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Long-Time Asymptotics for the Modified Camassa–Holm Equation with Nonzero Boundary Conditions

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We consider the modified Camassa–Holm (mCH) equation $m_t + \big((u^2-u_x^2)m\big)_x = 0$ with $m := u - u_{xx}$ on the line $-\infty < x < +\infty$, where $u(x,t)$ is subject to nonzero boundary conditions at infinity: $u(x,t) \to 1$ as $x \to \pm\infty$. The paper aims at studying the long-time asymptotics of solutions of the initial value problems for this problem, using the Riemann–Hilbert formalism recently developed in [3]. The emphasis is made on the asymptotics in two sectors of the (x,t) half-plane ($t > 0$), where the main asymptotic terms are given in terms of modulated, decaying (as $t^{-1/2}$) trigonometric oscillations, as well as in a sector where solitons dominate the long time behavior of the solution of the initial value problem.

Key words: Riemann–Hilbert problem, nonlinear steepest descent method, solitons

Mathematical Subject Classification 2010: 35Q53, 37K15, 35Q15, 35B40, 35Q51, 37K40

1. Introduction

In a recent paper [3], the Riemann–Hilbert formalism have been developed for studying initial value problems for the modified Camassa–Holm (mCH) equation

$$m_t + \big((u^2-u_x^2)m\big)_x = 0, \quad m := u - u_{xx}, \quad t > 0, \quad -\infty < x < +\infty, \tag{1.1a}$$
$$u(x,0) = u_0(x), \qquad -\infty < x < +\infty, \tag{1.1b}$$

in the case of nonzero boundary conditions: it is assumed that $u_0(x) \to 1$ as $x \to \pm\infty$ and that the solution $u(x,t)$ satisfies these boundary conditions for all $t > 0$: $u(x,t) \to 1$ as $x \to \pm\infty$. An important additional assumption on the data for this problem adopted in [3] is that $m_0(x) := (1-\partial_x^2)u_0(x) > 0$ for all x; then it can be shown that $m(x,t) > 0$ for all t as long as the solution exists [28]. In the present paper we study the behavior of $u(x,t)$ as $t \to +\infty$.

Equation (1.1a) is a modification, involving a cubic nonlinearity, of the Camassa–Holm (CH) equation [9, 10]

$$m_t + (um)_x + u_x m = 0, \quad m := u - u_{xx}. \tag{1.2}$$

The Camassa–Holm equation has been studied intensively due to its rich mathematical structure as well as applications for modeling the unidirectional propagation of shallow water waves over a flat bottom [16, 31]. The CH and mCH equations are both have Lax pair representations, which makes it possible to develop the Inverse Scattering Transform method to study various properties of solutions of initial (Cauchy) value problems for these equations. In particular, to study the large-time behavior of solutions of initial value problems for the CH equation, the Riemann–Hilbert (RH) problem formalism (which is a particular form of the inverse scattering method) has been developed in [6]. This formalism was subsequently used for developing an appropriate adaptation of the nonlinear steepest descent method [18] for studying the long-time behavior of solutions of these problems [2, 5, 7].

Over the last few years various modifications and generalizations of the CH equation have been introduced, see, e.g., [43] and references therein. In an equivalent form, equation (1.1a) appeared already in [23] (see also [35] and [26]). Remarkably, equation (1.1a) can be considered as a dual to the modified Korteweg–de Vries (mKdV) equation [39]. Qiao [36] provided an equivalent Lax pair for (1.1a), which is also referred to as the Fokas–Olver–Rosenau–Qiao (FORQ) equation [29].

Being considered for functions satisfying zero boundary conditions at infinity, equation (1.1a) possesses solutions in the form of localized, peaked traveling waves – peakons [28] (see also [12], where multipeakon solutions are discussed using the inverse spectral method for an associated peakon system of ordinary differential equations).

Existence properties of the mCH equation and its generalizations have also been studied intensively, see [13, 14, 25, 28, 34], where the focus was on the local well-posedness and wave-breaking mechanisms. The local well-posedness for classical solutions and for global weak solutions to (1.1a) in Lagrangian coordinates are discussed in [27]. Quasi-periodic solutions of algebro-geometric nature are studied in [29].

The Hamiltonian structure and Liouville integrability of peakon systems are discussed in [1, 11, 28, 35]. In [32], a Liouville-type transformation was presented relating the isospectral problems for the mKdV equation and for the mCH equation, and a Miura-type map from the mCH equation to the CH equation was introduced. The Bäcklund transformation for the mCH equation and a related nonlinear superposition formula are presented in [42].

Notice that in the case of the CH equation with zero boundary conditions at infinity, the inverse scattering transform method (particularly, in the form of a Riemann–Hilbert factorization problem) has been developed for the version of the CH equation involving an additional linear dispersion term. A linear change of variables for such equation allows reducing it to the standard form (1.2), but the solution has to be considered on a nonzero, constant background (with nonzero boundary conditions). On the other hand, the asymptotic analysis of the dispersionless CH equation (1.2) on the zero background requires a different tool (although having a certain analogy with the Riemann–Hilbert method), namely the analysis of a coupling problem for entire functions [20–22].

In the case of the mCH equation, the situation is somewhat similar: the inverse scattering method for the Cauchy problem can be developed when equation (1.1a) is considered on a nonzero background. The Riemann–Hilbert formalism for this problem has been developed in [3, 4]. On the other hand, reducing to a zero background leads to an equation of different form (see Section 2).

In the present paper, we discuss the long-time behavior of the solution of the initial value problem for the mCH equation (1.1) on a nonzero background. Starting form the RH formalism developed for this situation in [3] and proceeding with a series of RH problem transformations, we obtain main terms of the asymptotics in various sectors of the space-time half-plane. Focusing on the solitonless case, Section 2 deals with reducing the original (singular) RH problem representation for the solution of (1.1), proposed for the first time in [3], to a regular RH problem. In Section 3, the obtained RH problem is treated asymptotically, assuming $t \to +\infty$. The resulting slowly decaying, oscillating asymptotics are given in Theorems 3.2 and 3.4. Finally, the soliton asymptoics is discussed in Section 4.

Notations. Furthermore, $\sigma_1 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_3 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ denote the standard Pauli matrices. We let $\mathbb{C}^+ = \{\operatorname{Im}\mu > 0\}$ and $\mathbb{C}^- = \{\operatorname{Im}\mu < 0\}$ denote the open upper and lower complex half-planes. We also let $f^*(\mu) := \overline{f(\bar{\mu})}$ denote the Schwarz conjugate of a function $f(\mu)$, $\mu \in \mathbb{C}$. If M is a 2×2 matrix we denote by $M^{(1)}$ and $M^{(2)}$ its first and second columns, respectively.

2. RH problem formalism: from non-regular to regular problem

As we noticed in Introduction, reducing the mCH equation on a nonzero background to that with zero boundary conditions at infinity leads to a nonlinear equation of different form. Namely, introducing $\widetilde{u}(x,t)$ by

$$\widetilde{u}(x,t) := u(x+t,t) - 1, \tag{2.1}$$

the mCH equation (1.1a) reduces to

$$\widetilde{m}_t + (\widetilde{\omega}\widetilde{m})_x = 0, \tag{2.2a}$$
$$\widetilde{m} := \widetilde{u} - \widetilde{u}_{xx} + 1, \tag{2.2b}$$
$$\widetilde{\omega} := \widetilde{u}^2 - \widetilde{u}_x^2 + 2\widetilde{u}. \tag{2.2c}$$

Here the solution $\widetilde{u}$ is considered on zero background: $\widetilde{u}(x,t) \to 0$ as $x \to \pm\infty$ for all $t \geq 0$. Particularly, the initial data $\widetilde{u}_0(x)$ is also assumed to decay to 0 as $x \to \pm\infty$. In accordance with (1.1), it is further assumed that $\widetilde{m}_0(x) := (1 - \partial_x^2)u_0(x) = (1 - \partial_x^2)\widetilde{u}_0(x) + 1 > 0$ for all $x > 0$. The Riemann–Hilbert formalism for the Cauchy problem for the system (2.2) has recently been developed in [3]. The formalism provides a parametric representation for $\widetilde{u}(x,t)$ in terms of the solution of an associated RH problem according to the following algorithm:

(a) Given $u_0(x)$, determine (by solving the Lax pair equations associated with (2.2), whose coefficients are determined in terms of $u_0(x)$) the reflection coefficient $r(\mu)$, $\mu \in \mathbb{R}$, the spectral function $a(\mu)$, $\mu \in \mathbb{C}^+$, and, if applicable (i.e., if $a(\mu)$ have simple zeros at μ_j and $-\bar{\mu}_j$, $j = 1, \dots, n$, where $\mu_j = \mathrm{e}^{\mathrm{i}\theta_j}$ with $\theta_j \in (0, \frac{\pi}{2})$, the discrete data $\{\theta_j, \delta_j\}_{j=1}^n$, .

(b) Construct the jump matrix $J(y, t, \mu)$, $\mu \in \mathbb{R}$ by

$$J(y,t,\mu) := \mathrm{e}^{-p(y,t,\mu)\sigma_3} J_0(\mu) \mathrm{e}^{p(y,t,\mu)\sigma_3} \tag{2.3}$$

where

$$p(y,t,\mu) := -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(-y + \frac{8\mu^2}{(\mu^2+1)^2} t\right) \tag{2.4}$$

and $J_0(\mu)$ is defined by

$$J_0(\mu) := \begin{pmatrix} 1 - r(\mu)r^*(\mu) & r(\mu) \\ -r^*(\mu) & 1 \end{pmatrix}. \tag{2.5}$$

(c) Solve the following *RH problem* parametrized by y and t: Find a 2×2-matrix valued function $M(y, t, \mu)$ meromorphic with respect to μ in $\mathbb{C}^+$ and $\mathbb{C}^-$, which satisfies the following conditions:

- The jump condition

$$M_+(y,t,\mu) = M_-(y,t,\mu) J(y,t,\mu), \qquad \mu \in \mathbb{R}, \quad \mu \neq \pm 1, \tag{2.6}$$

where $M_\pm(\cdot,\cdot,\mu)$ are the limiting values of M as μ is approached from $\mathbb{C}^\pm$ respectively.

- The residue conditions

$$\begin{aligned} \operatorname{Res}_{\mu_j} M^{(1)}(y,t,\mu) &= \frac{1}{\varkappa_j(y,t)} M^{(2)}(y,t,\mu_j), \\ \operatorname{Res}_{-\bar{\mu}_j} M^{(1)}(y,t,\mu) &= \frac{1}{\overline{\varkappa_j}(y,t)} M^{(2)}(y,t,-\bar{\mu}_j), \\ \operatorname{Res}_{\bar{\mu}_j} M^{(2)}(y,t,\mu) &= \frac{1}{\overline{\varkappa_j}(y,t)} M^{(1)}(y,t,\overline{\mu_j}), \\ \operatorname{Res}_{-\mu_j} M^{(2)}(y,t,\mu) &= \frac{1}{\varkappa_j(y,t)} M^{(1)}(y,t,\overline{-\mu_j}), \end{aligned} \tag{2.7}$$

with $\varkappa_j(y,t) := \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_j}\delta_j \mathrm{e}^{-2p(y,t,\mu_j)}$.

- The normalization condition

$$M(y,t,\mu) \to I \text{ as } \mu \to \infty. \tag{2.8}$$

- The symmetries

$$M(\mu) = \overline{M(\bar{\mu}^{-1})} = \sigma_3 \overline{M(-\bar{\mu})} \sigma_3 = \sigma_1 \overline{M(\bar{\mu})} \sigma_1, \tag{2.9}$$

where $M(\mu) \equiv M(y,t,\mu)$.

- The singularity conditions

$$M(y,t,\mu) = \frac{\mathrm{i}\alpha_+(y,t)}{2(\mu-1)}\begin{pmatrix} -c & 1 \\ -c & 1 \end{pmatrix} + \mathrm{O}(1) \quad \text{as } \mu \to 1,\ \operatorname{Im}\mu > 0, \tag{2.10a}$$

$$M(y,t,\mu) = -\frac{\mathrm{i}\alpha_+(y,t)}{2(\mu+1)}\begin{pmatrix} c & 1 \\ -c & -1 \end{pmatrix} + \mathrm{O}(1) \quad \text{as } \mu \to -1,\ \operatorname{Im}\mu > 0, \tag{2.10b}$$

where $c = 1 + r(1)$ (generically, $c = 0$) whereas $\alpha_+(y,t) \in \mathbb{R}$ is not specified.

