M\lambda_i^{\beta_i}$ for any $i\in I$ where M is a fixed positive constant which depends only on K. Since λ_i does not move under the action of X, the Palais–Smale condition then satisfied along the flow lines of X. Moreover, by (3.17), we get

$$\langle\partial J(u),X(u)\rangle\le-\frac{c}{\lambda_p^{\beta_p}}+\sum_{i\in A_p\setminus I}O(\varepsilon_{ip})+o\left(\sum_{i=1}^{p-1}\frac{1}{\lambda_i^{\beta_i}}\right).$$

From (3.14), we have

$$\langle\partial J(u),X(u)\rangle\le-c\sum_{i\in I}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in A_p\setminus I}O(\varepsilon_{ip})+o\left(\sum_{i\notin I}\frac{1}{\lambda_i^{\beta_i}}\right).$$

Observe that for any $i\in I$, we have $\varepsilon_{ip}\sim\frac{1}{\lambda_i^{\beta_i}}$. We therefore have

$$\langle\partial J(u),X(u)\rangle\le-c\sum_{i\in I}\left(\frac{1}{\lambda_i^{\beta_i}}+\varepsilon_{ip}\right)+\sum_{i\in A_p\setminus I}O(\varepsilon_{ip})+o\left(\sum_{i\notin I}\frac{1}{\lambda_i^{\beta_i}}\right).\tag{3.18}$$

We now take care for the indices $i\in A_p\setminus I$ For any $i\in A_p\setminus I$, we have

$$\frac{2^{\frac{n-2\sigma}{2}}\tilde{c}}{(K(y_i)K(y_p))^{\frac{n-2\sigma}{4}}}\frac{G(y_i,y_p)}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}\ge\frac{(1+2\gamma_0)\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}}\frac{|\sum_{k=1}^{n}b_k(y_i)|}{\lambda_i^{\beta_i}}.\tag{3.19}$$

We decrease all $\lambda_i, i\in A_p\setminus I$ according to the differential equation $\dot{\lambda}_i=-\lambda_i$. The related pseudo-gradient is

$$X'(u)=-\sum_{i\in A_p\setminus I}Z_i(u).$$

Using estimate (3.6), we have

$$\begin{aligned}\langle \partial J(u), X'(u)\rangle = -(n-2\sigma)J(u)^{1-\frac{n}{2}} \Bigg(&\sum_{i\in A_p\setminus I} \frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta_i}} \\ &+ 2^{\frac{n-2\sigma}{2}}\tilde{c} \sum_{i\in A_p\setminus I}\sum_{j\neq i} \frac{G(y_i,y_j)}{(K(y_i)K(y_j))^{\frac{n-2\sigma}{4}}} \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2\sigma}{2}}}\Bigg) \\ &+ \sum_{j=1}^{p} o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\sum_{i\in A_p\setminus I, j\neq i} \varepsilon_{ij}\right).\end{aligned}$$

Since $\varepsilon_{ij} = o\Big(\frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\frac{1}{\lambda_j^{\beta_j}}\Big)$ for all $i\in A_p\setminus I, j\neq i$ and $j\neq p$, we get

$$\begin{aligned}\langle \partial J(u), X'(u)\rangle = -(n-2\sigma)J(u)^{1-\frac{n}{2}} \Bigg(&\sum_{i\in A_p\setminus I} \frac{\beta_i c_i}{nK(y_i)^{1+\frac{n-2\sigma}{2}}} \frac{\sum_{k=1}^n b_k(y_i)}{\lambda_i^{\beta_i}} \\ &+ 2^{\frac{n-2\sigma}{2}}\tilde{c} \sum_{i\in A_p\setminus I} \frac{G(y_i,y_p)}{(K(y_i)K(y_p)\,)^{\frac{n-2\sigma}{4}}} \frac{1}{(\lambda_i\lambda_p)^{\frac{n-2\sigma}{2}}}\Bigg) + \sum_{j=1}^{p} o\left(\frac{1}{\lambda_i^{\beta_i}}\right).\end{aligned}$$

