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General Decay Result for a Weakly Damped Thermo-Viscoelastic System with Second Sound

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In this paper, an n-dimensional thermo-viscoelastic system with second sound with a weak frictional damping is considered. We establish an explicit and general decay rate result using some properties of convex functions. Our result is obtained without imposing any restrictive growth assumptions on the frictional damping term.

Key words: general decay, weak frictional damping, thermo-viscoelastic system with second sound, convexity

Mathematical Subject Classification 2010: 35B37, 35L55, 74D05, 93D15, 93D20

1. Introduction

In this paper, we are concerned with the following problem:

$$\begin{aligned}
&u_{tt} - k_0 \triangle u(t) - (\mu+\lambda)\nabla(\operatorname{div} u) \\
&\qquad + \int_0^t g(t-s)\,\Delta u(s)\,ds + \beta\nabla\theta + \alpha(t)\,w(u_t) = 0 && \text{in } \Omega\times(0,\infty), && (1.1)\\
&c\theta_t + k\operatorname{div} q + \beta\operatorname{div} u_t = 0 && \text{in } \Omega\times(0,\infty), && (1.2)\\
&\tau_0 q_t + q + k\nabla\theta = 0 && \text{in } \Omega\times(0,\infty), && (1.3)\\
&k_0\frac{\partial u}{\partial\nu} + (\mu+\lambda)\operatorname{div} u\times\nu - \int_0^t g(t-s)(\nabla u(s))\times\nu\,ds \\
&\qquad\qquad + h(u_t) = 0 && \text{on } \Gamma_1\times(0,\infty), && (1.4)\\
&u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ \theta(\cdot,0) = \theta_0,\ q(\cdot,0) = q_0 && \text{in } \Omega, && (1.5)\\
&u = 0 && \text{on } \Gamma_0\times(0,\infty), && (1.6)\\
&\theta = 0 && \text{on } \partial\Omega\times(0,\infty), && (1.7)
\end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$, such that $\{\Gamma_0\cup\Gamma_1\}$ is a partition of $\partial\Omega$, with meas $(\Gamma_0)>0$, ν is the unit outward normal to $\partial\Omega$, $u = u(x,t)\in\mathbb{R}^n$ is the displacement vector, $\theta=\theta(x,t)$ is the difference temperature, $q = q(x,t)\in\mathbb{R}^n$ is the heat flux vector and α, w are specific positive functions. The coefficients $k_0, \beta, c, \mu, \lambda$, τ_0 are positive constants, where τ_0 is the thermal relaxation time, k is the heat conductivity and μ, λ are Lame moduli. The relaxation function g is a positive and uniformly decaying function, h is a function satisfying some conditions given in $(G3)$. The third equation of system (1.1)–(1.7) represents Cattaneo's law of heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. In this work, we study the decay properties of the solutions of (1.1)–(1.7).

This type of problem without viscoelastic term in the first equations (thermoelastic system with second sound) has been considered by many mathematicians during the past decades and many results have been established. In this regard, Racke [19] established the existence result for the following n-dimensional problem:

$$
\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0 && \text{in } \Omega\times(0,+\infty), && (1.8)\\
&\theta_t + \gamma \operatorname{div} q + \delta \operatorname{div} u_t = 0 && \text{in } \Omega\times(0,+\infty), && (1.9)\\
&\tau q_t + q + \kappa\nabla\theta = 0 && \text{in } \Omega\times(0,+\infty), && (1.10)\\
&u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ \theta(\cdot,0) = \theta_0,\ q(\cdot,0) = q_0 && \text{in } \Omega, && (1.11)\\
&u = \theta = 0 && \text{on } \partial\Omega\times[0,+\infty), && (1.12)
\end{aligned}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $u = u(x,t)$, $q = q(x,t) \in \mathbb{R}^n$, and μ, λ , β, γ, δ, τ, κ, are positive constants, where μ, λ are Lame moduli and τ is the relaxation time, a small parameter compared to the others. He also proved, under the conditions $\operatorname{rot} u = \operatorname{rot} q = 0$, an exponential decay result for (1.8)–(1.12). This result is applied automatically to the radially symmetric solution since it is only a special case. Messaoudi [12] considered (1.8)–(1.12) in the presence of a source term and proved a blow up result for solutions with negative initial energy. This result was extended later to certain solutions with positive energy by Messaoudi and Said-Houari [15]. It is also worth mentioning the work of Messaoudi and Madani [13] in which they considered a system similar to (1.8)–(1.12) in the presence of a viscoelastic term acting in the domain and established a general uniform stability result for kernels of general decay type for which the usual exponential and polynomial decays are special cases.

Concerning stabilization by boundary feedback, we mention the recent work by Messaoudi and Al-Shehri [16], where a system of the form

$$
\begin{aligned}
&u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0 && \text{in } \Omega\times(0,+\infty), && (1.13)\\
&c\theta_t + \kappa \operatorname{div} q + \beta \operatorname{div} u_t = 0 && \text{in } \Omega\times(0,+\infty), && (1.14)\\
&\tau_0 q_t + q + \kappa\nabla\theta = 0 && \text{in } \Omega\times(0,+\infty), && (1.15)\\
&u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ \theta(\cdot,0) = \theta_0,\ q(\cdot,0) = q_0 && \text{in } \Omega, && (1.16)\\
&u = 0 && \text{on } \Gamma_0\times[0,+\infty), && (1.17)\\
&u(x,t) = -\int_0^t \mathrm{g}(t-s)(\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu)(s)\,ds && \text{on } \Gamma_1\times[0,+\infty), && (1.18)\\
&\theta = 0 && \text{on } \partial\Omega\times[0,+\infty) && (1.19)
\end{aligned}
$$

was considered. Here $\{\Gamma_0, \Gamma_1\}$ is a partition of $\partial\Omega$, ν is the outward normal to $\partial\Omega$ and the kernel g is the relaxation function, which is positive and of general decay. Under suitable conditions on the boundary and for kernels of general type, a general decay result was established.

