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Existence and Multiplicity of Solutions for a Class of Fractional Kirchhoff Type Problems with Variable Exponents

M. Ben Mohamed Salah, A. Ghanmi, and K. Kefi

In this paper, we consider some class of Kirchhoff type problems involving the fractional operator with variable exponents. By using direct variational method, we obtain some existence result. Moreover, by combing Mountain pass theorem with Ekeland's variational principle, we prove multiplicity results. The main results of this paper improve and generalize the previous ones introduced in the literature.

Key words: fractional $p(x)$-laplacian, variational methods, generalized Sobolev spaces

Mathematical Subject Classification 2010: 35J35, 35J60, 46E35

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with smooth boundary $\partial\Omega$. In this paper, we are interested in the existence and multiplicity of solutions for the following Kirchhoff type problem

$$\begin{cases} M\left(J(u)\right)(-\Delta)^s_{p(x,\cdot)}u(x) + |u(x)|^{q(x)-2}u(x) \\ \qquad\qquad = \lambda\Big(V_1(x)|u|^{l(x)-2}u - V_2(x)|u(x)|^{\beta(x)-2}u(x)\Big) \quad \text{in } \Omega, \\ \qquad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $0 < s < 1$, p, q, V_1, V_2, l, β and M are functions satisfying some suitable conditions which will be given later. $J(u)$ is given by

$$J(u) = \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy. \tag{1.2}$$

The operator $(-\Delta)^s_{p(\cdot,\cdot)}$ is the so called fractional $p(x,y)$-laplacian which is defined by

$$(-\Delta)^s_{p(x,y)}u(x) = \text{P.V.} \int_{\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy, \quad x \in \Omega,$$

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where P.V. is a commonly used abbreviation in the principal value sense. Note that this operator is a natural generalization of the well known $p(x)$-Laplace operator. These type of operators arise in many different contexts, such as fluids, nonlinear elasticity theory and image processing (see [1,31,36]). Recently, a great deal of attention has been focused on studying problems involving $p(x)$-laplacian operator, we refer the reader to [4,6,10,19,20,24,28–30,30,34]. Also, many papers deal with Dirichlet problems of Kirchhoff type, such problems are introduced by Kirchhoff in [25] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Meanwhile, elliptic problems involving the Kirchhoff type equation involving $p(x)$-Laplace operator can be found in [5,18,27].

Motivated by the above mentioned works, in this paper, we study the existence and the multiplicity of solutions for a new class of Kirchhoff type problems. Precisely, we use a direct variational method in order to prove the existence of at least one solution. Moreover, mountain pass theorem is combined with the Ekeland's variational principle in order to prove the existence of at least two solution. In order to present the main results of this article, We impose the following conditions:

(H_1) The function $M \in C(\mathbb{R}, [0, \infty))$ is such that there exist $0 < m_1 \leq m_2$ and $\alpha > 1$ for which

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}, \quad t \in [0, \infty),$$

and

$$1 < \alpha p(x, x) < \frac{N}{s}, \quad x \in \overline{\Omega}.$$

(H_2) V_1 and V_2 are nonnegative bounded functions in Ω, moreover, there exist $0 < r_0 \leq R_0$, $x_0 \in \Omega$ with $\overline{B_{R_0}(x_0)} \subset \Omega$, and

$$\begin{cases} V_1(x) = 0 & \text{if } x \in \overline{B_{R_0}(x_0) \setminus B_{r_0}(x_0)}, \\ V_1(x) > 0 & \text{if } x \in \Omega \setminus \overline{B_{R_0}(x_0) \setminus B_{r_0}(x_0)}. \end{cases}$$

(H_3) The functions p and q are such that $q(x) \leq p(x, x)$ for all $x \in \overline{\Omega}$, moreover, we have

$$1 < \frac{m_1}{m_2} \left(\frac{p^-}{p^+}\right)^{\alpha-1} l(x) \leq \frac{m_1}{m_2} \left(\frac{p^-}{p^+}\right)^{\alpha-1} \beta(x) \leq p^*(x) := \frac{Np(x, x)}{N - sp(x, x)}.$$

(H_4) Either

$$\begin{aligned} \max_{x \in B_{r_0}(x_0)} l(x) \leq \max_{x \in B_{r_0}(x_0)} \beta(x) &\leq \min\{\alpha p^-, q^-\} \leq \max\{\alpha p^+, q^+\} \\ &< \frac{m_1}{m_2} \left(\frac{p^-}{p^+}\right)^{\alpha-1} \min_{x \in \Omega \setminus B_{R_0}(x_0)} l(x) \\ &< \frac{m_1}{m_2} \left(\frac{p^-}{p^+}\right)^{\alpha-1} \min_{x \in \Omega \setminus B_{R_0}(x_0)} \beta(x), \end{aligned}$$

or

$$\begin{aligned}\max_{x\in\Omega\setminus B_{R_0}(x_0)} l(x) \le \max_{x\in\Omega\setminus B_{R_0}(x_0)} \beta(x) \le \min\{\alpha p^-, q^-\}\\ \le \max\{\alpha p^+, q^+\} < \frac{m_1}{m_2}\left(\frac{p^-}{p^+}\right)^{\alpha-1} \min_{x\in B_{r_0}(x_0)} l(x)\\ < \frac{m_1}{m_2}\left(\frac{p^-}{p^+}\right)^{\alpha-1} \min_{x\in B_{r_0}(x_0)} \gamma(x).\end{aligned}$$

Remark 1.1. We notice that from hypothesis (H$_3$) and (H$_4$), we deduce

$$1 < l(x) < \beta(x) < \alpha p(x,x) < \frac{N}{s} < \infty \tag{H}$$

and

$$\alpha p(x,x) \le q(x) < p^*(x) := \frac{Np(x,x)}{N - sp(x,x)}, \quad x \in \overline{\Omega},$$

where α is the constant given in the assertion (H$_1$).