Using inequality (3.19), we obtain

$$\langle \partial J(u), X'(u)\rangle \leq -c\sum_{i\in A_p\setminus I} \varepsilon_{ip} + o\left(\sum_{j=1}^{p} \frac{1}{\lambda_j^{\beta_j}}\right).$$

Again by (3.19), we have

$$\langle \partial J(u), X'(u)\rangle \leq -c\sum_{i\in A_p\setminus I} \left(\varepsilon_{ip} + \frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\sum_{i\in A_p\setminus I} \frac{1}{\lambda_j^{\beta_j}}\right). \tag{3.20}$$

Now for $m>0$ being small enough, let

$$Y_1(u) = X'(u) + mX(u).$$

From (3.18) and (3.20), we get

$$\langle \partial J(u), Y_1(u)\rangle \leq -c\sum_{i\in A_p} \left(\frac{1}{\lambda_j^{\beta_j}} + \varepsilon_{ip}\right) + o\left(\sum_{j\notin A_p} \frac{1}{\lambda_j^{\beta_j}}\right).$$

The proof of Lemma 3.5 is thereby completed. □

In order to complete the construction of the required pseudo-gradient V_2 in this statement 2, we denote the index of A_p such that $\lambda_{i_1} = \min_{i\in A_p} \lambda_i$ by i_1. Set

$$I_1 = \left\{ i \mid 1\leq i\leq p \text{ and } \lambda_i^{\beta_i} \geq \frac{1}{2}\lambda_{i_1}^{\beta_{i_1}} \right\}.$$

Since $\beta_i = \beta_{i_1}$ for all $i \in A_p$, A_p is then included in I_1 and thus,

$$\langle \partial J(u), Y_1(u) \rangle \leq -c \left(\sum_{i \in I_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in A_p} \varepsilon_{ip} \right) + o \left(\sum_{i \notin I_1} \frac{1}{\lambda_i^{\beta_i}} \right). \tag{3.21}$$

We now introduce the following lemma.

Lemma 3.6. *There exists a bounded pseudo-gradient $Y_2(u)$ satisfying*

$$\langle \partial J(u), Y_2(u) \rangle \leq -c \left(\sum_{i \in I_1^c} \left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j \in I_1^c} \varepsilon_{ij} \right) + \sum_{i \in I_1^c, j \in I_1} O(\varepsilon_{ij}).$$

Moreover, $\max_{1 \leq i \leq p} \lambda_i(s)$ *remains bounded along the associated flow-line*

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) \delta_{(a_i(s), \lambda_i(s))}.$$

Proof. Setting $\hat{u} = \sum_{i \in I_1^c} \alpha_i \delta_{(a_i, \lambda_i)}$. Consider the following three cases for $\hat{u}$.

Case 1: $I_1^c = \{p\}$. Let in this case $Y_2(u) = W_2(\hat{u})$ where W_2 is the pseudo-gradient of Proposition 3.4. It satisfies

$$\langle \partial J(u), Y_2(u) \rangle \leq -c \left(\frac{1}{\lambda_p^{\beta_p}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i=1}^{p-1} O(\varepsilon_{ip}).$$

Case 2: $p \notin I_1^c$. In this case, let $Y_2(u) = W_1(\hat{u})$ where W_1 is the pseudo-gradient defined in Proposition 3.3. It satisfies

$$\langle \partial J(u), Y_2(u) \rangle \leq -c \left(\sum_{i \in I_1^c} \left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j \in I_1^c} \varepsilon_{ij} \right) + \sum_{i \in I_1^c, j \in I_1} O(\varepsilon_{ij}).$$

Case 3: $p \in I_1^c$ and $\sharp I_1^c \geq 2$. In this case, $\hat{u}$ satisfies the condition of statement 1. Namely, for any $i \in I_1^c$ and $i \neq p$, we have $\beta_i + \beta_p - \frac{2\beta_i \beta_p}{n - 2\sigma} > 0$. We apply then $Y_2(u) = V_1(\hat{u})$ where V_1 is the pseudo-gradient defined in statement 1. It satisfies

$$\langle \partial J(u), Y_2(u) \rangle \leq -c \left(\sum_{i \in I_1^c} \left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \neq j \in I_1^c} \varepsilon_{ij} \right) + \sum_{i \in I_1^c, j \in I_1} O(\varepsilon_{ij}).$$