(d) Having found the solution $M(y,t,\mu)$ of this RH problem (which is unique, if it exists, see [3]), extract the real-valued functions $a_j(y,t)$, $j = 1,2,3$ from the expansion of $M(y,t,\mu)$ at $\mu = \mathrm{i}$:

$$M(y,t,\mu) = \begin{pmatrix} a_1(y,t) & 0 \\ 0 & a_1^{-1}(y,t) \end{pmatrix} + \begin{pmatrix} 0 & a_2(y,t) \\ a_3(y,t) & 0 \end{pmatrix}(\mu - \mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \quad \mu \to \mathrm{i}. \tag{2.11}$$

(e) Obtain $\widetilde{u}(x,t)$ in parametric form as follows:

$$\widetilde{u}(x,t) = \widehat{u}(y(x,t),t),$$

where

$$\widehat{u}(y,t) := -a_2(y,t)a_1(y,t) - a_3(y,t)a_1^{-1}(y,t), \tag{2.12a}$$

$$x(y,t) := y + 2\ln a_1(y,t). \tag{2.12b}$$

Remark 2.1. Comparing with [3], M_+ and M_- are interchanged in the jump relation (2.6) so that here the jump is the inverse of that in [3]: $J_0 = \widehat{J}_0^{-1}$ and $J = \widehat{J}^{-1}$.

Notice that the symmetries (2.9) are consistent with the symmetries of $r(\mu)$:

$$r(\mu) = -\overline{r(-\mu)} = \overline{r(\mu^{-1})}. \tag{2.13}$$

In turn, (2.13) follows from the construction of the RH problem above in terms of the dedicated (Jost) solutions of the Lax pair equations associated with the mCH equation, see [3]. In particular, the symmetries (2.9) imply the specific matrix structure of the terms in (2.11).

Remark 2.2. In the case of the Camassa–Holm (CH) equation, the condition $m_0(x) := (1 - \partial_x^2)u_0(x) > 0$ for all x provides the existence of a global solution (i.e., for all $0 < t < \infty$) to the corresponding initial value problem (see, e.g., [15]). In the case of the modified Camassa–Holm (mCH) equation, the situation is different: even if the initial potential m_0 does not change sign the solution

$u(x,t)$ may blow-up in finite time [28]. With this respect we notice that our asymptotic analysis is applicable assuming that the solution of problem (1.1) exists globally. On the other hand, the RH formalism, being intrinsically local in the corresponding variables (y and t in the case of the mCH equation), is well suited to present solutions that overcome finite time blowup, which allows discussing the long-time behavior of such (non-classical) solutions.

In the general context of nonlinear integrable equations, the RH problem formalism (i.e., the representation of the solution of the original problem — the Cauchy problem for a nonlinear integrable PDE — in terms of the solution of an associated RH problem) allows reducing the problem of the large time analysis of the solution of the nonlinear PDE to that of the RH problem. Residue conditions (if any) involved in the RH problem formulation generate a soliton-type, non-decaying contribution to the asymptotics whereas the jump conditions are responsible for the dispersive (decaying) part, details of which can be retrieved applying an appropriate modification of the nonlinear steepest descent method to the asymptotic analysis of a preliminarily regularized RH problem (i.e., a RH problem involving the jump and normalization conditions only).

As for the singularity conditions, we notice that in the case of the Camassa–Holm equation, where such a condition is also involved in the matrix RH problem formalism, an efficient way to handle it is to reduce the matrix RH problem to a vector one, multiplying from the left by the constant vector $(1,1)$. Indeed, the singularity condition for the CH equation has the form of (2.10b), and thus this multiplication removes the singularity, reducing the RH problem to a regular one. With this respect, we notice that the matrix RH problem for the modified Camassa–Holm equation is different: indeed, the singularity condition (2.10a) cannot be removed using the same trick.

Nevertheless, reducing to a regular RH problem is possible for the mCH as well. To fix ideas, in this section we proceed with the solitonless case assuming that there are no residue conditions. We are going to reduce the original RH problem (which is singular due to conditions (2.10)) to a regular one, proceeding in two steps.

In Step 1, we reduce the RH problem with the singularity conditions (2.10) at $\mu = \pm 1$ to a RH problem which is characterized by the following two conditions:

(i) the matrix entries of the solution of the RH problem are regular at $\mu = \pm 1$, but the determinant of the (matrix) solution vanishes at $\mu = \pm 1$ (notice that $\det M(\mu) \equiv 1$ for the solution of the original RH problem);

(ii) the solution of the RH problem is singular at $\mu = 0$.

Then, in Step 2, the latter RH problem is reduced to a regular one, i.e., to a RH problem with the jump and normalization conditions only.

Proposition 2.3. *Let $M(y,t,\mu)$ be a solution of the RH problem* (2.6), (2.8)–(2.10). *Define $\widetilde{M}$ by*

$$\widetilde{M}(y,t,\mu) := \left(I - \frac{1}{\mu}\sigma_1\right) M(y,t,\mu). \tag{2.14}$$

Then $\widetilde{M}(\mu) \equiv \widetilde{M}(y,t,\mu)$ *is the unique solution of the following RH problem:*

(i) $\widetilde{M}(\mu)$ *is analytic in* $\mathbb{C}^+$ *and* $\mathbb{C}^-$ *and continuous up to* $\mathbb{R}\setminus\{0\}$.

(ii) $\widetilde{M}(\mu)$ *satisfies the jump condition* (2.6) *with the jump defined by* (2.3)–(2.5).

(iii) $\widetilde{M}(\mu) \to I$ *as* $\mu \to \infty$.

(iv) $\widetilde{M}(\mu) = -\frac{1}{\mu}\sigma_1 + \mathrm{O}(1)$ *as* $\mu \to 0$.

(v) $\det \widetilde{M}(\pm 1) = 0$.

(vi) $\widetilde{M}(\mu^{-1}) = -\mu\widetilde{M}(\mu)\sigma_1$.

Proof. First, let's check that $\widetilde{M}(y,t,\mu)$ constructed from $M(y,t,\mu)$ satisfies the conditions above. The limiting properties (iii) and (iv) as $\mu \to \infty$ and as $\mu \to 0$ are obviously satisfied (by construction) whereas (ii) results from the fact that a multiplication from the left does not change the jump conditions. Further, since $\det M(y,t,\mu) \equiv 1$, it follows that $\det \widetilde{M}(y,t,\mu) = 1 - \frac{1}{\mu^2}$ and thus $\det \widetilde{M}(y,t,\pm 1) = 0$. Moreover, as $\mu \to 1$ we have

$$\begin{aligned}\left(\widetilde{M}_{11}(\mu), \widetilde{M}_{12}(\mu)\right) &= (M_{11}(\mu), M_{12}(\mu)) - \frac{1}{\mu}(M_{21}(\mu), M_{22}(\mu)) \\ &= (M_{11}(\mu) - M_{21}(\mu), M_{12}(\mu) - M_{22}(\mu)) + \mathrm{O}(1) = \mathrm{O}(1)\end{aligned}$$

due to (2.10a). Similarly, as $\mu \to -1$ we have

$$\left(\widetilde{M}_{11}(\mu), \widetilde{M}_{12}(\mu)\right) = (M_{11}(\mu) + M_{21}(\mu), M_{12}(\mu) + M_{22}(\mu)) + \mathrm{O}(1) = \mathrm{O}(1)$$

due to (2.10b). Similarly for $\left(\widetilde{M}_{21}(\mu), \widetilde{M}_{22}(\mu)\right)$. Thus $\widetilde{M}(y,t,\mu)$ is non-singular at $\mu = \pm 1$. Finally, (C6) follows from the symmetry relation $M(\mu^{-1}) = \sigma_1 M(\mu)\sigma_1$ from (2.9).

Now, let's prove that the solution of the RH problem (i)–(vi) above is unique (if it exists). First, we notice that if $\widetilde{M}(y,t,\mu)$ solves the RH problem (i)–(vi), then

$$\det \widetilde{M}(y,t,\mu) = 1 - \frac{1}{\mu^2}. \tag{2.15}$$

Indeed, since $\det J(y,t,\mu) \equiv 1$ and $\det M(y,t,\mu)$ is bounded at $\mu = \infty$, it follows that $\det M(\mu)$ is a rational function. Moreover, from (C4) we have that $\det M(\mu) = -\frac{1}{\mu^2} + \frac{c}{\mu} + \mathrm{O}(1)$ as $\mu \to 0$, with some $c \equiv c(y,t)$. Taking into account (iii) we have that $\zeta(y,t,\mu) := \det M(y,t,\mu) - 1 + \frac{1}{\mu^2} - \frac{c}{\mu}$ is a bounded entire function of μ, which, by Liouville's theorem and (iii), vanishes for all (y,t). Finally, evaluating $\zeta(y,t,\mu)$ at $\mu = \pm 1$ and using (v), it follows that $c(y,t) \equiv 0$ and thus (2.15) follows.

Now let's assume that $\widetilde{\widetilde{M}}$ is another solution of the RH problem (i)–(vi) and define $N(\mu) := \widetilde{M}(\mu)\widetilde{\widetilde{M}}^{-1}(\mu)$. Since $\widetilde{M}$ and $\widetilde{\widetilde{M}}$ satisfy the same jump conditions,

$N(\mu)$ is a rational function, with possible singularities at $\mu = 0, -1, 1$. In view of (2.15) and (iii), $\widetilde{\widetilde{M}}^{-1}(\mu) = \frac{\mu^2}{\mu^2-1}(\frac{1}{\mu}\sigma_1 + \mathrm{O}(1)) = \mathrm{O}(\mu)$ as $\mu \to 0$ and thus $N(\mu)$ is non-singular at $\mu = 0$. In order to prove that $N(\mu)$ is non-singular at $\mu = \pm 1$, we use relation (vi). In particular, we have $\widetilde{M}(1) = -\widetilde{M}(1)\sigma_1$ and thus $\widetilde{M}(\mu) = \left(\begin{smallmatrix} g_1 & -g_1 \\ g_2 & -g_2 \end{smallmatrix}\right) + \mathrm{O}(\mu - 1)$ as $\mu \to 1$, with some g_j, $j = 1, 2$. Consequently, $\widetilde{\widetilde{M}}^{-1}(\mu) = \frac{\mu^2}{\mu^2-1}\left(\left(\begin{smallmatrix} -\tilde{g}_2 & \tilde{g}_1 \\ -\tilde{g}_2 & \tilde{g}_1 \end{smallmatrix}\right) + \mathrm{O}(\mu - 1)\right)$ as $\mu \to 1$, with some $\tilde{g}_j$, $j = 1, 2$, which implies that $N(\mu)$ is bounded as $\mu \to 1$. Similarly for $\mu \to -1$. Therefore, $N(\mu)$ is an entire function such that $N(\infty) = I$ and thus $N(\mu) \equiv I$ by Liouville's theorem. □

Remark 2.4. Assuming $r(\mu) = -\overline{r(-\mu)}$ (see (2.13)), we have that $J(\mu)$ satisfies the symmetries

$$J(\mu) = \sigma_3 \overline{J(-\mu)} \sigma_3 = \sigma_1 \overline{J^{-1}(\mu)} \sigma_1,$$

which, due to uniqueness, imply for $\widetilde{M}$ the same symmetries as for M:

$$\widetilde{M}(\mu) = \sigma_3 \overline{\widetilde{M}(-\bar{\mu})} \sigma_3 = \sigma_1 \overline{\widetilde{M}(\bar{\mu})} \sigma_1 \tag{2.16}$$

(taking also into account that the symmetries (2.16) are consistent with all conditions in the RH problem in Proposition 2.3).

Step 2 in the reduction of the RH problem is formulated in the following proposition (see [30, 40, 41] for the case of the nonlinear Schrödinger equation with "finite density" boundary conditions).