Since the $p(x,y)$-Laplacian operator is a new and interesting topic, then, problems of type (1.1) are rare, we refer the interested reader to [3,7–9,12,22,37]. Note that in [3,26], the authors have studied a similar problem in the whole space $\mathbb{R}^N$, but the weight functions are in $L^\infty \cap L^r$ for some $r > 1$. While, in [37], the authors have considered a positive bounded weight function. So, in our paper we consider a more general class of weight functions V_1 and V_2, which can be zero in a nontrivial subset of Ω. Also, compared with the paper of Hamdani et al. [22], we have considered the same operator perturbed by $|u(x)|^{q(x)-2}u(x)$, this means that some complicated analysis has to be carefully carried out in this paper. On the other hand, in [22], there is no weight function in the source term. Hence, this paper extend and generalize some papers in the literature.

The main results of this paper are summarized as follows.

Theorem 1.2. *Assume that conditions* (H$_1$)–(H$_4$) *are satisfied. Then, for each* $\lambda > 0$, *problem* (1.1) *has at least one nontrivial weak solution with negative energy.*

Theorem 1.3. *If hypothesis* (H$_1$)–(H$_4$) *hold. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$, *problem* (1.1) *has at least two nontrivial weak solutions with negative energy.*

The rest of this paper is organized as follows, in Section 2, we present some preliminary and important results related to new fractional Sobolev spaces. Section 3, is devoted to the proofs of Theorems 1.2 and 1.3.

2. Notation and Background

In this section, we recall some definitions and basic properties of variable exponent fractional Sobolev spaces. For a deeper treatment on these spaces, we refer the reader to [14,16,23,30], for more details and properties on these spaces.

Put

$$C_+(\overline{\Omega}) := \{h \mid \forall x \in \overline{\Omega} \quad h \in C(\overline{\Omega}) \text{ and } h(x) > 1\},$$

and let $p \in C_+(\overline{\Omega} \times \overline{\Omega})$ and $q \in C_+(\overline{\Omega})$ such that

$$1 < q^- := \min_{x \in \overline{\Omega}} q(x) \leq q(x) \leq q^+ := \max_{x \in \overline{\Omega}} q(x) < +\infty, \tag{2.1}$$

$$1 < p^- := \min_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) \leq p(x,y) \leq p^+ := \max_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) < +\infty. \tag{2.2}$$

Let us define the Lebesgue space with variable exponent as

$$L^{q(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega |u(x)|^{q(x)} dx < \infty \right\},$$

which is equipped with the so-called Luxemburg norm

$$|u|_{q(x)} = \inf \left\{ \mu > 0 \mid \int_\Omega \left| \frac{u(x)}{\mu} \right|^{q(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces are like classical Lebesgue spaces in many respects: they are Banach spaces, they are reflexive if and only if $1 < q^- \leq q^+ < \infty$. Moreover, the inclusion between Lebesgue spaces is generalized naturally, that is, if q_1, q_2 are such that $q_1(x) \leq q_2(x)$ for a.a. $x \in \Omega$, then there exists a continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1}(x)(\Omega)$.

For $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, the Hölder inequality

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{q^-} + \frac{1}{(q')^-} \right) |u|_{q(x)} |v|_{q'(x)}. \tag{2.3}$$

holds true, where q' is such that $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$.

The modular on the space $L^{q(x)}(\Omega)$ is the map $\rho_{q(x)} : L^{q(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{q(x)}(u) := \int_\Omega |u|^{q(x)} dx.$$

This modular satisfies the following results.

Proposition 2.1 (See [30]). *For all $u \in L^{q(x)}(\Omega)$, we have*

1. $|u|_{q(x)} < 1$ (*respectively,* $= 1, > 1$) $\Leftrightarrow \rho_{q(x)}(u) < 1$ (*respectively,* $= 1, > 1$).
2. *If* $|u|_{q(x)} > 1$, *then we have* $|u|_{q(x)}^{q^-} \leq \rho_{q(x)}(u) \leq |u|_{q(x)}^{q^+}$.
3. *If* $|u|_{q(x)} < 1$, *then we have* $|u|_{q(x)}^{q^+} \leq \rho_{q(x)}(u) \leq |u|_{q(x)}^{q^-}$.