Observe that in all three cases above, $Y_2(u)$ acts on the indices $i \in I^c$. Therefore, it satisfies the Palais–Smale condition on its flow-lines, since for any $i \in I_1^c$, we have $\lambda_i^{\beta_i} \leq \frac{1}{2} \lambda_{i_1}^{\beta_{i_1}}$ and λ_{i_1} does not move. This finishes the proof of Lemma 3.6. □

From (3.7) it follows that

$$\varepsilon_{ij} = o \left(\frac{1}{\lambda_i^{\beta_i}} \right) + o \left(\frac{1}{\lambda_j^{\beta_j}} \right), \quad i, j = 1, \ldots, p-1, \ i \neq j, \tag{3.22}$$

$$\varepsilon_{ip} = o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + o\left(\frac{1}{\lambda_p^{\beta_p}}\right), \quad i \notin A_p. \tag{3.23}$$

Therefore,

$$\langle \partial J(u), Y_2(u)\rangle \leq -c\left(\sum_{i\in I_1^c}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j\in I_1^c}\varepsilon_{ij}\right) + \sum_{i\in I_1} o\left(\frac{1}{\lambda_i^{\beta_i}}\right) + \sum_{i\in A_p} O(\varepsilon_{ip}). \tag{3.24}$$

Let $m>0$ be small enough. From (3.21) and (3.24), we obtain

$$\langle \partial J(u), Y_1(u) + mY_2(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\in A_p}\varepsilon_{ip}\right),$$

and from (3.22) and (3.23), we get

$$\langle \partial J(u), Y_1(u) + mY_2(u)\rangle \leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{i\neq j}\varepsilon_{ij}\right).$$

In this statement, we set $V_2 = Y_1 + mY_2$. No concentration phenomenon happens along the flow lines of V_2.

Statement 3:

$$u_1 = \sum_{i=1}^{p-1}\alpha_i\delta_{(a_i,\lambda_i)} \text{ and } u_2 = \alpha_p\delta_{(a_p,\lambda_p)}$$

and there exists at least $i_0 \in \{1,\dots,p-1\}$ such that $\beta_{i_0} + \beta_p - 2\dfrac{\beta_{i_0}\beta_p}{n-2\sigma} < 0$, or u_2 is a sum of at least two bubbles.

It is easy to check that for any $y_i, y_j \in \mathcal{K}_{>n-2\sigma}$, $i\neq j$, we have $\beta_i + \beta_j - \frac{2\beta_i\beta_j}{n-2\sigma} < 0$. Therefore, for $u = u_1 + u_2$, there exist $1 \leq j_0 \neq j_0 \leq p$ such that

$$\beta_{i_0} + \beta_{j_0} - \frac{2\beta_{i_0}\beta_{j_0}}{n-2\sigma} < 0. \tag{3.25}$$

We order all $\lambda_i^{\beta_i}$, $i = 1,\dots,p$. Without loss of generality, we assume that

$$\lambda_{i_1}^{\beta_{i_1}} \leq \lambda_{i_2}^{\beta_{i_2}} \leq \cdots \lambda_{i_p}^{\beta_{i_p}}.$$

For $M>0$ being a large enough, we define

$$I = \left\{i \mid 1 \leq i \leq p \text{ and } \lambda_i^{\beta_i} \leq M\lambda_{i_1}^{\beta_{i_1}}\right\}.$$

Three cases may occurs.