Proposition 2.5 (regular RH problem)**.** *The solution $\widetilde{M}$ of the RH problem from Proposition 2.3 can be represented in terms of the solution of a regular RH problem as follows:*

$$\widetilde{M}(y, t, \mu) = \left(I - \frac{1}{\mu}\Delta(y, t)\right) M^R(y, t, \mu), \tag{2.17}$$

where $M^R(\mu) \equiv M^R(y, t, \mu)$ is the solution of the following RH problem:

Find $M^R(\mu)$ such that

(a) *$M^R(\mu)$ is analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous up to the real axis.*

(b) *$M^R(\mu)$ satisfies the jump condition (2.3)–(2.6).*

(c) *$M^R(\mu) \to I$ as $\mu \to \infty$.*

Here Δ in (2.17) is expressed in terms of the solution M^R of the RH problem above by:

$$\Delta(y, t) = \sigma_1 [M^R(y, t, 0)]^{-1}.$$

Proof. Let $M^R(\mu)$ be the solution of the regular RH problem (a)-(c) above. Then $\widetilde{M}(y, t, \mu)$ defined by (2.17) obviously (by construction) satisfies conditions (i)-(iv) of the RH problem from Proposition 2.3. In order to check conditions (v) and (vi), we use the matrix structure of Δ that follows from the symmetries of $M^R(\mu)$.

(i) Since $M^R(\mu)$ and $M(\mu)$ satisfy the same jump condition, the uniqueness of the solution of the regular RH problem implies that $M^R(\mu)$ satisfies the same symmetries (see (2.9)) (generated by the symmetry $r(\mu) = -\overline{r(-\mu)}$):

$$M^R(\mu) = \sigma_3 \overline{M^R(-\bar{\mu})} \sigma_3 = \sigma_1 \overline{M^R(\bar{\mu})} \sigma_1. \tag{2.18}$$

Considering this for $\mu = 0$ it follows that $M^R(y,t,0) = \begin{pmatrix} \alpha(y,t) & \mathrm{i}\beta(y,t) \\ -\mathrm{i}\beta(y,t) & \alpha(y,t) \end{pmatrix}$ with some $\alpha(y,t) \in \mathbb{R}$ and $\beta(y,t) \in \mathbb{R}$. Moreover, $\alpha^2(y,t) - \beta^2(y,t) \equiv 1$ since $\det M^R(\mu) \equiv 1$. Consequently, $\Delta(y,t)$ has the structure

$$\Delta = \begin{pmatrix} \mathrm{i}\beta & \alpha \\ \alpha & -\mathrm{i}\beta \end{pmatrix} \tag{2.19}$$

with $\alpha^2 - \beta^2 = 1$ and thus $\det(I - \mu^{-1}\Delta(y,t)) = 1 - \frac{\alpha^2-\beta^2}{\mu^2} = 1 - \frac{1}{\mu^2}$, which implies (v). Notice that $\Delta^2 \equiv I$.

(ii) Now consider the symmetry $\mu \mapsto \mu^{-1}$. From $r(\mu) = \overline{r(\mu^{-1})}$ it follows that $J(\mu) = \sigma_1 J^{-1}(\mu^{-1})\sigma_1$ and thus $\check{M}(\mu) := \sigma_1 M^R(\mu^{-1})\sigma_1$ satisfies the same jump condition as $M^R(\mu)$ does. Taking into account that $\check{M}(\infty) = \sigma_1 M^R(0)\sigma_1$, Liouville's theorem implies that $\check{M}^{-1}(\infty)\check{M}(\mu) \equiv \sigma_1 [M^R(0)]^{-1} M^R(\mu^{-1})\sigma_1 = M(\mu)$, or, in terms of Δ,

$$M^R(\mu^{-1}) = \Delta M^R(\mu)\sigma_1. \tag{2.20}$$

Now, combining (2.17) with (2.20) we can express $\widetilde{M}(\mu^{-1})$ in terms of $\widetilde{M}(\mu)$ as follows:

$$\widetilde{M}(\mu^{-1}) = (I - \Delta\mu)M^R(\mu^{-1}) = (I - \Delta\mu)\Delta M^R(\mu)\sigma_1 = Q(\mu)\widetilde{M}(\mu)\sigma_1 \tag{2.21}$$

with

$$Q(\mu) = (I - \Delta\mu)\Delta \left(I - \Delta\mu^{-1}\right)^{-1}.$$

Using (2.19), direct calculations give $Q(\mu) = -\mu I$ and thus the symmetry (2.20) takes the form of (vi) in Proposition 2.3. □

2.1. From M^R back to $\widetilde{u}$. Now, we can obtain a parametric representation of the solution $\widetilde{u}(x,t)$ of the Cauchy problem (2.2) in terms of the solution $M^R(y,t,\mu)$ of the regular RH problem from Proposition 2.5. First, using (2.14) and (2.17), we get M from M^R:

$$M(\mu) = \left(I - \frac{1}{\mu}\sigma_1\right)^{-1} \left(I - \frac{1}{\mu}\Delta\right) M^R(\mu). \tag{2.22}$$

Then, by (2.11) and (2.12) we find

$$M(y,t,\mu) \rightsquigarrow \{a_1(y,t), a_2(y,t), a_3(y,t)\} \rightsquigarrow \{\widehat{u}(y,t), x(y,t)\},$$

and finally $\widetilde{u}(x,t) = \widehat{u}(y(x,t),t)$.

3. Long-time asymptotics

In this section, we study the long-time asymptotics of the solution $M^R(y,t,\mu)$ of the regular RH problem from Proposition 2.5 using the ideas and tools of the nonlinear steepest descent method [18], which will finally lead us to the asymptotic formulas for the solution of the mCH equation. The method consists of successive transformations of RH problems, in order to arrive at a RH problem that can be solved explicitly. The transformation steps include

(a) appropriate triangular factorizations of the jump matrix;

(b) "absorption" of the triangular factors with good large-time behavior;

(c) reduction, after rescaling, to a RH problem which is solvable in terms of certain special functions;

(d) analysis of the approximation errors.

The information on L^p-RH problems and their applications to the asymptotics can be found in [17, 19, 24, 44]. Here we focus on deriving the leading terms of the long-time asymptotics, while for error estimates we refer to [33].

3.1. Transformations of the regular RH problem.

Introduce

$$\theta(\mu,\xi) := \widehat{\theta}(k(\mu),\xi),$$

where

$$\xi := \frac{y}{t}, \quad k(\mu) := \frac{1}{4}\left(\mu - \frac{1}{\mu}\right), \quad \widehat{\theta}(k,\xi) := k\xi - \frac{2k}{1+4k^2}. \tag{3.1}$$

Hence, $p(y,t,\mu) = \mathrm{i}t\theta(\mu,\xi)$. The jump matrix $J(y,t,\mu)$ in (2.6) which is defined by (2.3)–(2.5) allows two triangular factorizations:

$$J(y,t,\mu) = \begin{pmatrix} 1 & r(\mu)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r^*(\mu)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \tag{3.2a}$$

$$J(y,t,\mu) = \begin{pmatrix} 1 & 0 \\ -\frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix} \begin{pmatrix} 1-r(\mu)r^*(\mu) & 0 \\ 0 & \frac{1}{1-r(\mu)r^*(\mu)} \end{pmatrix} \times \begin{pmatrix} 1 & \frac{r(\mu)}{1-r(\mu)r^*(\mu)}\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}. \tag{3.2b}$$

Following the basic idea of the nonlinear steepest descent method [18], the factorizations (3.2) can be used in such a way that the (oscillating) jump matrix on $\mathbb{R}$ for a modified RH problem reduces (see the RH problem for M_2 below) to the identity matrix whereas the arising jumps outside $\mathbb{R}$ are exponentially small as $t \to +\infty$. The use of one or another form of the factorization is dictated by the "signature table" for θ, i.e., the distribution of signs of $\operatorname{Im}\theta(\mu,\xi)$ (that depends on ξ) in the μ-complex plane.

a) The factorization (3.2a) is appropriate for the (open) intervals of $\mathbb{R}$ for which $\operatorname{Im}\theta(\mu)$ is positive for $\mu \in \mathbb{C}^+$ close to these intervals (and negative for $\mu \in \mathbb{C}^-$ close to the same intervals). We denote by $\Sigma_a \equiv \Sigma_a(\xi)$ the union of these intervals.

b) On the other hand the factorization (3.2b) is appropriate for the (open) intervals of $\mathbb{R}$ for which $\operatorname{Im}\theta(\mu)$ is negative for $\mu \in \mathbb{C}^+$ close to these intervals. We denote their union by $\Sigma_b(\xi) = \mathbb{R} \setminus \overline{\Sigma_a(\xi)}$.

In turn, one can get rid of the diagonal factor in (3.2b) using the solution of the following *scalar RH problem*: Find a scalar function $\delta(\mu,\xi)$ (ξ being a parameter) analytic in $\mu \in \mathbb{C} \setminus \overline{\Sigma_b(\xi)}$ and such that

$$\delta_+(\mu,\xi) = \delta_-(\mu,\xi)(1-|r(\mu)|^2), \quad \mu \in \Sigma_b(\xi), \tag{3.3a}$$
$$\delta(\mu,\xi) \to 1, \quad \mu \to \infty. \tag{3.3b}$$

The solution of the RH problem (3.3) is given by the Cauchy integral:

$$\delta(\mu,\xi) = \exp\left\{\frac{1}{2\pi \mathrm{i}}\int_{\Sigma_b(\xi)} \frac{\ln(1-|r(s)|^2)}{s-\mu}\mathrm{d}s\right\}. \tag{3.4}$$

Define $M_1(y,t,\mu) := M^R(y,t,\mu)\delta^{-\sigma_3}(\mu,\xi)$. Then M_1 can be characterized as the solution of the RH problem including the standard normalization condition $M_1(\mu) \to I$ as $\mu \to \infty$ and the jump condition

$$M_{1+}(y,t,\mu) = M_{1-}(y,t,\mu)J_1(y,t,\mu), \quad \mu \in \mathbb{R}, \tag{3.5}$$

where the jump matrix is factorized as

$$J_1(y,t,\mu) = \begin{pmatrix} 1 & r(\mu)\delta^2(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -r^*(\mu)\delta^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \quad \mu \in \Sigma_a(\xi) \tag{3.6a}$$

$$J_1(y,t,\mu) = \begin{pmatrix} 1 & 0 \\ -\frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\delta_-^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{r(\mu)}{1-r(\mu)r^*(\mu)}\delta_+^2(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, \quad \mu \in \Sigma_b(\xi). \tag{3.6b}$$

Now let us discuss the structure of $\Sigma_a(\xi)$ and $\Sigma_b(\xi)$. First, we notice that $\widehat{\theta}(\xi,k)$ is exactly the same as in the case of the CH equation [7]. Taking into account the relation between μ and k (see (3.1)), the "signature table" for the CH equation near the real axis suggests that for the mCH equation (the latter being, additionally, symmetric with respect to $\mu \mapsto 1/\mu$) while the ranges of values of ξ for which the "signature table" keeps the same structure are the same. Namely, one can distinguish four ranges of values of ξ for which $\Sigma_a(\xi)$ and $\Sigma_b(\xi)$ have qualitatively different structures (which, consequently, implies four qualitatively different types of large-time asymptotics): (I) $\xi > 2$, (II) $0 < \xi < 2$, (III) $-\frac{1}{4} < \xi < 0$, and (IV) $\xi < -\frac{1}{4}$. Each range of values of ξ is characterized by the structure of $\Sigma_a(\xi)$ (or $\Sigma_b(\xi)$): $\Sigma_a(\xi)$ is the union of disjoint intervals whose (finite) end points are (real) stationary points of $\theta(\mu,\xi)$, i.e., points $\mu\in\mathbb{R}$ where $\frac{\mathrm{d}\theta}{\mathrm{d}\mu}(\mu,\xi)=0$, and similarly for $\Sigma_b(\xi)$. More precisely,

$$\Sigma_b(\xi)=\begin{cases}\emptyset, & \xi>2\\ (-\mu_0,-\frac{1}{\mu_0})\cup(\frac{1}{\mu_0},\mu_0), & 0<\xi<2\\ (-\infty,-\mu_1)\cup(-\mu_0,-\frac{1}{\mu_0})\cup(-\frac{1}{\mu_1},\frac{1}{\mu_1}) & \\ \qquad\qquad\cup(\frac{1}{\mu_0},\mu_0)\cup(\mu_1,+\infty), & -\frac14<\xi<0\\ (-\infty,+\infty), & \xi<-\frac14\end{cases}. \tag{3.7}$$

Here the values of $\mu_0(\xi)>1$ and $\mu_1(\xi)>1$ are those associated (via $\kappa_j=\frac14(\mu_j-\frac{1}{\mu_j})$, $j=0,1$) with the (real) stationary points $\kappa_0(\xi)$ and $\kappa_1(\xi)$ of $\widehat{\theta}(k)$, i.e., the end points in the case of the CH equation. They are determined by the relation $\xi=\frac{2-8\kappa^2}{(1+4\kappa^2)^2}$ (see [7]):

$$\kappa_0^2(\xi)=\frac{\sqrt{1+4\xi}-1-\xi}{4\xi},\qquad \kappa_1^2(\xi)=-\frac{\sqrt{1+4\xi}+1+\xi}{4\xi}$$

($\kappa_0(\xi)$ is relevant for ranges II and III whereas $\kappa_1(\xi)$ is relevant for range III only). In analogy with the case of the CH equation, for ξ in ranges I and IV, the solution M_2 of the RH problem (see below) decays rapidly (as $t\to+\infty$) to the identity matrix, which corresponds (in the case without discrete spectrum) to rapid decay of the resulting $\widehat{u}(y,t)$. On the other hand, ranges II and III are those where the large-time asymptotics in the case of the CH equation are of Zakharov–Manakov type (trigonometric oscillations decaying as $t^{-1/2}$), see [5,7]. Our main goal in the present paper is the derivation of analogous asymptotic formulas, for ranges II and III, in the case of the mCH equation.