For $0 < s < 1$, we define the fractional Sobolev space with variable exponents via the Gagliardo approach as follows:

$$W^{s,q(x),p(x,y)}(\Omega) = \left\{ u \in L^{q(x)}(\Omega) \mid \right.$$

$$\exists t > 0 \quad \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{t^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty \Bigg\}.$$

Note that $W^{s,q(x),p(x,y)}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{W^{s,q(x),p(x,y)}(\Omega)} = \|u\|_{L^{q(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where the variable exponent seminorm $[u]_{s,p(x,y)}$, is given by

$$[u]_{s,p(x,y)} = \inf_{t>0}\left\{\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{t^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \le 1\right\}.$$

Similarly to the discussion of the norm in variable exponent space, we can prove the following lemma:

Lemma 2.2. *The following statements hold true:*

1. *If* $1 \le [u]_{s,p(x,y)} < \infty$, *then*

$$[u]^{p^-}_{s,p(x,y)} \le \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \le [u]^{p^+}_{s,p(x,y)}.$$

2. *If* $[u]_{s,p(x,y)} \le 1$, *then*

$$[u]^{p^+}_{s,p(x,y)} \le \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy \le [u]^{p^-}_{s,p(x,y)}.$$

In the sequel, we denoted by $E := W_0^{s,q(x),p(x,y)}(\Omega)$ the subspace of $W^{s,q(x),p(x,y)}(\Omega)$ which is the closure of compactly supported functions in Ω with respect to the norm $\|u\|_{W^{s,q(x),p(x,y)}(\Omega)}$.

For $u \in W^{s,q(x),p(x,y)}(\Omega)$, we define:

$$\rho(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega |u(x)|^{q(x)}dx,$$

and

$$\|u\|_\rho = \inf_{t>0}\left\{\rho(\frac{u}{t}) \le 1\right\}.$$

Then, $\|\cdot\|_\rho$ is a norm which is equivalent to the norm $\|\cdot\|_{W^{s,q(x),p(x,y)}(\Omega)}$.

Moreover, $\big(W^{s,q(x),p(x,y)}(\Omega), \|\cdot\|_\rho\big)$ is a uniformly convex reflexive Banach space.

Theorem 2.3 (See [16]). *Assume that the functions $q(x)$ and $p(x,y)$ are continuous such that for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$, we have*

$$sp(x,y) < N \quad \text{and} \quad q(x) > p(x,x). \tag{2.4}$$

Let $\beta \in C_+(\Omega)$ and, for all $x \in \overline{\Omega}$, we have

$$1 < \beta^- \leq \beta(x) < \frac{Np(x,x)}{N - sp(x,x)} := p^*(x). \tag{2.5}$$

If (2.1) and (2.2) are satisfied, then there exists $C = C(N, s, p, q, \beta, \Omega) > 0$ such that for every $f \in W^{s,q(x),p(x,y)}(\Omega)$,

$$\|f\|_{L^{\beta(x)}} \leq C\|f\|_{W^{s,q(x),p(x,y)}(\Omega)}.$$

Thus, for any $\beta \in (1, p^*)$, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embedded in $L^{\beta(x)}(\Omega)$. Moreover, this embedding is compact.

Proposition 2.4 (See [15]). *Let q_1 be a measurable function in $L^\infty(\Omega)$, and q_2 be a measurable function such that $1 \leq q_1(x)q_2(x) \leq \infty$, for a.e. $x \in \Omega$. If u is a nontrivial function in $L^{q_2(x)}(\Omega)$, then*

$$\min\left(|u|^{q_1^+}_{q_1(x)q_2(x)}, |u|^{q_1^-}_{q_1(x)q_2(x)}\right) \leq \left||u|^{q_1^+}\right|_{q_1(x)} \leq \max\left(|u|^{q_1^+}_{q_1(x)q_2(x)}, |u|^{q_1^-}_{q_1(x)q_2(x)}\right).$$

In addition, note that the above properties remain true if we replace the space $W^{s,q(x),p(x,y)}(\Omega)$ by $W_0^{s,q(x),p(x,y)}(\Omega)$.

3. Proofs of the main results

In order to formulate the variational approach of problem (1.1), let us recall the definition of weak solutions.

Definition 3.1. We say that $u \in E := W_0^{s,q(x),p(x,y)}(\Omega)$ is a weak solution of problem (1.1) if

$$\begin{aligned} M\left(J(u)\right) &\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)}{|x-y|^{N+sp(x,y)}} dxdy \\ &+ \int_\Omega |u(x)|^{q(x)-2}u(x)v(x)dx \\ &- \lambda \int_\Omega \left(V_1(x)|u|^{l(x)-2} - V_2(x)|u(x)|^{\beta(x)-2}\right) u(x)v(x)dx = 0, \quad v \in E. \end{aligned}$$

Firstly, let us denote by

$$\phi(u) = \lambda\left(\phi_1(u) - \phi_2(u)\right),$$

$$\phi_1(u) = \int_\Omega \frac{V_1(x)}{l(x)}|u|^{l(x)}\, dx \quad \text{and} \quad \phi_2(u) = \int_\Omega \frac{V_2(x)}{\beta(x)}|u|^{\beta(x)}\, dx.$$

The Euler–Lagrange functional corresponding to problem (1.1), is defined by $\psi_\lambda : E_0 \to \mathbb{R}$, where

$$\psi_\lambda(u) = \widehat{M}(J(u)) - \phi(u) + \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx, \quad u \in E_0,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$. In the rest of this article, we need the following lemma.

Lemma 3.2. *If hypothesis* (H$_3$) *and* (H$_4$) *are fulfilled, then we have*

$$l(x) < \beta(x) < p^*(x), \quad x \in \overline{\Omega}.$$

So, we deduce that the embeddings

$$W_0^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{l(x)}(\Omega) \text{ and } W_0^{s,q(x),p(x,y)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega),$$

are compact and continuous.