Case 1: $\sharp I = 1$. In this statement, we have

$$\frac{1}{\lambda_i^{\beta_i}} = o\left(\frac{1}{\lambda_{i_1}^{\beta_{i_1}}}\right) \quad \text{if } M \text{ is large}, \ i \neq i_1.$$

Using expansion (3.2) and the fact that

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\varepsilon_{ij}, \quad i \neq j,$$

we obtain

$$\begin{aligned}\left\langle \partial J(u), -\sum_{i\neq i_1} Z_i(u)\right\rangle &\leq -c\sum_{i\neq j}\varepsilon_{ij} + \sum_{i\neq i_1} O\left(\frac{1}{\lambda_i^{\beta_i}}\right)\\ &\leq -c\sum_{i\neq j}\varepsilon_{ij} + o\left(\frac{1}{\lambda_{i_1}^{\beta_{i_1}}}\right). \qquad (3.26)\end{aligned}$$

We now move λ_{i_1} according to the differential equation

$$\dot{\lambda}_{i_1} = (-\sum_{k=1}^{n} b_k(y_{i_1}))\lambda_{i_1}.$$

The corresponding vector field is

$$X_{i_1}(u) = (-\sum_{k=1}^{n} b_k(y_{i_1}))Z_{i_1}(u),$$

where X_{i_1} satisfies the Palais–Smale condition along its flow lines, and by (3.2), we have

$$\langle \partial J(u), X_{i_1}(u)\rangle \leq -\frac{c}{\lambda_{i_1}^{\beta_{i_1}}} + \sum_{j\neq i_1} O(\varepsilon_{1j}). \qquad (3.27)$$

In this case, let

$$V_3^1(u) = mX_{i_1}(u) + \sum_{i\neq i_1} Z_i(u),$$

where $m > 0$ and small. From the inequalities (3.26) and (3.27), we deduce that

$$\langle \partial J(u), V_3^1(u)\rangle \leq -c\left(\sum_{i=1}^{p}\frac{1}{\lambda_1^{\beta_1}} + \sum_{i\neq j}\varepsilon_{1j}\right)$$

$$\leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i}\right) + \sum_{j\neq i}\varepsilon_{1j}\right).$$

Case 2: $\sharp I \geq 2$ and $\beta_i + \beta_j - \frac{2\beta_i\beta_j}{n-2\sigma} \geq 0$ for all $i \neq j \in I$. We introduce the following Lemma.

Lemma 3.7. *There exists a bounded pseudo-gradient $Y_3(u)$ satisfying*

$$\langle \partial J(u), Y_3(u)\rangle \leq -c\left(\sum_{i\in I}\left(\frac{1}{\lambda_i^{\beta_i}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i\in I}\varepsilon_{ij}\right)+O\left(\sum_{i\in I,j\notin I}\varepsilon_{ij}\right).$$

Moreover, $\max_{1\leq i\leq p}\lambda_i(s)$ *remains bounded along the associated flow line*

$$u(s)=\sum_{i=1}^{p}\alpha_i(s)\delta_{(a_i(s),\lambda_i(s))}.$$

Proof. Let $\hat{u}=\sum_{i\in I}\alpha_i\delta_i$. Then, $\hat{u}$ has to satisfy one of the following situations.

Situation 1: $\beta_i\leq n-2\sigma$ for all $li\in I$. Let in this case, $Y_3(u)=W_1(\hat{u})$ where W_1 is the pseudo-gradient of Proposition 3.3. It satisfies

$$\langle \partial J(u), Y_3(u)\rangle \leq -c\left(\sum_{i\in I}\Big(\frac{1}{\lambda_i^{\beta_i}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i\in I}\varepsilon_{ij}\right)+O\left(\sum_{i\in I,j\notin I}\varepsilon_{ij}\right).$$

Situation 2: There exists only one index $i\in I$ such that $\beta_i>n-2\sigma$. In this case, $\hat{u}$ satisfies either the condition of statement 1 or the condition of statement 2. Let $Y_3(u)=V_i(\hat{u})$ where V_i is the pseudo-gradient defined in the above statement, $i=1,2$. Therefore, it satisfies

$$\langle \partial J(u), Y_3(u)\rangle \leq -c\left(\sum_{i\in I}\left(\frac{1}{\lambda_i^{\beta_i}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i\in I}\varepsilon_{ij}\right)+O\left(\sum_{i\in I,j\notin I}\varepsilon_{ij}\right).$$