The next step in the transformation of the RH problem is the "absorption" of the triangular factors in (3.6a) and (3.6b) into the solution of a deformed RH problem, with an enhanced jump contour (having parts outside $\mathbb{R}$). This absorption requires the triangular factors in (3.6a) and (3.6b) to have analytic continuation at least into a band surrounding $\mathbb{R}$. With this respect we notice that, as in the case of other integrable equations (in particular, the CH equation), the reflection coefficient $r(\mu)$ is defined, in general, for $\mu\in\mathbb{R}$ only. However, one can approximate $r(\mu)$ and $\frac{r(\mu)}{1-r(\mu)r^*(\mu)}$ by some rational functions with well-controlled errors (see, e.g., [33]). Alternatively, if we *assume* that the initial data $\widetilde{u}(x,0)$ decays exponentially to 0 as $x\to\pm\infty$ (or that $\widetilde{u}(x,0)$ has finite support in $\mathbb{R}$), then $r(\mu)$ turns out to be analytic in a band containing the real axis (or analytic in the whole plane) and thus there is no need to use rational approximations in order to be able to perform this absorption (see the transformation $M_1\rightsquigarrow M_2$ below). Henceforth, in order to avoid technicalities and to keep the presentation of our main result as simple as possible, we *assume* that $r(\mu)$ (and thus $1-r(\mu)r^*(\mu)$) is analytic in a domain of the complex plane containing the contours of the successive RH problems (and refer to [33] for details related to the rational approximations).

For $0 < \xi < 2$ and for $-\frac{1}{4} < \xi < 0$, we define a contour $\Sigma \equiv \Sigma(\xi)$ consistent with the signature table for $\theta(\mu, \xi)$, see Figures 3.1 and 3.2, respectively.

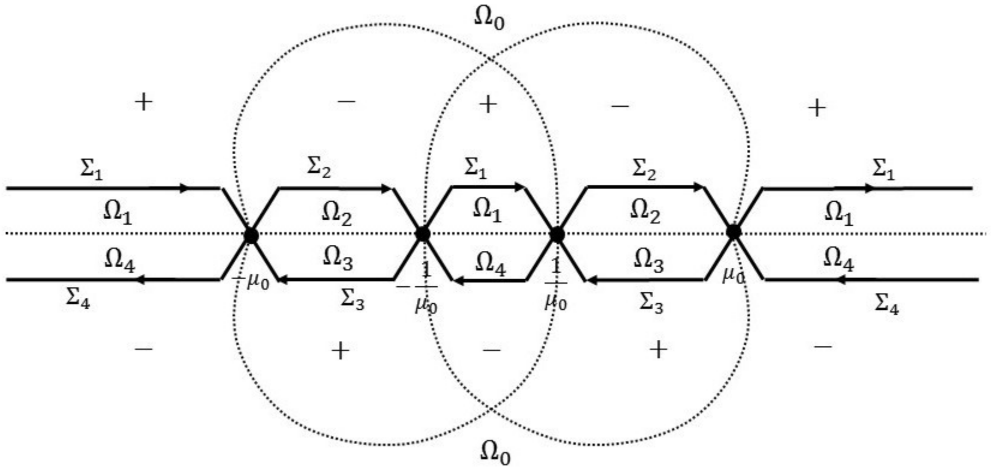

Fig. 3.1: Signature table (dotted lines), contour $\Sigma(\xi) = \cup_{j=1}^{4} \Sigma_j$ (solid lines) and domains $\Omega_j(\xi)$ for $0 < \xi < 2$.

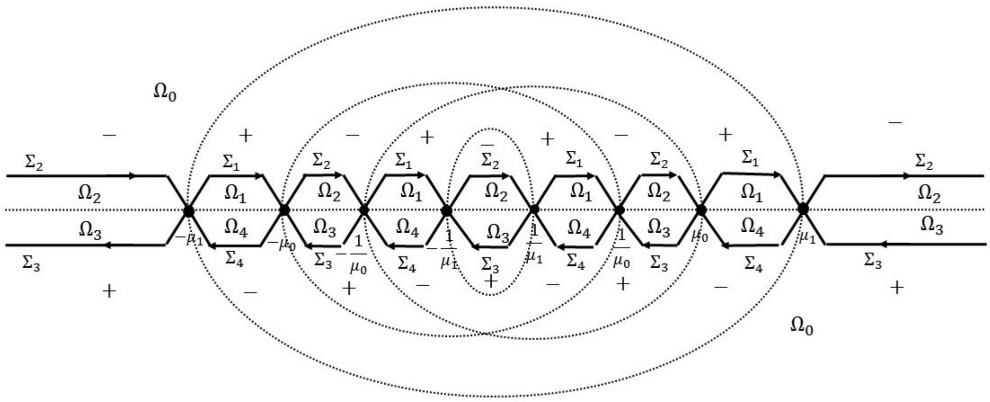

Fig. 3.2: Signature table (dotted lines), contour $\Sigma(\xi) = \cup_{j=1}^{4} \Sigma_j$ (solid lines) and domains $\Omega_j(\xi)$ for $-\frac{1}{4} < \xi < 0$.

Further, define M_2 by $M_2(y, t, \mu) := M_1(y, t, \mu) P(y, t, \mu)$, where

$$P(y, t, \mu) = I, \qquad \mu \in \Omega_0, \tag{3.8a}$$

$$P(y, t, \mu) = \begin{pmatrix} 1 & 0 \\ r^*(\mu)\delta^{-2}(\mu, \xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \qquad \mu \in \Omega_1, \tag{3.8b}$$

$$P(y, t, \mu) = \begin{pmatrix} 1 & -\frac{r(\mu)}{1 - r(\mu)r^*(\mu)}\delta^2(\mu, \xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, \qquad \mu \in \Omega_2, \tag{3.8c}$$

$$P(y, t, \mu) = \begin{pmatrix} 1 & 0 \\ -\frac{r^*(\mu)}{1 - r(\mu)r^*(\mu)}\delta^{-2}(\mu, \xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \qquad \mu \in \Omega_3, \tag{3.8d}$$

$$P(y,t,\mu)=\begin{pmatrix}1 & r(\mu)\delta^2(\mu,\xi)e^{-2it\theta}\\ 0 & 1\end{pmatrix},\qquad \mu\in\Omega_4. \tag{3.8e}$$

Then $M_2(y,t,\mu)$ can be characterized as the solution of the RH problem with the standard normalization condition $M_2(\mu)\to I$ as $\mu\to\infty$ and the jump condition

$$M_{2+}(y,t,\mu)=M_{2-}(y,t,\mu)J_2(y,t,\mu),\quad \mu\in\Sigma:=\cup_{j=1}^4\Sigma_j, \tag{3.9}$$

where $\Sigma_j:=\overline{\Omega_0}\cap\overline{\Omega_j}$ and

$$J_2(y,t,\mu)=\begin{pmatrix}1 & 0\\ -r^*(\mu)\delta^{-2}(\mu,\xi)e^{2it\theta} & 1\end{pmatrix},\qquad \mu\in\Sigma_1, \tag{3.10a}$$

$$J_2(y,t,\mu)=\begin{pmatrix}1 & \frac{r(\mu)}{1-r(\mu)r^*(\mu)}\delta^2(\mu,\xi)e^{-2it\theta}\\ 0 & 1\end{pmatrix},\qquad \mu\in\Sigma_2, \tag{3.10b}$$

$$J_2(y,t,\mu)=\begin{pmatrix}1 & 0\\ \frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\delta^{-2}(\mu,\xi)e^{2it\theta} & 1\end{pmatrix},\qquad \mu\in\Sigma_3, \tag{3.10c}$$

$$J_2(y,t,\mu)=\begin{pmatrix}1 & -r(\mu)\delta^2(\mu,\xi)e^{-2it\theta}\\ 0 & 1\end{pmatrix},\qquad \mu\in\Sigma_4. \tag{3.10d}$$

The RH problem for M_2 is such that uniform decay (as $t\to+\infty$) of the jump matrix is violated only near the stationary phase points of $\theta(\mu)$. The large-time analysis, with appropriate estimates, of such problems involves the "comparison" of the RH problem with that modified in small vicinities of the stationary phase points, using rescaled spectral parameters as well as approximations of the jump matrices in these vicinities [18].

In our large-time analysis for M_2, we follow the strategy presented in [33].

Step (i). Add to Σ small circles γ_j ($j=0,1$) surrounding μ_j, together with their images $-\gamma_j$ (surrounding $-\mu_j$) and $\pm\gamma_j^{-1}$ (surrounding $\pm 1/\mu_j$) under the mappings $\mu\mapsto-\mu$ and $\mu\mapsto 1/\mu$, respectively.

Step (ii). Inside the circles around μ_0 and μ_1, define (explicitly) a function $m_0(y,t,\mu)$ which exactly satisfies the jump condition across Σ obtained from (3.10) by replacing $r(\mu)$ with $r(\mu_0)$ and $r(\mu_1)$, respectively, and by replacing $\delta^2(\mu,\xi)e^{-2it\theta(\mu,\xi)}$ with its large-time approximation.

Step (iii). Define $m_0(y,t,\mu)$ inside the other small contours using the symmetries $m_0(\mu)=\overline{m_0(1/\bar\mu)}$ and $m_0(\mu)=\sigma_3\overline{m_0(-\bar\mu)}\sigma_3$ (which are consistent with the symmetries of $M_2(\mu)$).

Step (iv). Define $\widehat{m}(\mu)$ by

$$\widehat{m}(y,t,\mu)=\begin{cases}M_2(y,t,\mu)m_0^{-1}(y,t,\mu), & \text{inside } \pm\gamma_j \text{ and } \pm\gamma_j^{-1},\\ M_2(y,t,\mu), & \text{otherwise,}\end{cases}$$

Then $\widehat{m}(\mu)$ satisfies the conditions of the RH problem

$$\begin{cases}\widehat{m}_+(y,t,\mu)=\widehat{m}_-(y,t,\mu)\widehat{J}(y,t,\mu), & \mu\in\widehat{\Sigma}:=\Sigma\cup_j\{\pm\gamma_j\}\cup_j\{\pm\gamma_j^{-1}\},\\ \widehat{m}(y,t,\mu)\to I, & \mu\to\infty,\end{cases}$$

where

$$\widehat{J}(y,t,\mu)=\begin{cases} m_0^{-1}(y,t,\mu), & \mu\in\cup_j\{\pm\gamma_j\}\cup_j\{\pm\gamma_j^{-1}\},\\ m_{0-}^{-1}(y,t,\mu)J_2(y,t,\mu)m_{0+}(y,t,\mu), & \mu\in\Sigma\cap\{\mu \text{ in } \cup_j\{\pm\gamma_j^{\pm1}\}\},\\ J_2(y,t,\mu) & \text{otherwise.}\end{cases}$$

On the other hand, the unique solution of this problem can be expressed in terms of the solution $\Theta(\mu)$ of the singular integral equation (see [33]*Lemma 2.9):

$$\widehat{m}(y,t,\mu)=I+\frac{1}{2\pi\mathrm{i}}\int_{\widehat{\Sigma}}\Theta(y,t,s)\widehat{w}(y,t,s)\frac{\mathrm{d}s}{s-\mu}. \tag{3.11}$$

Here $\widehat{w}(y,t,s):=\widehat{J}(y,t,s)-I$ and $\Theta\in I+L^2(\widehat{\Sigma})$ is the solution of the integral equation

$$\Theta(\mu)-\mathcal{C}_{\widehat{w}}\Theta(\mu)=I,$$

where $\mathcal{C}_{\widehat{w}}\colon L^2(\widehat{\Sigma})+L^\infty(\widehat{\Sigma})\to L^2(\widehat{\Sigma})$ is an integral operator defined with the help of the singular Cauchy operator: $\mathcal{C}_{\widehat{w}}f:=\mathcal{C}_-(f\widehat{w})$, where $\mathcal{C}_-=\frac{1}{2}(-I+S_{\widehat{\Sigma}})$ and $S_{\widehat{\Sigma}}$ is the operator associated with $\widehat{\Sigma}$ and defined by the principal value of the Cauchy integral:

$$(S_{\widehat{\Sigma}}f)(\mu)=\frac{1}{2\pi\mathrm{i}}\int_{\widehat{\Sigma}}\frac{f(s)}{s-\mu}ds,\quad \mu\in\widehat{\Sigma}.$$

Here $L^2(\widehat{\Sigma})+L^\infty(\widehat{\Sigma})$ denotes the space of all functions that can be written as the sum of a function in $L^2(\widehat{\Sigma})$ and a function in $L^\infty(\widehat{\Sigma})$.