Remark 3.3. From Lemma 3.2, for any $u \in E$, we have

$$|\phi_1(u)| \le \frac{1}{l^-}|V_1|_\infty ||u|^{l(x)}|_1 \le \begin{cases} \frac{1}{l^-}|V_1|_\infty |u|_{l(x)}^{l^-} & \text{if } |u|_{l(x)} \le 1 \\ \frac{1}{l^-}|V_1|_\infty |u|_{l(x)}^{l^+} & \text{if } |u|_{l(x)} > 1 \end{cases},$$

and

$$|\phi_2(u)| \le \frac{1}{\alpha^-}|V_2|_\infty ||u|^{\beta(x)}|_1 \le \begin{cases} \frac{1}{\beta^-}|V_2|_\infty |u|_{\beta(x)}^{\beta^-} & \text{if } |u|_{\beta(x)} \le 1 \\ \frac{1}{\beta^-}|V_2|_\infty |u|_{\beta(x)}^{\beta^+} & \text{if } |u|_{\beta(x)} > 1 \end{cases}.$$

Moreover, for all $u \in E$, we get

$$|u|_{l(x)} \le c\|u\|_E \quad \text{and} \quad |u|_{\beta(x)} \le c_1\|u\|_E. \tag{3.1}$$

Similarly, using assumption (H$_1$), we have

$$\begin{aligned} |\widehat{M}(J(u))| &\le \frac{m_2}{\alpha}\left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy\right)^{\alpha} \\ &\le \begin{cases} \frac{m_2}{\alpha p^-}\|u\|_E^{\alpha p^-} & \text{if } \|u\|_E \le 1, \\ \frac{m_2}{\alpha p^-}\|u\|_E^{\alpha p^+} & \text{if } \|u\|_E > 1. \end{cases} \end{aligned}$$

Therefore, using Proposition 2.4, we deduce that ψ_λ is well defined on E.

3.1. Proof of Theorem 1.2. In this subsection, using direct variational method, we will present the proof of Theorem 1.2. First let as recall from [8] the following important result.

Proposition 3.4. *The energy functional* $J : E \to \mathbb{R}$ *given by* (1.2) *is sequentially weakly lower semi-continuous and of class* C^1. *Moreover, the mapping* $J' : E \to E^*$ *is a strictly monotone bounded homeomorphism and is of type* (S_+), *that is, if* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty} J'(u_n)(u_n - u) \le 0$, *then* $u_n \to u \in E$.

We note that from Proposition 3.4 and assumption (H$_1$), we can prove that J and $\widehat{M} \circ J$ are in $C^1(E, \mathbb{R})$. Moreover, using assumption (H$_2$) and Proposition 2 in [10], we see that $\phi_1, \phi_2 \in C^1(E, \mathbb{R})$. Thus, $\psi_\lambda \in C^1(E, \mathbb{R})$, and we can demonstrate that for all $u, v \in E$, we have

$$\begin{aligned}\langle d\psi_\lambda(u), v\rangle = M\left(J(u)\right) & \\ & \times \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)}{|x-y|^{N+sp(x,y)}}\, dx\, dy \\ & - \lambda \int_\Omega \left(V_1(x)|u|^{l(x)-2} - V_2(x)|u(x)|^{\beta(x)-2}\right) u(x)v(x)\, dx \\ & + \int_\Omega |u(x)|^{q(x)-2} u(x)v(x)\, dx.\end{aligned}$$

In order to present other properties for the functional ψ_λ, let us introduce some notations and elementary inequalities. From (H$_4$), we know that

$$\max_{x\in B_{r_0}(x_0)} l(x) \le \max_{x\in B_{r_0}(x_0)} \beta(x) \le \min\{\alpha p^-, q^-\} \le \max\{\alpha p^+, q^+\},$$

and

$$\max\{\alpha p^+, q^+\} < \frac{m_1}{m_2}\left(\frac{p^-}{p^+}\right)^{\alpha-1} \min_{x\in\Omega\setminus B_{R_0}(x_0)} l(x) < \frac{m_1}{m_2}\left(\frac{p^-}{p^+}\right)^{\alpha-1} \min_{x\in\Omega\setminus B_{R_0}(x_0)} \beta(x).$$

We denote by l_1 and l_2, the restriction of the function l to $\overline{B_{r_0}(x_0)}$ and $\overline{\Omega \setminus B_{R_0}(x_0)}$, respectively. Also, we introduce the notations

$$\begin{aligned}\bar{l}_1 &:= \max_{x\in B_{r_0}(x_0)} l(x), & \underline{l}_1 &:= \min_{x\in B_{r_0}(x_0)} l(x), \\ \bar{l}_2 &:= \max_{x\in\Omega\setminus B_{R_0}(x_0)} l(x), & \underline{l}_2 &:= \min_{x\in\Omega\setminus B_{R_0}(x_0)} l(x).\end{aligned}$$

From conditions (H$_3$) and (H$_4$), we get for each $x \in \overline{\Omega}$

$$\begin{aligned}1 < \underline{l}_1 \le \bar{l}_1 \le \min\{\alpha p^-, q^-\} \le \max\{\alpha p^+, q^+\} & \\ < \frac{m_1}{m_2}\left(\frac{p^-}{p^+}\right)^{\alpha-1} \underline{l}_2 < \frac{m_1}{m_2}\left(\frac{p^-}{p^+}\right)^{\alpha-1} \bar{l}_2 < p^*(x).\end{aligned}$$