Observe that in the above two situations $Y_3(u)$ satisfies the Palais–Smale condition on its flow-lines $u(s)$, since it acts only on the indices $i\in I$ and by (3.25), there exists at least an index $j_0\in\{1,\dots,p\}$ such that $j_0\notin I$. Notice that

$$\lambda_i^{\beta_i}\leq\frac{1}{M}\lambda_j^{\beta_j},\quad i\in I,\ j\notin I.$$

This conclude the proof of Lemma 3.7. □

We now decrease all $\lambda_i, i\notin I$. Using (3.2), we have

$$\begin{aligned}\left\langle \partial J(u), -\sum_{i\notin I}Z_i(u)\right\rangle &\leq -c\sum_{i\notin I,j\neq i}\varepsilon_{ij}+\sum_{i\notin I}O\left(\frac{1}{\lambda_i^{\beta_i}}\right)\\ &\leq -c\sum_{i\notin I,j\neq i}\varepsilon_{ij}+o\left(\frac{1}{\lambda_{i_1}^{\beta_{i_1}}}\right).\end{aligned}$$

Let in this case

$$V_3^2(u)=mY_3(u)-\sum_{i\notin I}Z_i(u),$$

where $m > 0$ and small. Using Lemma 3.7 and the above inequality, we have

$$< \langle \partial J(u), V_3^2(u) \rangle \leq -c \sum_{i \in I} \frac{1}{\lambda_i^{\beta_i}} + \sum_{j \neq i} \varepsilon_{ij} \leq -c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{j \neq i} \varepsilon_{ij} \right)$$

$$\leq -c \left(\sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\beta_i}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

Case 3: $\sharp I \geq 2$ and there exist $i_0 \neq j_0 \in I$ such that

$$\beta_{i_0} + \beta_{j_0} - \frac{2\beta_{i_0}\beta_{j_0}}{n - 2\sigma} < 0.$$

In this case, we claim that

$$\frac{1}{\lambda_{i_0}^{\beta_{i_0}}} = o(\varepsilon_{i_0 j_0}) \quad \text{as } \varepsilon \to 0. \tag{3.28}$$

Indeed,

$$\frac{1}{\lambda_{i_0}^{\beta_{i_0}}} \varepsilon_{i_0 j_0}^{-1} \sim \frac{(\lambda_{i_0}\lambda_{j_0})^{\frac{n-2\sigma}{2}}}{\lambda_{i_0}^{\beta_{i_0}}} = \frac{\lambda_{j_0}^{\frac{n-2\sigma}{2}}}{\lambda_{i_0}^{\beta_{i_0} - \frac{n-2\sigma}{2}}}.$$

Since $i_0, j_0 \in I$, we get $\frac{1}{M} \leq \frac{\lambda_{j_0}^{\beta_{j_0}}}{\lambda_{i_0}^{\beta_{i_0}}} \leq M$. Therefore,

$$\frac{1}{\lambda_{i_0}^{\beta_{i_0}}} \varepsilon_{i_0 j_0}^{-1} \leq M \frac{\lambda_{i_0}^{\frac{\beta_{i_0}}{\beta_{j_0}} \frac{n-2\sigma}{2}}}{\lambda_{i_0}^{\beta_{i_0} - \frac{n-2\sigma}{2}}} \leq M \frac{1}{\lambda_{i_0}^{-\frac{n-2\sigma}{2\beta_{j_0}} \left(\beta_{i_0} + \beta_{j_0} - \frac{2\beta_{i_0}\beta_{j_0}}{n-2\sigma} \right)}} \to 0 \text{ as } \varepsilon \to 0.$$

Hence claim (3.28) is valid. We now decrease all the λ_i, $i = 1, \dots, p$. Let

$$V_3^3(u) = -\sum_{i=1}^{p} Z_i(u).$$

By (3.2), we have

$$\langle \partial J(u), V_3^3(u) \rangle \leq -c \sum_{j \neq i} \varepsilon_{ij} + \sum_{i=1}^{p} O\left(\frac{1}{\lambda_i^{\beta_i}} \right).$$