Step (v). Estimate the large-time behavior of $\widehat{m}(y,t,\mu)$ at $\mu=\mathrm{i}$ and $\mu=0$ taking into account the following facts:

- The main contribution to the right hand side of (3.11) comes from the integrals over the small contours, where $\widehat{w}(y,t,\mu)=m_0^{-1}(y,t,\mu)-I$:

$$\widehat{m}(y,t,\mu)=I+\frac{1}{2\pi\mathrm{i}}\int_{\cup_j\{\pm\gamma_j\}\cup_j\{\pm\gamma_j^{-1}\}}\frac{m_0^{-1}(y,t,s)-I}{s-\mu}\mathrm{d}s+\mathrm{o}(t^{-1/2}). \tag{3.12}$$

  Henceforth the error estimates are uniform for $\varepsilon<\xi<2-\varepsilon$ and $-\frac{1}{4}+\varepsilon<\xi<-\varepsilon$, for any small $\varepsilon>0$. For detailed estimates, see [33].

- In turn, the main contribution to $m_0^{-1}(y,t,\mu)-I$ comes from the asymptotics of the RH problem for parabolic cylinder functions (involved in the construction of $m_0(y,t,\mu)$), see [33]*Appendix B, which can be given explicitly.

3.2. Range $0<\xi<2$. This range is characterized by four real critical points: $\pm\mu_0$ and $\pm\mu_0^{-1}$.

3.2.1. Construction of m_0. First, we approximate $\mathrm{i}t\theta(\mu,\xi)$ using (3.1), the relation

$$\kappa_0 = \frac{1}{4}\left(\mu_0 - \frac{1}{\mu_0}\right) \tag{3.13}$$

between μ_0 and κ_0, and the approximation for $\widehat{\theta}(k,\xi)$ near κ_0, see [7]:

$$\widehat{\theta}(k,\xi) \approx \widehat{\theta}(\kappa_0) + 8f_0(\kappa_0)(k-\kappa_0)^2,$$

where

$$f_0(\kappa_0) = \frac{\kappa_0(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3}, \qquad \widehat{\theta}(\kappa_0) = -\frac{16\kappa_0^3}{(1+4\kappa_0^2)^2}. \tag{3.14}$$

Here and below we use the symbol $\approx$ somewhat loosely to express that the left-hand side is approximated by the right-hand side as a function of the spectral parameter with an error term that we are able to control in the subsequent error estimates (see, e.g., (3.21) and (3.24)–(3.26)). We have $-\mathrm{i}t\theta(\mu,\xi) \approx -\mathrm{i}t\widehat{\theta}(\kappa_0) - \frac{\mathrm{i}\widehat{\mu}^2}{4}$, where the scaled spectral variable $\widehat{\mu}$ is introduced by

$$\mu - \mu_0 = \frac{\widehat{\mu}}{(1+\mu_0^{-2})\sqrt{2f_0 t}}. \tag{3.15}$$

Now we approximate $\delta(\mu,\xi)$ near $\mu=\mu_0$. From (3.4) we have

$$\delta(\mu,\xi) = \left(\frac{\mu-\mu_0}{\mu-1/\mu_0}\right)^{\mathrm{i}h_0}\left(\frac{\mu+1/\mu_0}{\mu+\mu_0}\right)^{\mathrm{i}h_0}\mathrm{e}^{\chi(\mu)},$$

where

$$h_0 = -\frac{1}{2\pi}\ln(1-|r(\mu_0)|^2),$$
$$\chi(\mu) = \frac{1}{2\pi\mathrm{i}}\left(\int_{-\mu_0}^{-1/\mu_0} + \int_{1/\mu_0}^{\mu_0}\right)\ln\frac{1-|r(s)|^2}{1-|r(\mu_0)|^2}\frac{\mathrm{d}s}{s-\mu}$$

(notice that $|r(\mu)| = |r(-\mu)| = |r(1/\mu)|$). Therefore (cf. [7]),

$$\delta(\mu,\xi) \approx (\mu-\mu_0)^{\mathrm{i}h_0}\left(\frac{\mu_0+1/\mu_0}{2\mu_0(\mu_0-1/\mu_0)}\right)^{\mathrm{i}h_0}\mathrm{e}^{\chi(\mu_0)} = \widehat{\mu}^{\mathrm{i}h_0}(128f_0\kappa_0^2 t)^{-\frac{\mathrm{i}h_0}{2}}\mathrm{e}^{\chi(\mu_0)}$$

and thus

$$\delta(\mu,\xi)\mathrm{e}^{-\mathrm{i}t\theta(\mu,\xi)} \approx \delta_{\mu_0}(\xi,t)\widehat{\mu}^{\mathrm{i}h_0}\mathrm{e}^{-\frac{\mathrm{i}\widehat{\mu}^2}{4}}, \tag{3.16}$$

where

$$\delta_{\mu_0}(\xi,t) = \mathrm{e}^{-\mathrm{i}t\widehat{\theta}(\kappa_0(\mu_0))}\mathrm{e}^{\chi(\mu_0)}(128f_0(\kappa_0(\mu_0))\kappa_0^2(\mu_0)t)^{-\frac{\mathrm{i}h_0}{2}}. \tag{3.17}$$

The approximation (3.16) suggests introducing $m_0(y,t,\mu)$ (near $\mu=\mu_0$) as follows:

$$m_0(y,t,\mu) = D(\xi,t)m^X(\xi,\widehat{\mu})D^{-1}(\xi,t), \tag{3.18}$$

where $D(\xi,t)=\delta_{\mu_0}^{\sigma_3}(t)$ and $m^X(\xi,\widehat{\mu})$ is the solution of the RH problem, in the $\widehat{\mu}$-complex plane, whose solution is given in terms of parabolic cylinder functions [33] (with $q=-\bar{r}(\mu_0)$).

Since (see (3.15)) finite values of μ correspond to growing (with t) values of $\widehat{\mu}$, the large-time asymptotics of $m_0(y,t,\mu)$ for μ on the small contours surrounding $\pm\mu_0$ and $\pm\frac{1}{\mu_0}$ involves the large-$\widehat{\mu}$ asymptotics of $m^X(\xi,\widehat{\mu})$, which is given by (see [33]*Appendix B)

$$m^X(\xi,\widehat{\mu})=I+\frac{\mathrm{i}}{\widehat{\mu}}\begin{pmatrix}0 & -\beta_{\mu_0}(\xi)\\ \bar{\beta}_{\mu_0}(\xi) & 0\end{pmatrix}+\mathrm{O}(\widehat{\mu}^{-2}) \tag{3.19}$$

with

$$\beta_{\mu_0}(\xi)=\sqrt{h_0}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}-\arg(-\bar{r}(\mu_0))+\arg\Gamma(\mathrm{i}h_0)\right)}, \tag{3.20}$$

where Γ is Euler's gamma function. From (3.15), (3.18) and (3.19) we have

$$\begin{aligned}m_0^{-1}(y,t,\mu)&=D(\xi,t)(m^X)^{-1}(\xi,\widehat{\mu}(\mu))D^{-1}(\xi,t)\\&=D(\xi,t)\left(I-\frac{\mathrm{i}}{\widehat{\mu}(\mu)}\begin{pmatrix}0 & -\beta_{\mu_0}(\xi)\\ \bar{\beta}_{\mu_0}(\xi) & 0\end{pmatrix}\right)D^{-1}(\xi,t)+\mathrm{O}(t^{-1})\\&=I+\frac{B(\xi,t)}{\sqrt{t}(\mu-\mu_0)}+\mathrm{O}(t^{-1}),\end{aligned} \tag{3.21}$$

where

$$B(\xi,t)=\begin{pmatrix}0 & B_0(\xi,t)\\ \bar{B}_0(\xi,t) & 0\end{pmatrix}\text{ with }B_0(\xi,t)=\frac{\mathrm{i}\delta_{\mu_0}^2(\xi,t)\beta_{\mu_0}(\xi)}{(1+\mu_0^{-2})\sqrt{2f_0(\kappa_0(\mu_0))}}. \tag{3.22}$$

Here the estimate $\mathrm{O}(t^{-1})$ is uniform for ξ and μ such that $\varepsilon_1<\xi<2-\varepsilon_1$ and $|\mu-\mu_0|=\varepsilon_2$ for any small positive ε_j, $j=1,2$.

3.2.2. Asymptotics for $\widehat{m}$. In view of our algorithm for representing u in terms of the solution of the associated regular RH problem, see (2.22), (2.11), (2.12), and (2.1), we need to know the asymptotics for $\widehat{m}(y,t,0)$, $\widehat{m}(y,t,\mathrm{i})$, and $\widehat{m}_1(y,t)$, where $\widehat{m}_1$ is extracted from the expansion $\widehat{m}(y,t,\mu)=\widehat{m}(y,t,\mathrm{i})+\widehat{m}_1(y,t)(\mu-\mu_0)+\mathrm{O}((\mu-\mu_0)^2)$ as $\mu\to\mu_0$. By (3.21) and the residue theorem, the leading contributions of the integral over γ_0 into (3.12) for these quantities are, respectively,

$$\frac{B}{\mu_0\sqrt{t}},\quad\frac{B}{(\mu_0-\mathrm{i})\sqrt{t}},\quad\text{and }\frac{B}{(\mu_0-\mathrm{i})^2\sqrt{t}}. \tag{3.23}$$

In order to take into account the contributions of all small contours, we extend the definition of m_0 by symmetries (as indicated in Step (iii)). This gives

$$\begin{aligned}\widehat{m}(y,t,0)&=I+\left(\frac{B}{\mu_0}-\frac{\bar{B}}{\mu_0}-\frac{1}{\mu_0^2}\frac{\bar{B}}{\mu_0^{-1}}+\frac{1}{\mu_0^2}\frac{B}{\mu_0^{-1}}\right)\frac{1}{\sqrt{t}}+\mathrm{o}(t^{-1/2})\\&=I+\frac{4i\operatorname{Im}B_0(\xi,t)}{\mu_0\sqrt{t}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+\mathrm{o}(t^{-1/2}),\end{aligned} \tag{3.24}$$

$$\widehat{m}(y,t,\mathrm{i}) = I + \left(\frac{B}{\mu_0-\mathrm{i}} + \frac{\bar{B}}{-\mu_0-\mathrm{i}} - \frac{1}{\mu_0^2}\frac{\bar{B}}{\mu_0^{-1}-\mathrm{i}} - \frac{1}{\mu_0^2}\frac{B}{-\mu_0^{-1}-\mathrm{i}}\right)\frac{1}{\sqrt{t}} + \mathrm{o}(t^{-1/2})$$
$$= I + \frac{2\mathrm{i}\,\mathrm{Im}\,B_0(\xi,t)}{\mu_0\sqrt{t}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + \mathrm{o}(t^{-1/2}), \tag{3.25}$$

and

$$\widehat{m}_1(y,t) = \left(\frac{B}{(\mu_0-\mathrm{i})^2} + \frac{\bar{B}}{(-\mu_0-\mathrm{i})^2} - \frac{1}{\mu_0^2}\frac{\bar{B}}{(\mu_0^{-1}-\mathrm{i})^2} - \frac{1}{\mu_0^2}\frac{B}{(-\mu_0^{-1}-\mathrm{i})^2}\right)\frac{1}{\sqrt{t}} + \mathrm{o}(t^{-1/2})$$
$$= \frac{4}{\sqrt{t}}\begin{pmatrix}0 & \mathrm{Re}\,\frac{B_0}{(\mu_0-\mathrm{i})^2}\\ \mathrm{Re}\,\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2} & 0\end{pmatrix} + \mathrm{o}(t^{-1/2}). \tag{3.26}$$

3.2.3. From $\widehat{m}$ back to M^R. In Section 3.2.2 we presented the large-time asymptotics of $\widehat{m}(y,t,\mu)$ (and thus of $M_2(y,t,\mu)$) for the dedicated values of μ. Since $P(y,t,0)=0$ whereas $P(y,t,\mu)$ tends to I exponentially fast, as $t\to+\infty$ for all μ close to i, in order to obtain the leading terms of the asymptotics for $M^R(y,t,\mu) = M_1(y,t,\mu)\delta^{\sigma_3}(\mu,\xi) = M_2(y,t,\mu)P^{-1}(y,t,\mu)\delta^{\sigma_3}(\mu,\xi)$, we need to know $\delta(\mu,\xi)$ (3.4) for $\mu=0$ and μ near i.