So E is continuously embedded either in $L^{\bar{l}_i}(\Omega)$ and in $L^{\underline{l}_i}(\Omega)$, for $i = 1, 2$. Therefore, there exists $c_0 > 0$ such that

$$\max\left(\int_\Omega |u|^{\bar{l}_i} dx, \int_\Omega |u|^{\underline{l}_i} dx\right) \le c_0 \|u\|_E, \quad u \in E,\ i = 1, 2. \tag{3.2}$$

From (3.2), there exists $c_1 > 0$ such that

$$\int_{B_{r_0}(x_0)} |u|^{l_1}(x) dx \le \int_{B_{r_0}(x_0)} |u|^{\bar{l}_1} dx + \int_{B_{r_0}(x_0)} |u|^{\underline{l}_1}\, dx$$

$$\leq \int_\Omega |u|^{\bar{l}_1} dx + \int_\Omega |u|^{\underline{l}_1} dx$$
$$\leq c_1 \left(\|u\|_E^{\bar{l}_1} + \|u\|_E^{\underline{l}_1} \right), \quad u \in E \tag{3.3}$$

and

$$\int_{\Omega \setminus B_{R_0}(x_0)} |u|^{l_2}(x) dx \leq \int_{\Omega \setminus B_{R_0}(x_0)} |u|^{\bar{l}_2} dx + \int_{\Omega \setminus B_{R_0}(x_0)} |u|^{\underline{l}_2} dx$$
$$\leq \int_\Omega |u|^{\bar{l}_2} dx + \int_\Omega |u|^{\underline{l}_2} dx$$
$$\leq c_1 \left(\|u\|_E^{\bar{l}_2} + \|u\|_E^{\underline{l}_2} \right), \quad u \in E. \tag{3.4}$$

Lemma 3.5. *If hypothesis* (H_1), (H_3), *and* (H_4) *are fulfilled, then the functional ψ_λ is coercive on E.*

Proof. Let (H_1), (H_3), and (H_4) be fulfilled. Then, using Proposition 2.4 for $u \in E$ with $\|u\|_E > 1$, we have

$$\psi_\lambda(u) = \widehat{M}\left(J(u)\right) - \lambda \int_\Omega \frac{V_1(x)}{l(x)} |u|^{l(x)} \, dx + \lambda \int_\Omega \frac{V_2(x)}{\beta(x)} |u|^{\beta(x)} \, dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)} \, dx$$
$$\geq \frac{m_1}{\alpha} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} \, dx \, dy \right)^\alpha - \lambda \int_\Omega \frac{V_1(x)}{l(x)} |u|^{l(x)} \, dx$$
$$\geq \frac{m_1}{\alpha {p^+}^\alpha} \|u\|_E^{\alpha p^+} - \frac{\lambda C}{l^-} |V_1|_\infty |u|_l^{l^-} \geq \frac{m_1}{\alpha {p^+}^\alpha} \|u\|_E^{\alpha p^+} - \lambda C_2 |V_1|_\infty c_1^{l^-} \|u\|^{l^-}.$$

Since $l^- < \alpha p^+$ we infer that $\psi_\lambda(u) \to \infty$ as $\|u\|_E \to \infty$, in other words, ψ_λ is coercive on E □

The following result asserts the existence of a valley for ψ_λ near the origin.

Lemma 3.6. *Assume that conditions* (H_1)–(H_4) *are satisfied. Then there exists $u_0 \in E$ such that $u_0 > 0$ and $\psi_\lambda(tu_0) < 0$ for $t > 0$ small enough.*

Proof. Let $u_0 \in C_0^\infty(\Omega, (0, \infty))$. Then, there exist $x_1 \in \Omega \setminus B_{R_0}(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_1) \subset (\Omega \setminus B_{R_0}(x_0)) \cap \operatorname{supp}(u_0)$, since $l_2^- < \beta_1^+ < \min\{\alpha p^-, q^-\}$, we have

$$\psi_\lambda(tu_0) = \widehat{M}\left(J(tu_0))\right) - \lambda \int_\Omega \frac{V_1(x)}{l(x)} |tu_0(x)|^{l(x)} \, dx$$
$$+ \lambda \int_\Omega \frac{V_2(x)}{\beta(x)} |tu_0(x)|^{\beta(x)} dx + \int_\Omega \frac{|tu_0(x)|^{q(x)}}{q(x)} \, dx$$
$$\leq \frac{m_2}{\alpha} (t_0)^\alpha - \lambda \int_\Omega \frac{V_1(x)}{l(x)} |tu_0(x)|^{l(x)} \, dx$$
$$+ \lambda \int_\Omega \frac{V_2(x)}{\beta(x)} |tu_0(x)|^{\beta(x)} dx + \int_\Omega \frac{|tu_0(x)|^{q(x)}}{q(x)} dx$$