Observe that for M large we have $\frac{1}{\lambda_i^{\beta_i}} = o\left(\frac{1}{\lambda_{i_1}^{\beta_{i_1}}} \right)$ for all $i \notin I$ and for ε being small enough, $\frac{1}{\lambda_i^{\beta_i}} \sim \frac{1}{\lambda_{i_0}^{\beta_{i_0}}} = o(\varepsilon_{i_0 j_0})$ for all $i \in I$. Therefore,

$$\langle \partial J(u), V_3^3(u) \rangle \leq -c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{\beta_i}} + \sum_{j \neq i} \varepsilon_{ij} \right)$$

$$\leq -c\left(\sum_{i=1}^{p}\left(\frac{1}{\lambda_i^{\beta_i}}+\frac{|\nabla K(a_i)|}{\lambda_i}\right)+\sum_{j\neq i}\varepsilon_{1j}\right).$$

This finishes the construction of the required pseudo-gradient V_3 in this statement. It is defined by a convex combination of V_3^1, V_3^2, and V_3^3.

The global vector field W in $V_\delta(p,\varepsilon)$ will be a convex combination of V_1, V_2 and V_3. It satisfies conditions (i) and (iii) of Theorem 3.1. Concerning (ii) it follows from (i) and the estimate of $\|\bar{v}\|$ given in Proposition 2.2. The proof of Theorem 3.1 is thereby completed. □

Remark 3.8. Let $p \geq 1$. On $V(p,\varepsilon)$, we define a global pseudogradient $\widetilde{W}$ as a convex combination of W_0 and W where W_0 and W are the vector fields defined in Proposition 2.4 and Theorem 3.1 respectively. Of course the flow lines of $\widetilde{W}$ do not preserve Σ^+ for any time. In order to obtain solutions of (1.2) in Σ^+, we reduce our study to a small neighborhood of Σ^+ as is done in [8]. For η_0 a fixed positive constant small enough, we set

$$J_{\eta_0} = \left\{u \in \Sigma \mid J(u)^{\frac{n}{2\sigma}}\|u^-\| < \eta_0\right\},$$

where $u^- = \max(-u, 0)$. Let V be the vector field on $V_{\eta_0}(\Sigma^+)$ defined by

$$\begin{aligned} V(u) &= \widetilde{W}(u) && \text{if } u \in V(p,\varepsilon), \quad p \geq 1,\\ V(u) &= -\partial J(u) && \text{if } u \in V_{\eta_0}(\Sigma^+) \setminus V\left(p, \frac{\varepsilon}{2}\right), \quad p \geq 1. \end{aligned}$$

Observe that for small ε, $V(p,\varepsilon) \subset V_{\frac{\eta_0}{2}}(\Sigma^+)$. Therefore, V coincide with $-\partial J$ on $V_{\eta_0}(\Sigma^+) \setminus V_{\frac{\eta_0}{2}}(\Sigma^+)$. Reasoning by analogy with [8, Lemma 4.1], we can see that any flow line generated by $-\partial J$ with initial condition in $V_{\eta_0}(\Sigma^+)$ remains in $V_{\eta_0}(\Sigma^+)$. Thus $V_{\eta_0}(\Sigma^+)$ in invariant under the flow of V.

4. Proof of Theorem 1.3

Assume that J has no critical point in $V_{\eta_0}(\Sigma^+)$. Under the assumptions **(H₁)**, **(H₂)**, and $(f)_\beta$, $\beta \in (1,n)$, the critical points at infinity of J are $(y_1,\dots,y_p)_\infty = \delta_{(y_i,\infty)}$, $p \geq 1$, and $(y_1,\dots,y_p) \in (\mathcal{K}_{>n-2\sigma} \cap \mathcal{K}^+) \cup \mathcal{C}^\infty_{\leq n-2\sigma} \cup \mathcal{C}^\infty$. To compute the index of J at $(y_1,\dots,y_p)_\infty$, we proceed as in [8, Lemma 4.2]. For any $u = \sum_{i=1}^p \alpha_i\delta_{(a_i,\lambda_i)}$ near $(y_1,\dots,y_p)_\infty$ the following generalized Morse Lemma holds. We have