Due to the symmetry $|r(\mu)| = |r(-\mu)|$ we have

$$\delta(0,\xi) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_b(\xi)}\frac{\ln(1-|r(s)|^2)}{s}\mathrm{d}s\right\} \equiv 1. \tag{3.27}$$

Let I_0 and I_1 be such that $\delta(\mu,\xi) = \mathrm{e}^{I_0+I_1(\mu-\mathrm{i})+\dots}$ as $\mu\to\mathrm{i}$. Then, using again the symmetry $|r(\mu)| = |r(-\mu)|)$,

$$I_0 = \frac{1}{2\pi\mathrm{i}}\int_{\Sigma_b(\xi)}\frac{\ln(1-|r(s)|^2)}{s-\mathrm{i}}\mathrm{d}s = \frac{1}{\pi}\int_{1/\mu_0}^{\mu_0}\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s.$$

On the other hand,

$$I_1 = \frac{1}{2\pi\mathrm{i}}\int_{1/\mu_0}^{\mu_0}\ln(1-|r(s)|^2)\left(\frac{1}{(s-\mathrm{i})^2} + \frac{1}{(-s-\mathrm{i})^2}\right)\mathrm{d}s$$
$$= \frac{1}{\pi\mathrm{i}}\int_{1/\mu_0}^{\mu_0}\ln(1-|r(s)|^2)\frac{s^2-1}{(s^2+1)^2}\mathrm{d}s \equiv 0,$$

the latter equality being due to the symmetry $|r(\mu)| = |r(\mu^{-1})|$. Thus, as $\mu\to\mathrm{i}$,

$$\delta(\mu,\xi) = \delta(\mathrm{i},\xi) + \mathrm{O}((\mu-\mathrm{i})^2) \text{ with } \delta(\mathrm{i},\xi) = \exp\left\{\frac{1}{\pi}\int_{1/\mu_0}^{\mu_0}\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s\right\}. \tag{3.28}$$

Therefore, if $M^R(y,t,\mu) = M^R(y,t,\mathrm{i}) + M_1^R(y,t)(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2)$ we have the following asymptotics for $M^R(y,t,0)$, $M^R(y,t,\mathrm{i})$, and $M_1^R(y,t)$:

$$M^R(y,t,0) = \widehat{m}(y,t,0) = I + \frac{4\mathrm{i}\,\mathrm{Im}\,B_0(\xi,t)}{\mu_0\sqrt{t}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + \mathrm{o}(t^{-1/2}), \tag{3.29a}$$

$$\begin{aligned} M^R(y,t,\mathrm{i}) &= \widehat{m}(y,t,\mathrm{i})\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{O}(\mathrm{e}^{-\varepsilon t}) \\ &= \left(I + \frac{2\mathrm{i}\,\mathrm{Im}\,B_0(\xi,t)}{\mu_0\sqrt{t}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}), \end{aligned} \tag{3.29b}$$

$$\begin{aligned} M_1^R(y,t) &= \widehat{m}_1(y,t)\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{O}(\mathrm{e}^{-\varepsilon t}) \\ &= \frac{4}{\sqrt{t}}\begin{pmatrix}0 & \mathrm{Re}\,\frac{B_0}{(\mu_0-\mathrm{i})^2}\\ \mathrm{Re}\,\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2} & 0\end{pmatrix}\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}), \end{aligned} \tag{3.29c}$$

where $B_0(\xi,t)$ is given by (3.22) and $\delta(\mathrm{i},\xi)$ is given by (3.28).

3.2.4. Long-time asymptotics of u. Combining the asymptotics (3.29) for $M^R(y,t,\mu)$ with (2.11), (2.12), (2.14), and (2.17), we can obtain the leading term of the long-time asymptotics of $u(x,t)$.

Introducing $\eta := \frac{2\,\mathrm{Im}\,B_0}{\mu_0\sqrt{t}}$, from (3.29a) we have:

$$\Delta(y,t) = \sigma_1[M^R(y,t,0)]^{-1} = \begin{pmatrix}2\mathrm{i}\eta & 1\\ 1 & -2\mathrm{i}\eta\end{pmatrix} + \mathrm{o}(t^{-1/2}). \tag{3.30}$$

Therefore, for

$$M(\mu) = \left(I - \frac{1}{\mu}\sigma_1\right)^{-1}\left(I - \frac{1}{\mu}\Delta\right)M^R(\mu) \tag{3.31}$$

we have $M(\mu) = I_1(\mu)I_2(\mu)M^R(\mu) + \mathrm{o}(t^{-1/2})$, where

$$I_1(\mu) = \begin{pmatrix}\frac{\mu^2}{\mu^2-1} & \frac{\mu}{\mu^2-1}\\ \frac{\mu}{\mu^2-1} & \frac{\mu^2}{\mu^2-1}\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{\mathrm{i}}{2}\\ -\frac{\mathrm{i}}{2} & \frac{1}{2}\end{pmatrix} - \frac{\mathrm{i}}{2}I(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \tag{3.32a}$$

$$\begin{aligned} I_2(\mu) &= \begin{pmatrix}1-\frac{2\mathrm{i}\eta}{\mu} & -\frac{1}{\mu}\\ -\frac{1}{\mu} & 1+\frac{2\mathrm{i}\eta}{\mu}\end{pmatrix} \\ &= \begin{pmatrix}1-2\eta & \mathrm{i}\\ \mathrm{i} & 1+2\eta\end{pmatrix} + \begin{pmatrix}-2\mathrm{i}\eta & -1\\ -1 & 2\mathrm{i}\eta\end{pmatrix}(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \end{aligned} \tag{3.32b}$$

$$M^R(\mu) = \begin{pmatrix}1 & \mathrm{i}\eta\\ -\mathrm{i}\eta & 1\end{pmatrix}\delta^{\sigma_3}(\mathrm{i}) + \begin{pmatrix}0 & \beta_1\\ \beta_2 & 0\end{pmatrix}\delta^{\sigma_3}(\mathrm{i})(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \tag{3.32c}$$

with

$$\beta_1 = \frac{4}{\sqrt{t}}\mathrm{Re}\,\frac{B_0}{(\mu_0-\mathrm{i})^2}, \quad \beta_2 = \frac{4}{\sqrt{t}}\mathrm{Re}\,\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2}. \tag{3.33}$$

Substituting (3.32) into (3.31) and keeping the terms of order $t^{-1/2}$ we have

$$\begin{aligned} M(\mu) &= \begin{pmatrix}(1-\eta)\delta(\mathrm{i}) & 0\\ 0 & (1+\eta)\delta^{-1}(\mathrm{i})\end{pmatrix} \\ &\quad + \begin{pmatrix}0 & (\beta_1+\eta)\delta^{-1}(\mathrm{i})\\ (\beta_2-\eta)\delta(\mathrm{i}) & 0\end{pmatrix}(\mu-\mathrm{i}) + \mathrm{o}((\mu-\mathrm{i})t^{-1/2}) \end{aligned}$$

and thus (see (2.11)) $a_1 = (1-\eta)\delta(\mathrm{i}) + \mathrm{o}(t^{-1/2})$, $a_2 = (\beta_1+\eta)\delta^{-1}(\mathrm{i}) + \mathrm{o}(t^{-1/2})$, and $a_3 = (\beta_2-\eta)\delta(\mathrm{i}) + \mathrm{o}(t^{-1/2})$. It follows (see (2.12)) that

$$\widehat{u}(y,t) = -(\beta_1+\beta_2) + \mathrm{o}(t^{-1/2}) = \frac{8(1-\mu_0^2)}{(1+\mu_0^2)^2\sqrt{t}}\operatorname{Re} B_0 + \mathrm{o}(t^{-1/2}), \tag{3.34a}$$

$$x(y,t) = y + 2\ln((1-\eta)\delta(\mathrm{i})) + \mathrm{o}(t^{-1/2}) = y + y_0(\xi) + \mathrm{O}(t^{-1/2}), \tag{3.34b}$$

where (see (3.28)) $y_0(\xi) = \frac{2}{\pi}\int_{1/\mu_0}^{\mu_0} \frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s$.

Recalling the definition (3.22) of B_0 and introducing the real-valued functions $\varphi_\delta(\xi,t)$ and $\varphi_\beta(\xi)$ (see (3.20) and (3.17)) by

$$\beta_{\mu_0}(\xi) = \sqrt{h_0}\mathrm{e}^{\mathrm{i}\varphi_\beta(\xi)}, \quad \delta_{\mu_0}^2(\xi,t) = \mathrm{e}^{\mathrm{i}\varphi_\delta(\xi,t)},$$

we have $B_0 = \frac{\sqrt{h_0}}{(1+\mu_0^{-2})\sqrt{2f_0}}\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\varphi_\delta(\xi,t))+\varphi_\beta(\xi))}$ and thus

$$\operatorname{Re} B_0(\xi,t) = \frac{\sqrt{h_0}}{(1+\mu_0^{-2})\sqrt{2f_0}}\cos\left\{\frac{\pi}{2} + \varphi_\delta(\xi,t) + \varphi_\beta(\xi)\right\}. \tag{3.35}$$

Substituting (3.35) into (3.34a) gives the asymptotics of the solution of the Cauchy problem for the mCH equation (in the form (2.2)) expressed parametrically, in the (y,t) variables. Recalling the definitions of f_0, φ_δ, φ_β, β_{μ_0} (see (3.14), (3.17), (3.20)) and the relationship (3.13) between μ_0 and κ_0 we obtain the following large-time asymptotics along the rays $\frac{y}{t} = \xi$ for $0 < \xi < 2$:

$$\widehat{u}(y,t) = \frac{C_1(\xi)}{\sqrt{t}}\cos\{C_2(\xi)t + C_3(\xi)\ln t + C_4(\xi)\} + \mathrm{o}(t^{-1/2}), \tag{3.36}$$

where

$$C_1(\xi) = -\left(\frac{8h_0\kappa_0}{3-4\kappa_0^2}\right)^{\frac{1}{2}}, \tag{3.37a}$$

$$C_2(\xi) = \frac{32\kappa_0^3}{(1+4\kappa_0^2)^2}, \tag{3.37b}$$

$$C_3(\xi) = -h_0, \tag{3.37c}$$

$$\begin{aligned} C_4(\xi) = {} & \frac{3\pi}{4} - \frac{1}{\pi}\left(\int_{-\mu_0}^{-1/\mu_0} + \int_{1/\mu_0}^{\mu_0}\right)\ln\frac{1-|r(s)|^2}{1-|r(\mu_0)|^2}\frac{\mathrm{d}s}{s-\mu_0} \\ & - h_0\ln\frac{128\kappa_0^3(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} - \arg(-\bar{r}(\mu_0)) + \arg\Gamma(\mathrm{i}h_0), \end{aligned} \tag{3.37d}$$

taking into account that h_0, κ_0, and μ_0 are defined as functions of ξ.