$$
\begin{aligned}
&\leq \frac{m_2}{\alpha(p^-)^\alpha} t^{\alpha p^-} \left(\int_{\Omega\times\Omega} \frac{|u_0(x)-u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \right)^\alpha \\
&\quad - \lambda t^{\underline{l}_2} \int_{\Omega\setminus B_{R_0}(x_0)} \frac{V_1(x)}{l(x)} |u_0(x)|^{l(x)}\, dx \\
&\quad + \lambda t^{\beta_1^+} \int_{B_{r_0}(x_0)} \frac{V_2(x)}{\beta(x)} |u_0(x)|^{\beta(x)}\, dx + \frac{t^{q^-}}{q^-} \int_\Omega |u_0(x)|^{q(x)}\, dx \\
&\leq t^{\alpha p^-} \max\{\|u_0\|^{\alpha p^+}, \|u_0\|^{\alpha p^-}\} - \lambda \frac{t^{\underline{l}_2}}{\underline{l}_2} \int_{\Omega\setminus B_{R_0}(x_0)} V_1(x)|u_0(x)|^{l(x)}\, dx \\
&\quad + \frac{t^{r_1^+}}{r_1^-} \lambda \int_{B_{r_0}(x_0)} V_2(x)|u_0(x)|^{\beta(x)}\, dx + \frac{t^{q^-}}{q^-} \max\{\|u_0\|^{q^-}, \|u_0\|_E^{q^+}\} \\
&\leq t^{\underline{l}_2} \left(\frac{t^{\beta_1^+ - \underline{l}_2}}{\beta_1^-} [c_3 \max\{\|u_0\|^{\alpha p^+}, \|u_0\|_E^{\alpha p^-}\} + c_4 \max\{\|u_0\|_E^{q^-}, \|u_0\|_E^{q^+}\} \right. \\
&\quad \left. + \lambda \int_{B_{r_0}(x_0)} V_2(x)|u_0(x)|^{\beta(x)}\, dx] \right) \\
&\quad - t^{\underline{l}_2} \left(\frac{\lambda}{\underline{l}_2} \int_{\Omega\setminus B_{R_0}(x_0)} V_1(x)|u_0(x)|^{l(x)}\, dx \right) < 0, \quad t < \min\{1, \delta\},
\end{aligned}
$$

where

$$
\begin{aligned}
\delta &= \left(\frac{1}{\delta_1} \beta_1^- \lambda \int_{\Omega\setminus B_{R_0}(x_0)} V_1(x)|u_0|^{l(x)} dx) \right)^{\frac{1}{\beta_1^+ - l_2^-}}, \\
\delta_1 &= \underline{l}_2 \Bigg(c_3 \max(\|u_0\|_E^{\alpha p^+}, \|u_0\|_E^{\alpha p^-}) + c_4 \max(\|u_0\|_E^{q^+}, \|u_0\|_E^{q^-}) \\
&\qquad\qquad + \lambda \int_{B_{r_0}(x_0)} V_2(x)|u_0|^{\beta(x)} dx \Bigg)
\end{aligned}
$$

Finally, if $\lambda \in (0, \lambda^*)$ and hypothesis (H$_1$)–(H$_4$) are satisfied, then, the functional ψ_λ is coercive on E and weakly lower semi-continuous. So, there exists a global minimizer u. Since ψ_λ is of class C^1, then, u is a critical point of ψ_λ. Therefore, u is a weak solution of problem (1.1). Moreover, Lemma 3.6 ensures that u is non-trivial. □

3.2. Proof of Theorem 1.3. In this subsection, we establish the existence of multiple solutions to problem (1.1). The proof is related to Ekeland's variational principle combined with mountain pass theorem. So we assume that hypothesis (H$_1$)–(H$_4$) are satisfied.

Lemma 3.7. *The functional ψ_λ satisfies the Palais–Smale condition in E.*

Proof. Let $\{u_n\}$ be a sequence in E such that

$$
\psi_\lambda(u_n) \to \underline{c} \quad \text{and} \quad d\psi_\lambda(u_n) \to 0_{E^*} \quad \text{as } n \to \infty, \tag{3.5}
$$

where E^* is the dual space of E.

We will prove that $\{u_n\}$ is bounded in E. By contradiction, up to a subsequence, we assume that $\|u_n\|_E \to \infty$ as $n \to \infty$. From (H$_4$), (3.5) and Proposition 2.1, we have for n large enough

$$
\begin{aligned}
1 + \underline{c} + \|u_n\|_E &\geq \psi_\lambda(u_n) - \frac{1}{\underline{l}_2} d\psi_\lambda(u_n)(u_n) \\
&= \widehat{M}\left(J(u_n)\right) - \lambda \int_\Omega \frac{V_1(x)}{l(x)} |u_n(x)|^{l(x)}\, dx \\
&+ \lambda \int_\Omega \frac{V_2(x)}{\beta(x)} |u_n(x)|^{\beta(x)}\, dx + \int_\Omega \frac{|u_n(x)|^{q(x)}}{q(x)}\, dx \\
&+ M\left(t_n\right) \int_{\Omega\times\Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy - \frac{1}{l_2^-} \int_\Omega |u_n(x)|^{q(x)}\, dx \\
&+ \frac{\lambda}{l_2^-} \int_\Omega V_1(x)|u_n(x)|^{l(x)}\, dx - \frac{\lambda}{l_2^-} \int_\Omega V_2(x)|u_n(x)|^{\beta(x)}\, dx \\
&\geq c_5 \int_{\Omega\times\Omega} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy + c_6 \int_\Omega |u_n(x)|^{q(x)}\, dx \\
&- c_7 \int_{B_{r_0}(x_0)} \frac{V_1(x)}{l(x)} |u_n(x)|^{l_1(x)}\, dx + c_8 \int_{B_{r_0}(x_0)} \frac{V_2(x)}{l_1(x)} |u_n(x)|^{\beta(x)}\, dx \\
&\geq c_5 \|u_n\|_E^{\alpha p^-} - C_6 \left(\|u_n\|_E^{\underline{l}_1} + \|u_n\|_E^{\overline{l}_1} \right) + C_7 \left(\|u_n\|_E^{\beta^-} + \|u_n\|_E^{\beta^+} \right) \\
&\geq c_5 \|u_n\|_E^{\alpha p^-} - C_6 \left(\|u_n\|_E^{\underline{l}_1} + \|u_n\|_E^{\overline{l}_1} \right) + C_7 \left(\|u_n\|_E^{\beta^-} + \|u_n\|_E^{\beta^+} \right)
\end{aligned}
$$