$$J\left(\sum_{i=1}^{p}\alpha_i\delta_{(a_i,\lambda_i)}+\bar{v}\right) = S\left(\sum_{i=1}^{p}\frac{1}{K(y_i)^{\frac{n-2\sigma}{2}}}\right)^{\frac{2}{n}} \times\left(1-|H|^2+\sum_{i=1}^{p}\left(\sum_{k=1}^{n}b_k(y_i)\right)|(a_i-y_i)_k|^{\beta_i}+c\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\right),$$

where $H \in \mathbb{R}^{p-1}$ is the coordinate related to the expansion of J with respect to the α_i's variables and S and c are two positive constants.

Using the fact that $b_k(y_i) \neq 0$ for all $k = 1, \dots, n$, the index of J at $(y_1, \dots, y_p)_\infty$ is then given by

$$i(y_1, \dots, y_p)_\infty = p - 1 + \sum_{i=1}^{p} (n - \widetilde{i}(y_i)),$$

where

$$\widetilde{i}(y_i) = \sharp\{b_k(y_i) \mid 1 \leq k \leq n \text{ and } b_k(y_i) < 0\}.$$

We apply now the deformation Lemma of [7]. Since J has no critical point in $V_{\eta_0}(\Sigma^+)$,

$$V_{\eta_0}(\Sigma^+) \simeq \bigcup_{(y_1,\dots,y_p) \in (\mathcal{K}_{>n-2\sigma} \cap \mathcal{K}^+) \cup \mathcal{C}^\infty_{\leq n-2\sigma} \cup \mathcal{C}^\infty} W_u^\infty(y_1, \dots, y_p)_\infty, \tag{4.1}$$

where $W_u^\infty(y_1, \dots, y_p)_\infty$ denote the unstable manifold of $(-\partial J)$ at $(y_1, \dots, y_p)_\infty$ and $\simeq$ denotes retract by deformation.

By applying the Euler–Poincaré characteristic on the both side of (4.1) after recalling that $V_{\eta_0}(\Sigma^+)$ is a contractible space and $\dim W_u^\infty(y_1, \dots, y_p)_\infty = i(y_1, \dots, y_p)_\infty$, we get

$$1 = \sum_{(y_1,\dots,y_p) \in (\mathcal{K}_{>n-2\sigma} \cap \mathcal{K}^+) \cup \mathcal{C}^\infty_{\leq n-2\sigma} \cup \mathcal{C}^\infty} (-1)^{i(y_1,\dots,y_p)_\infty}$$

This contradicts the assumption of Theorem 1.3. Therefore, J admits a critical point u_0 in $V_{\eta_0}(\Sigma^+)$. Using the same argument of ([8], pages 659-660), we obtain that ${u_0}^- = 0$ and therefore $u_0 \in \sum^+$. This completes the proof of our result.

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References

[1] W. Abdelhedi and H. Chtioui, *On a Nirenberg-type problem involving the square root of the Laplacian*, J. Funct. Anal. **265** (2013), 2937–2955.

[2] W. Abdelhedi, H. Chtioui, and H. Hajaiej, *A complete study of the lack of compactness and existence results of a Fractional Nirenberg Equation via a flatness hypothesis: Part I*, Anal. PDE, **9** (2016), No. 6, 1285–1315.

[3] W. Abdelhedi and H. Chtioui, *On a fractional Nirenberg problem on n-dimensional spheres: Existence and multiplicity results*, Bull. Sc. Math. **140** (2016), No. 6, 617–628.

[4] A. Bahri, *Critical Point at Infinity in Some Variational Problems*, Pitman Res. Notes Math, **182**, Longman Sci. Tech., Harlow, 1989.

[5] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar curvature problems in high dimensions,* A celebration of J. F. Nash Jr., Duke Math. J. **81** (1996), 323–466.

[6] A. Bahri and J.-M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of topology of the domain,* Comm. Pure Appli. Math. **41** (1988), 255–294.