In order to express the asymptotics of $\widetilde{u}(x,t) = \widehat{u}(y(x,t),t)$ in the (x,t) variables, we notice that (3.34b) reads

$$\frac{y}{t} = \frac{x}{t} - \frac{y_0}{t} + \mathrm{O}(t^{-3/2})$$

and thus introducing $\zeta := \frac{x}{t}$ gives $C_j(\xi) = C_j(\zeta) + \mathrm{O}(t^{-1})$, $j = 1, \dots, 4$ and

$$C_2(\xi)t = C_2(\zeta)t - \frac{\mathrm{d}C_2}{\mathrm{d}\zeta}(\zeta)y_0(\zeta) + \mathrm{o}(1).$$

It follows that the leading term of the asymptotics for $\widetilde{u}(x,t)$ can be obtained from the right hand side of (3.36), where

(i) $C_j(\xi)$ is replaced by $C_j(\zeta)$ for $j = 1, 2, 3$, and

(ii) $C_4(\xi)$ is replaced by $\widetilde{C}_4(\zeta) := C_4(\zeta) - C_2'(\zeta)y_0(\zeta)$.

In turn, calculating $C_2'(\zeta)$ in terms of $\kappa_0(\zeta)$ and using (3.37b) and $\zeta = \frac{2-8\kappa_0^2}{(1+4\kappa_0^2)^2}$, we get $C_2'(\zeta) = -2\kappa_0$ and thus

$$\widetilde{C}_4(\zeta) = C_4(\zeta) + \frac{4\kappa_0(\zeta)}{\pi} \int_{1/\mu_0}^{\mu_0} \frac{\ln(1 - |r(s)|^2)}{s^2 + 1} \mathrm{d}s. \tag{3.38}$$

The asymptotic analysis we have presented above can be summarized in the following

Theorem 3.1. *In the solitonless case, the solution $\widetilde{u}(x,t)$ of the Cauchy problem for the mCH equation in the form (2.2) has the following large-time asymptotics along the rays $\frac{x}{t} =: \zeta$ in the sector of the (x,t) half-plane $0 < \zeta < 2$:*

$$\widetilde{u}(x,t) = \frac{C_1(\zeta)}{\sqrt{t}} \cos\left\{C_2(\zeta)t + C_3(\zeta)\ln t + \widetilde{C}_4(\zeta)\right\} + \mathrm{o}(t^{-1/2}) \tag{3.39}$$

with C_1, C_2, C_3 defined by (3.37a)–(3.37c), and $\widetilde{C}_4$ defined by (3.38)–(3.37d). Moreover, in these definitions $h_0 = -\frac{1}{2\pi}\ln(1 - |r(\mu_0)|^2)$, $\kappa_0(\zeta) = \left(\frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}\right)^{\frac{1}{2}}$, and $\mu_0(\zeta) > 1$ is characterized by the relation $\kappa_0(\zeta) = \frac{1}{4}(\mu_0(\zeta) - \mu_0(\zeta)^{-1})$.

By using the relation (2.1) between $\widetilde{u}$ and u we immediately obtain, as a corollary, the large-time asymptotics for $u(x,t)$ in the sector $1 < \frac{x}{t} < 3$.

Theorem 3.2. *Let $u_0(x)$ be a smooth function which tends sufficiently fast to 1 as $x \to \pm\infty$ and satisfies $(1 - \partial_x^2)u_0(x) > 0$ for all x. Assume we are in the solitonless case, i.e., assume that the spectral function associated with $u_0(x)$ has no zeros in the upper half-plane and thus the "discrete spectrum" is empty.*

Then the solution $u(x,t)$ of the Cauchy problem (1.1) for the mCH equation has the following large-time asymptotics in the sector of the (x,t) half-plane defined by $1 < \zeta := \frac{x}{t} < 3$:

$$u(x,t) = 1 + \frac{C_1(\zeta-1)}{\sqrt{t}} \cos\left\{C_2(\zeta-1)t + C_3(\zeta-1)\ln t + \widetilde{C}_4(\zeta-1)\right\} + \mathrm{o}(t^{-1/2}). \tag{3.40}$$

The error term is uniform in any sector $1 + \varepsilon < \zeta < 3 - \varepsilon$ where ε is a small positive number.

3.3. Range $-\frac{1}{4} < \xi < 0$. This range is characterized by the presence of eight real critical points: $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_0^{-1}$, and $\pm\mu_1^{-1}$, see Figure 3.2. Similarly to the range $0 < \xi < 2$, we proceed, first, by evaluating the contribution to (3.12) from γ_0 and $-\gamma_1$ and then by using the symmetries $\mu \mapsto -\mu$ and $\mu \mapsto 1/\mu$. Notice that choosing $-\gamma_1$ surrounding $-\mu_1$ is suggested by the structure (3.7) of $\Sigma_b(\xi)$: the parts of $\Sigma_b(\xi)$ ending at μ_0 and at $-\mu_1$ are located to the left of these points. This implies that the construction of the local approximation near $-\mu_1$ follows exactly the same lines as for μ_0, the only difference being in the contributions to the right hand side of (3.2.1) from other critical points.

Namely, from (3.4) we have

$$\delta(\mu,\xi) = \left(\frac{\mu-\mu_0}{\mu-\mu_0^{-1}}\right)^{\mathrm{i}h_0} \left(\frac{\mu+\mu_0^{-1}}{\mu+\mu_0}\right)^{\mathrm{i}h_0} \left(\frac{\mu-\mu_1^{-1}}{\mu+\mu_1^{-1}}\right)^{\mathrm{i}h_1} \left(\frac{\mu+\mu_1}{\mu_1-\mu}\right)^{\mathrm{i}h_1} \mathrm{e}^{\chi(\mu)}, \quad (3.41)$$

where $h_j = -\frac{1}{2\pi}\ln(1-|r(\mu_j)|^2)$, $j = 0, 1$ and

$$\begin{aligned}\chi(\mu) = \frac{1}{2\pi\mathrm{i}}\Bigg\{&-\int_{-\infty}^{-\mu_1} \ln(\mu-s)\mathrm{d}\ln(1-|r(s)|^2) \\ &+\left(\int_{-\mu_0}^{-\mu_0^{-1}} + \int_{\mu_0^{-1}}^{\mu_0}\right)\ln\frac{1-|r(s)|^2}{1-|r(\mu_0)|^2}\frac{\mathrm{d}s}{s-\mu} \\ &+\int_{-\mu_1^{-1}}^{\mu_1^{-1}} \ln\frac{1-|r(s)|^2}{1-|r(\mu_1)|^2}\frac{\mathrm{d}s}{s-\mu} - \int_{\mu_1}^{+\infty}\ln(s-\mu)\mathrm{d}\ln(1-|r(s)|^2)\Bigg\}. \quad (3.42)\end{aligned}$$

Thus, using $\kappa_0(\mu_0)$, $f_0(\kappa_0(\mu_0))$, (see (3.13), (3.14)), and similarly for $\kappa_1(\mu_1)$ and $f_1(\kappa_1(\mu_1))$

$$\delta(\mu,\xi) \approx \widehat{\mu}^{\mathrm{i}h_0}(128 f_0\kappa_0^2 t)^{-\frac{\mathrm{i}h_0}{2}} \left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_1} \mathrm{e}^{\chi(\mu_0)}$$

with $\widehat{\mu} = (\mu-\mu_0)\left(1+\frac{1}{\mu_0^2}\right)\sqrt{2f_0 t}$ for μ near μ_0 and

$$\delta(\mu,\xi) \approx \widehat{\mu}^{\mathrm{i}h_1}(-128 f_1\kappa_1^2 t)^{-\frac{\mathrm{i}h_1}{2}} \left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_0} \mathrm{e}^{\chi(-\mu_1)}$$

with $\widehat{\mu} = (\mu+\mu_1)\left(1+\frac{1}{\mu_1^2}\right)\sqrt{-2f_1 t}$ for μ near $-\mu_1$ (notice that $f_0(\kappa_0) = \frac{\kappa_0(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} > 0$ whereas $f_1(\kappa_1) = \frac{\kappa_1(3-4\kappa_1^2)}{(1+4\kappa_1^2)^3} < 0$). Consequently, the coefficients $\delta_{\mu_0}(\xi,t)$ and $\delta_{\mu_1}(\xi,t)$ to be used in the construction of m_0 (3.18) for μ near μ_0 and $-\mu_1$, respectively, are as follows:

$$\delta_{\mu_0}(\xi,t) = \mathrm{e}^{-\mathrm{i}t\widehat{\theta}(\kappa_0)}\mathrm{e}^{\chi(\mu_0)}\left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_1}(128 f_0\kappa_0^2(\mu_0)t)^{-\frac{\mathrm{i}h_0}{2}}, \quad (3.43\text{a})$$

$$\delta_{\mu_1}(\xi,t) = \mathrm{e}^{\mathrm{i}t\widehat{\theta}(\kappa_1)}\mathrm{e}^{\chi(-\mu_1)}\left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_0}(-128 f_1\kappa_1^2(\mu_1)t)^{-\frac{\mathrm{i}h_1}{2}}, \quad (3.43\text{b})$$

which implies (cf. (3.21))

$$m_0^{-1}(y,t,\mu) = I + \frac{B_{\mu_0}(\xi,t)}{\sqrt{t}(\mu-\mu_0)} + \mathrm{O}(t^{-1}), \qquad \text{for } \mu \text{ inside } \gamma_0,$$
$$m_0^{-1}(y,t,\mu) = I + \frac{B_{\mu_1}(\xi,t)}{\sqrt{t}(\mu+\mu_1)} + \mathrm{O}(t^{-1}), \qquad \text{for } \mu \text{ inside } -\gamma_1,$$

where (cf.(3.22))

$$B_{\mu_0}(\xi,t) = \begin{pmatrix} 0 & B_0(\xi,t) \\ \bar{B}_0(\xi,t) & 0 \end{pmatrix}, \quad B_{\mu_1}(\xi,t) = \begin{pmatrix} 0 & B_1(\xi,t) \\ \bar{B}_1(\xi,t) & 0 \end{pmatrix},$$

with

$$B_0(\xi,t) = \left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{2\mathrm{i}h_1} \frac{\mathrm{i}\delta^2_{\mu_0}(\xi,t)\beta_{\mu_0}(\xi)}{(1+\mu_0^{-2})\sqrt{2f_0(\kappa_0)}}, \tag{3.44a}$$
$$B_1(\xi,t) = \left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{2\mathrm{i}h_0} \frac{\mathrm{i}\delta^2_{\mu_1}(\xi,t)\beta_{\mu_1}(\xi)}{(1+\mu_1^{-2})\sqrt{-2f_1(\kappa_1)}}. \tag{3.44b}$$

Here $\beta_{\mu_0}(\xi)$ is given by (3.20) and

$$\beta_{\mu_1}(\xi) = \sqrt{h_1}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}-\arg(-\bar{r}(-\mu_1))+\arg\Gamma(\mathrm{i}h_1)\right)}.$$

In turn, due to the symmetries, the asymptotics for $\widehat{m}(y,t,0)$, $\widehat{m}(y,t,\mathrm{i})$, and $\widehat{m}_1(y,t)$ (and thus for $M^R(y,t,0)$, $M^R(y,t,\mathrm{i})$, and $M_1^R(y,t)$) in the present case (cf. (3.24)-(3.26) and (3.29)) involve two terms:

$$M^R(y,t,0) = I + \frac{4\mathrm{i}}{\sqrt{t}}\left(\frac{\operatorname{Im} B_0(\xi,t)}{\mu_0} - \frac{\operatorname{Im} B_1(\xi,t)}{\mu_1}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathrm{o}(t^{-1/2}),$$
$$M^R(y,t,\mathrm{i}) = \left(I + \frac{2\mathrm{i}}{\sqrt{t}}\left(\frac{\operatorname{Im} B_0(\xi,t)}{\mu_0} - \frac{\operatorname{Im} B_1(\xi,t)}{\mu_1}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}),$$
$$M_1^R(y,t) = \frac{4}{\sqrt{t}}\begin{pmatrix} 0 & \operatorname{Re}\left(\frac{B_0}{(\mu_0-\mathrm{i})^2} + \frac{B_1}{(\mu_1+\mathrm{i})^2}\right) \\ \operatorname{Re}\left(\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2} + \frac{\bar{B}_1}{(\mu_1+\mathrm{i})^2}\right) & 0 \end{pmatrix}\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}),$$

where $\delta(\mathrm{i},\xi)$ is now given by

$$\delta(\mathrm{i},\xi) = \exp\left\{\frac{1}{\pi}\left(\int_0^{\mu_1^{-1}} + \int_{\mu_0^{-1}}^{\mu_0} + \int_{\mu_1}^{+\infty}\right)\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s\right\}. \tag{3.45}$$

It follows that the asymptotics for the parametric representation of $\widetilde{u}$, see (3.34a) and (3.34b), takes the form

$$\widehat{u}(y,t) = \frac{8}{\sqrt{t}}\left(\frac{(1-\mu_0^2)}{(1+\mu_0^2)^2}\operatorname{Re} B_0 + \frac{(1-\mu_1^2)}{(1+\mu_1^2)^2}\operatorname{Re} B_1\right) + \mathrm{o}(t^{-1/2}), \tag{3.46a}$$

$$x(y,t) = y + y_{01}(\xi) + \mathrm{O}(t^{-1/2}), \tag{3.46b}$$

where $y_{01}(\xi) = \frac{2}{\pi}\left(\int_0^{\mu_1^{-1}} + \int_{\mu_0^{-1}}^{\mu_0} + \int_{\mu_1}^{+\infty}\right) \frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s$.