Since $1 < \max\left(\overline{l}_1, \beta^+\right) < p^-$, then, by dividing the above inequality by $\|u_n\|_E^{p^-}$, and by passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. Therefore, up to a subsequence, there is $u \in E$ such that $\{u_n\}$ converges weakly to u in E. Moreover, $\{\|u_n - u\|_E\}$ is bounded, so using Hölder inequality, we have

$$
\int_\Omega V_1(x)|u_n|^{l(x)-2} u_n(u - u_n) dx \leq \|V_1\|_\infty |u_n|_{l(x)} |u - u_n|_{l'(x)}.
$$

So from (H$_3$), and the Sobolev embedding, we obtain

$$
\lim_{n\to\infty} \int_\Omega V_1(x)|u_n|^{l(x)-2} u_n(u - u_n) dx = 0. \tag{3.6}
$$

Similar arguments show that

$$
\lim_{n\to\infty} \int_\Omega V_2(x)|u_n|^{\beta(x)-2} u_n(u - u_n) dx = 0, \tag{3.7}
$$

and

$$
\lim_{n\to\infty} \int_\Omega |u_n|^{q(x)-2} u_n(u - u_n) dx = 0. \tag{3.8}
$$

By combining (3.5)–(3.6) and (3.7) with the boundness of $\{u_n - u\}$ in E, we get $d\psi_\lambda(u_n) \to 0_{E^*}$ as $n \to \infty$, it follows that $\{u_n\}$ converges strongly to u in E. $\square$

The following property shows the existence of a mountain pass geometry for ψ_λ near the origin.

Lemma 3.8. *Suppose that the hypotheses of Theorem 1.2 are fulfilled, then there exist $\lambda^* > 0$, $\rho > 0$ and $a > 0$ such that for any $\lambda \in (0, \lambda^*)$, if $u \in E$ with $\|u\|_E = \rho$, then $\psi_\lambda(u) \geq a > 0$.*

Proof. Since the embedding $E \hookrightarrow L^{l(x)}(\Omega)$ is continuous, we have

$$|u|_l \leq c_1 \|u\|_E, \quad u \in E. \tag{3.9}$$

We fix $\rho \in (0,1)$ with $\rho < \frac{1}{c_1}$. Then, (3.9) implies that

$$|u|_l(x) < 1 \quad \text{for all } u \in E \text{ with } \|u\|_E = \rho.$$

Moreover, from hypothesis (H_1), we have $\widehat{M}(t) \geq \frac{m_1}{\alpha} t^\alpha$ for all $t \in [0, +\infty)$. Consequently, from Proposition 2.1, Equations (3.3), (3.4), (3.9) and using Hölder inequality for all $u \in E$ with $\|u\|_E < 1$, we obtain

$$\begin{aligned}
\psi_\lambda(u) &= \widehat{M}\left(J(u)\right) - \lambda \int_\Omega \frac{V_1(x)}{l(x)} |u|^{l(x)}\, dx + \lambda \int_\Omega \frac{V_2(x)}{\beta(x)} |u|^{\beta(x)} dx + \int_\Omega \frac{|u|^{q(x)}}{q(x)}\, dx \\
&\geq \frac{m_1}{\alpha} \left(J(u)\right)^\alpha - \lambda \int_{B_{r_0}(x_0)} \frac{V_1(x)}{l(x)} |u|^{l(x)}\, dx - \lambda \int_{\Omega \setminus B_{R_0}(x_0)} \frac{V_1(x)}{l(x)} |u|^{l(x)}\, dx \\
&\geq \frac{m_1}{\alpha p^+} \|u\|_E^{\alpha p^+} - \frac{\lambda C}{l^-} |V_1|_\infty \left(\|u\|_E^{\bar{l}_1} + \|u\|_{E_0}^{\underline{l}_1} + \|u\|_E^{\bar{l}_2} + \|u\|_E^{\underline{l}_2} \right) \\
&\geq \left[\frac{m_1}{2\alpha p^+} \|u\|^{\alpha p^+} - \frac{\lambda c_1}{l^-} |V_1|_\infty \left(\|u\|_E^{\bar{l}_1 - \alpha p^+} + \|u\|_E^{\underline{l}_1 - \alpha p^+} \right) \right] \|u\|_E^{\alpha p^+} \\
&\quad + \left[\frac{m_1}{2\alpha p^+} \|u\|^{\alpha p^+} - \frac{\lambda c_1}{l^-} |V_1|_\infty \left(\|u\|_E^{\bar{l}_2 - \alpha p^+} + \|u\|_E^{\underline{l}_2 - \alpha p^+} \right) \right] \|u\|_E^{\alpha p^+}.
\end{aligned}$$

Let $g : [0,1] \to \mathbb{R}$, be a function defined by

$$g(t) = \frac{m_1}{2\alpha p^+} - \frac{c_1}{l^-} |V_1|_\infty \left(t^{\bar{l}_2 - \alpha p^+} - t^{\underline{l}_2 - \alpha p^+} \right).$$

It is not difficult to prove the existence of $\rho \in (0,1)$ satisfying $g(\rho) > 0$.