[7] A. Bahri and P. H. Rabinowitz, *Periodic solutions of 3-body problems,* Ann. Inst. H. Poincaré Anal. Nonlin. **8** (1991), 561–649.

[8] M. Ben Ayed, Y. Chen, H. Chtioui, and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds,* Duke Math. J. **84** (1996), 633–677.

[9] C. Brändle, E. Colorado, A. de Pablo, and U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian,* Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), 39–71.

[10] X. Cabré, J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian,* Adv. Math. **224** (2010), 2052–2093.

[11] C.C. Chen and C.S. Lin, *Prescribing the scalar curvature on S^n, I. Apriori estimates* J. Differential Geom. **57** (2001), 67–171.

[12] W. Chen, C. Li, and B. Ou. *Classification of solutions for an integral equation,* Commun. Pure Appl. Math. **59** (2006), No. 3, 330–343.

[13] Y. Chen, C. Liu, and Y. Zheng, *Existence results for the fractional Nirenberg problem,* J. Funct. Anal. **270** (2016), No. 11, 4043–4086.

[14] H. Chtioui, *On the Chen-Lin conjecture for the prescribed scalar curvature problem,* https://arxiv.org/abs/2009.06262.

[15] A. Fiscella, *Infinitely many solutions for a critical Kirchhoff type problem involving a fractional operator,* Differential Integral Equations **29** (2016), No. 5-6, 513–530.

[16] M. Gonzalez and J. Qing, *Fractional conformal Laplacians and fractional Yamabe problems,* Anal. PDE **6** (2013), 1535–1576.

[17] T. Jin, Y. Li, and J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions,* J. Eur. Math. Soc. **16** (2014), 1111–1171.

[18] T. Jin, Y. Li and J. Xiong, *On a fractional Nirenberg problem, part II: existence of solutions,* Int. Math. Res. Not. **6** (2015), 1555–1589.

[19] T. Jin, Y. Li, and J. Xiong, *The Nirenberg problem and its generalizations: A unified approach,* Math. Ann. **369** (2017), 109–151.

[20] Y.Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres,* J. Eur. Math. Soc. **6** (2004),153–180.

[21] Y. Y. Li and M. Zhu, *Uniqueness theorems through the method of moving spheres,* Duke Math. J. **80** (1995), 383–417.

[22] C. Liu and Q. Ren, *Uniqueness of types of infinitely-many-bump solutions for the fractional Nirenberg problem,* J. Math. Anal. Appl. **468** (2018), No. 1, 1–37.

[23] R. Musina, A. I. Nazarov, *On fractional Laplacians,* Comm. Partial Diferential Equations **39** (2014), No. 9, 1780–1790.

[24] E. Di Nezza, J. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[25] R. Servadei and E. Valdinoci, *On the spectrum of two different fractional operators,* Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), No. 4, 831–855.

[26] K. Sharaf and H. Chtioui, *Conformal metrics with prescribed fractional Q-curvatures on the standard n-dimensional sphere,* Differential Geom. Appl. **68** (2020), 101562.

[27] M. Struwe, *A global compactness result for elliptic boundary value problem involving limiting nonlinearities,* Math. Z. **187** (1984), 511–517.

[28] J.Tan, *Positive solutions for non local elliptic problems,* Discrete Contin. Dyn. Syst. **33** (2013), 837–859.

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Повне дослідження відсутності компактності та теорем існування дробового рівняння Ніренберга за умови площинності. Частина II

Azeb Alghanemi, Wael Abdelhedi, and Hichem Chtioui

Ця стаття є продовженням досліджень статті [2], де вивчалась задача σ-кривини на стандартній сфері за умови, що порядок сплощення даної функції у критичних точках належить $(1, n-2\sigma]$. Наведено повний опис відсутності компактності задачі, коли порядок сплощення змінюється в $(1, n)$, і доведено теорему існування на основі формули типу Ейлера–Хопфа. Як наслідок, ми узагальнюємо результати робіт [2,17,18] та одержуємо новий.

Ключові слова: конформна геометрія, часткова кривина, варіаційне обчислення, критичні точки на нескінченності