Recalling the definitions (3.44) of B_j, $j = 0, 1$, and arguing as in the case $0 < \xi < 2$, we arrive at the asymptotics of $\widehat{u}(y,t)$ (cf. (3.36))

$$\widehat{u}(y,t) = \sum_{j=0,1} \frac{C_1^{(j)}(\xi)}{\sqrt{t}} \cos\left\{C_2^{(j)}(\xi)t + C_3^{(j)}(\xi)\ln t + C_4^{(j)}(\xi)\right\} + \mathrm{o}(t^{-1/2}), \tag{3.47}$$

where

$$C_1^{(j)}(\xi) = -\left(\frac{8h_j\kappa_j}{|3-4\kappa_j^2|}\right)^{\frac{1}{2}}, \tag{3.48a}$$

$$C_2^{(j)}(\xi) = \frac{(-1)^j 32\kappa_j^3}{(1+4\kappa_j^2)^2}, \tag{3.48b}$$

$$C_3^{(j)}(\xi) = -h_j, \tag{3.48c}$$

$$C_4^{(j)}(\xi) = \frac{3\pi}{4} - 2\mathrm{i}\chi((-1)^j\mu_j) - h_j \ln\frac{128\kappa_j^3|3-4\kappa_j^2|}{(1+4\kappa_j^2)^3} - \arg(-\bar{r}((-1)^j\mu_j)) + \arg\Gamma(\mathrm{i}h_j) + 2h_{1-j}\ln\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}, \tag{3.48d}$$

and $\chi(\mu)$ is given by (3.42).

Returning to the (x,t) variables, $C_4^{(j)}(\xi)$, $j = 0, 1$ are to be replaced, similarly to (3.38), by

$$\widetilde{C}_4^{(j)}(\zeta) = C_4^{(j)}(\zeta) + \frac{(-1)^j 4\kappa_j(\zeta)}{\pi}\left(\int_0^{\mu_1^{-1}} + \int_{\mu_0^{-1}}^{\mu_0} + \int_{\mu_1}^{+\infty}\right)\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s, \tag{3.49}$$

which finally leads us to

Theorem 3.3. *In the solitonless case, the solution $\widetilde{u}(x,t)$ of the Cauchy problem for the mCH equation in the form (2.2) has the following large-time asymptotics along the rays $\frac{x}{t} =: \zeta$ in the sector of the (x,t) half-plane $-\frac{1}{4} < \zeta < 0$:*

$$\widetilde{u}(x,t) = \sum_{j=0,1} \frac{C_1^{(j)}(\zeta)}{\sqrt{t}} \cos\left\{C_2^{(j)}(\zeta)t + C_3^{(j)}(\zeta)\ln t + \widetilde{C}_4^{(j)}(\zeta)\right\} + \mathrm{o}(t^{-1/2})$$

with an error term uniform in any sector $-\frac{1}{4} + \varepsilon < \zeta < -\varepsilon$ where ε is a small positive number. The coefficients $C_1^{(j)}, C_2^{(j)}, C_3^{(j)}$ are defined by (3.48a)–(3.48c) and $\widetilde{C}_4^{(j)}$ is defined by (3.49)–(3.48d). In these definitions

$$h_j = -\frac{1}{2\pi}\ln(1-|r(\mu_j)|^2),$$

$$\kappa_0(\zeta) = \left(\frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}\right)^{\frac{1}{2}}, \quad \kappa_1(\zeta) = \left(-\frac{\sqrt{1+4\zeta}+1+\zeta}{4\zeta}\right)^{\frac{1}{2}},$$

and $\mu_j(\zeta) > 1$, $j = 0, 1$ is characterized by the relation $\kappa_j(\zeta) = \frac{1}{4}(\mu_j(\zeta) - \mu_j(\zeta)^{-1})$.

Using again (2.1) we obtain, as a corollary, the large-time asymptotics of $u(x,t)$ in the sector $\frac{3}{4} < \frac{x}{t} < 1$.

Theorem 3.4. *Let $u_0(x)$ be a smooth function which tends sufficiently fast to 1 as $x \to \pm\infty$ and satisfies $(1-\partial_x^2)u_0(x) > 0$ for all x. Assume we are in the solitonless case, i.e., assume that the spectral function associated with $u_0(x)$ has no zeros in the upper half-plane and thus the "discrete spectrum" is empty.*

Then the solution $u(x,t)$ of the Cauchy problem (1.1) for the mCH equation has the following large-time asymptotics along the rays $\frac{x}{t} =: \zeta$ in the sector of the (x,t) half-plane defined by $\frac{3}{4} < \zeta < 1$:

$$u(x,t) = 1 + \sum_{j=0,1} \frac{C_1^{(j)}(\zeta-1)}{\sqrt{t}} \cos\left\{C_2^{(j)}(\zeta-1)t + C_3^{(j)}(\zeta-1)\ln t + \widetilde{C}_4^{(j)}(\zeta-1)\right\} + \mathrm{o}(t^{-1/2}). \quad (3.50)$$

The error term is uniform in any sector $\frac{3}{4} + \varepsilon < \zeta < 1 - \varepsilon$ where ε is small and positive.

4. Soliton asymptotics

As for other soliton equations, the soliton solutions of the mCH equation are associated with the residue conditions (2.7) (see also [37, 38] for the effects of residue conditions in the case of nonlocal nonlinear integrable equations). Accordingly, these conditions give rise to soliton asymptotics in a dedicated sector of the (x,t) plane. They can be handled by adding to the contour small circles around each μ_j and its symmetry counterparts and thus reducing the residue conditions to associated jump conditions across the circles and then proceeding as in the case without residue conditions [5].

The one-soliton solution $u \equiv u_{\theta,\delta}$ with parameters (θ, δ), where $\theta \in (0, \frac{\pi}{2})$, has the following parametric representation [3]:

$$u(x,t) = \widetilde{u}(x-t,t) + 1 = \widehat{u}(y(x-t,t),t) + 1, \quad (4.1a)$$

where

$$\widehat{u}(y,t) = 4\tan^2\theta \, \frac{z^2(y,t) + 2\cos^2\theta \cdot z(y,t) + \cos^2\theta}{(z^2(y,t) + 2z(y,t) + \cos^2\theta)^2} z(y,t), \quad (4.1b)$$

$$x(y,t) = t + y + 2\ln\frac{z(y,t) + 1 + \sin\theta}{z(y,t) + 1 - \sin\theta}, \quad (4.1c)$$

and

$$z(y,t) = 2\delta\sin\theta\, \mathrm{e}^{\sin\theta\left(y - \frac{2}{\cos^2\theta}t\right)}. \quad (4.1d)$$

Notice that if $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, then the x to y correspondence (4.1c) is not one-to-one and thus in this case (4.1) represent a loop-type multi-valued function of x [3]. On the other hand, if $\theta \in (0, \frac{\pi}{3})$, then (4.1) represent a smooth function, which dominates the long-time behavior of the solution of problem (1.1) in an associated sector. Similarly to [5] the following theorem holds:

Theorem 4.1 (soliton asymptotics)**.** *Assume that $a(\mu)$ associated with $u_0(x)$ has $2n$ simple zeros: $\mu_j = \mathrm{e}^{\mathrm{i}\theta_j}$ with $0 < \theta_1 < \cdots < \theta_n < \frac{\pi}{3}$ and $\mu_{n+l} = -\bar{\mu}_l$ for $l = 1, \dots, n$. Then the asymptotics of u (understood as a global solution of (1.1a) or a solution continued beyond possible blow-ups following the RH formalism) in the sector $3 < \frac{x}{t} < 9$ is given as follows:*

1. *In the sectors $\left|\frac{x}{t} - 1 - \frac{2}{\cos^2\theta_j}\right| < \varepsilon$ with any sufficiently small $\varepsilon > 0$,*

$$u(x,t) = u_j(x,t) + \mathrm{O}(t^{-l}), \qquad j = 1, \dots, n$$

with $l \geq 1$ depending on the rate of decay of $u_0(x) - 1$ as $|x| \to \infty$, where u_j is given, parametrically, by (4.1) with θ, δ, and z replaced by θ_j, δ_j, and z_j respectively, where

$$z_j(y,t) = 2\delta_j \sin\theta_j \, \mathrm{e}^{\sin\theta_j \left(y - \frac{2}{\cos^2\theta_j} t + y_j^0\right)}$$

and y_j^0 are constants determined by $\{\theta_m, \delta_m\}_{m=j+1}^n$.

2. *Outside these sectors, $u(x,t) = \mathrm{O}(t^{-l})$.*

Remark 4.2. Since it is the RH problem parametrized by y and t that undergoes the asymptotic analysis, and the soliton solutions (4.1b) are smooth in (y,t) variables, the asymptotic results of Theorem 4.1 hold true for the mCH equation written in (y,t) variables, see [3], even if $a(\mu)$ has zeros at some $\mu^* = \mathrm{e}^{\mathrm{i}\theta^*}$ with $\theta^* \in (\frac{\pi}{3}, \frac{\pi}{2})$. On the other hand, this allows deducing a sufficient condition for wave breaking of solutions of problem (1.1a) (in (x,t) variables): If $a(\mu)$ has a zero $\mu^* = \mathrm{e}^{\mathrm{i}\theta^*}$ with $\theta^* \in (\frac{\pi}{3}, \frac{\pi}{2})$, then wave breaking occurs at a certain finite time. In this case, the mechanism of wave breaking consists in breaking the one-to-one correspondence $x \leftrightarrow y$ (cf. [8]).

Remark 4.3 (other regions). $u(x,t)$ decays rapidly to 0 in the sectors $\frac{x}{t} > 9$ and $\frac{x}{t} < \frac{3}{4}$, cf. [7]. This is due to the fact that for these ranges of values of $\frac{x}{t}$, $\theta(\mu, \xi)$ has no real stationary points (lying on the contour of the original RH problem).

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Асимптотики за великим часом для модифікованого рівняння Камаси–Хольма з ненульовими крайовими умовами

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Ми розглядаємо модифіковане рівняння Камаси–Хольма (мКХ) $m_t + \left((u^2 - u_x^2)m\right)_x = 0$, $m := u - u_{xx}$ на осі $-\infty < x < +\infty$, де $u(x,t)$ задовільняє ненульові крайові умови на нескінченності: $u(x,t) \to 1$ при $x \to \pm\infty$. Метою роботи є дослідження асимптотики за великим часом розв'язків початкової задачі, застосовуючи формалізм задачі Рімана–Гільберта, що був нещодавно розроблений у [3]. Основна увага приділяється одержанню асимптотики у двох секторах півплощини (x,t) $(t > 0)$, де основні асимптотичні члени мають вигляд модульованих та спадаючих (як $t^{-1/2}$) тригонометричних осциляцій, а також асимптотиці у секторі, де у поведінці розв'язку початкової задачі домінують солітони.

Ключові слова: задача Рімана–Гільберта, нелінійний метод найшвидшого спуску, солітони