Put

$$\lambda^* = \min\left\{ 1, \frac{m_1 l^-}{4\alpha p^+ c_1 |V_1|_\infty} \min(\rho^{\alpha p^+ - \underline{l}_1}, \rho^{\alpha p^+ - \bar{l}_1}) \right\} > 0,$$

Then, for any $\lambda \in (0, \lambda^*)$, and any $u \in E$, with $\|u\|_E = \rho$ we have

$$\begin{aligned}
\psi_\lambda(u) &\geq \left[\frac{m_1}{2\alpha p^+} - \frac{\lambda c_1}{l^-} |V_1|_\infty \left(\rho^{\underline{l}_1 - \alpha p^+} + \rho^{\bar{l}_1 - \alpha p^+} \right) \right] \rho^{\alpha p^+} \\
&\quad + \left[\frac{m_1}{2\alpha p^+} - \frac{\lambda c_1}{l^-} |V_1|_\infty \left(\rho^{\underline{l}_2 - \alpha p^+} + \rho^{\bar{l}_2 - \alpha p^+} \right) \right] \rho^{\alpha p^+} \\
&\geq \left[\frac{m_1}{2\alpha p^+} - \frac{\lambda c_1}{l^-} |V_1|_\infty \left(\rho^{\underline{l}_1 - \alpha p^+} + \rho^{\bar{l}_1 - \alpha p^+} \right) \right] \rho^{\alpha p^+} + g(\rho) \rho^{\alpha p^+} \\
&\geq \frac{m_0}{4\alpha p^+} \rho^{\alpha p^+} := a > 0.
\end{aligned}$$

□

Now, we complete the proof of Theorem 1.3. Using Lemma 3.8, for all $u \in E$, with $\|u\|_E = \rho$ we have $\psi_\lambda(u) \geq a > 0$. Moreover, using Ekeland's variational principle, there exists $e \in E$ with $\|e\|_E > \rho$, and $\psi_\lambda(u) < 0$. Put

$$\Gamma := \{\gamma \in C([0,1], E) \mid \gamma(0) = 0,\ \gamma(1) = e\},$$

and define

$$\bar{c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \psi_\lambda(\gamma(t)).$$

Since $\|e\|_E > \rho$, then, every $\gamma \in \Gamma$ intercept $\{u \in E \mid \|u\|_E = \rho\}$, so we have

$$\bar{c} := \inf_{\|u\|_E=\rho} \psi_\lambda(u) \geq a > 0.$$

The mountain pass theorem (See [2]) implies the existence of a function $u_1 \in E$ as a non-trivial critical point of the functional ψ_λ with $\psi_\lambda(u_1) = \bar{c} > 0$. Therefore, we obtain the first nontrivial solution for the problem (1.1). On the other hand, from Lemma 3.8, we have

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} \psi_\lambda(u) < 0 \quad \text{and} \qquad \inf_{\partial B_\rho(0)} \psi_\lambda(u) > 0.$$

Moreover, there exists $u_0 \in E$, such that $\psi_\lambda(tu_0) < 0$ for all $t > 0$ small enough.

By using Lemma 3.7, we deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that as n tends to infinity, we have

$$\psi_\lambda(u_n) \to \underline{c} := \inf_{\overline{B_\rho(0)}} \psi_\lambda(u) < 0 \quad \text{and} \quad d\psi_\lambda(u_n) \to 0_{E^*}. \tag{3.10}$$

From Lemma 3.7, the sequence $\{u_n\}$ converges strongly to some $u_2 \in E$. Since $\psi_\lambda \in C^1(E, \mathbb{R})$, then, (3.10) implies that $\psi_\lambda(u_2) = \underline{c}$ and $d\psi_\lambda(u_2) = 0$. Thus, u_2 is a nontrivial solution of problem (1.1). Finally, since

$$\psi_\lambda(u_1) = \bar{c} > c > 0 > \underline{c} = \psi_\lambda(u_2),$$

then we deduce that u_1 and u_2 are distinct and nontrivial. The proof of Theorem 1.3 is now completed.

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M. Ben Mohamed Salah,
Faculté des sciences, université de Tunis el Manar, Tunis 2092, Tunisie,
E-mail: mohamed123salah@gmail.com

A. Ghanmi,
Faculté des Sciences de Tunis, LR10ES09 Modélisation mathématique, analyse harmonique et théorie du potentiel, Université de Tunis El Manar, Tunis 2092, Tunisie,
E-mail: abdeljabbar.ghanmi@lamsin.rnu.tn

K. Kefi,
Faculté des sciences, université de Tunis el Manar, Tunis 2092, Tunisie,
E-mail: khaled-kefi@yahoo.fr

Існування та множинність розв'язків для певного класу проблем типу Кірхгофа, які містять дробовий оператор зі змінними показниками

M. Ben Mohamed Salah, A. Ghanmi, and K. Kefi

У цій роботі ми розглядаємо певний клас проблем типу Кірхгофа, які містять дробовий оператор зі змінними показниками. Використовуючи прямий варіаційний метод, ми одержуємо результати про існування розв'язків. Крім того, комбінуючи теорему про гірський перевал і варіаційний принцип Екланда, ми доводимо множинність розв'язків. Основні результати цієї роботи посилюють і узагальнюють попередні результати у цій галузі.

Ключові слова: дробовий $p(x)$-лапласіан, варіаційні методи, узагальнені простри Соболєва