Drabla et al. [7] analyzed the behavior of the solution of the system

$$
\begin{aligned}
& u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta + \alpha(t)\mathrm{g}(u_t) = 0 && \text{in } \Omega\times(0,+\infty), && (1.20)\\
& c\theta_t + \kappa \operatorname{div} q + \beta \operatorname{div} u_t = 0 && \text{in } \Omega\times(0,+\infty), && (1.21)\\
& \tau_0 q_t + q + \kappa\nabla\theta = 0 && \text{in } \Omega\times(0,+\infty), && (1.22)\\
& u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ \theta(\cdot,0) = \theta_0,\ q(\cdot,0) = q_0 && \text{in } \Omega, && (1.23)\\
& u = \theta = 0 && \text{on } \partial\Omega\times[0,+\infty). && (1.24)
\end{aligned}
$$

They established an explicit and general decay result depending on g and α, for which the optimal exponential and polynomial decay rate estimates are only special cases. For more results, we refer the reader to Mustafa [17] and Boulanouar and Drabla [4]. Our aim in this work is to study (1.1)–(1.7), in which the frictional damping considered is modulated by a time dependent coefficient $\alpha(t)$. More precisely, we intend to obtain a general relation between the decay rate for the energy (when t goes to infinity) and the functions w and α without imposing any restrictive growth assumptions on the frictional damping term. The result of this paper provides an explicit energy decay formula that allows a larger class of functions w and α for which the energy decay rates are not necessarily of exponential or polynomial types (see the examples in Section 3). The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. These convexity arguments were introduced and developed by Lasiecka et al. ([6], [8]-[10]) and used by Liu and Zuazua [11] and Alabau-Boussouira [1]. The paper is organized as follows. In Section 2, we present some notations and material needed for our work. The statement and the proof of our main result are given in Section 3.

2. Preliminaries

As in [3], we consider the following hypotheses:

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$g(0) > 0, \quad k_0 - \int_0^\infty g(s)\,ds = l > 0. \tag{2.1}$$

(G2) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function such that

$$g'(t) \le -\alpha(t)\, g(t), \quad t \ge 0. \tag{2.2}$$

(G3) $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function with

$$h(s)\, s \ge \alpha\, |s|^2, \qquad s \in \mathbb{R}, \tag{2.3}$$

$$|h(s)| \le \gamma |s|, \qquad s \in \mathbb{R}, \tag{2.4}$$

where α, γ are positive constants.

(G4) $w : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exist constants ε, c_1, $c_2 > 0$, and an increasing function $G \in C^1([0,+\infty))$, with $G(0) = 0$, and G is a linear or strictly convex C^2 function on $[0,\varepsilon)$ such that

$$\begin{aligned} c_1 |s| \le |w(s)| \le c_2 \min\{|s|, |s|^p\} & \quad \text{if } |s| \ge \varepsilon, \\ s^2 + w^2(s) \le G^{-1}(sw(s)) & \quad \text{if } |s| \le \varepsilon, \end{aligned} \tag{2.5}$$

and p satisfies

$$\begin{aligned} & 1 \le p \le \frac{n+2}{n-2}, \quad n \ge 3, \\ & 1 \le p < \infty, \quad n = 1, 2. \end{aligned}$$

Remark 2.1. Hypothesis (G4) implies that $sw(s) > 0$ for all $s \neq 0$.

Theorem 2.2. *Let hypotheses (G1), (G3) hold, in the sequel we assume that* $(u_0, u_1, \theta_0, q_0) \in \mathcal{H} = \left[H^2(\Omega) \cap H^1_{\Gamma_0}\right]^n \times \left[H^1_0(\Omega)\right] \times W$, *where*

$$W = \left\{v \in \left[L^2(\Omega)\right]^n \mid \operatorname{div} v \in L^2(\Omega)\right\}.$$

Then there exists a strong unique solution u of (1.1)–(1.7) *satisfying*

$$\begin{aligned} & u \in \left[C\left([0,+\infty); H^2(\Omega) \cap H^1_{\Gamma_0}\right)\right]^n \cap \left[C^1\left([0,+\infty); H^1_{\Gamma_0}(\Omega)\right)\right]^n, \\ & \theta \in C\left([0,+\infty); H^1_0(\Omega)\right) \cap C^1\left([0,+\infty); L^2(\Omega)\right), \\ & q \in C^1\left([0,T); L^2(\Omega)\right). \end{aligned}$$

Proof. We can use [18, p. 185, Theorem 1.4]. □

Lemma 2.3 ([13]). *Let $v \in \left[L^2(\Omega)\right]^n$. Then* $\|\operatorname{div} v\|_{H^{-1}(\Omega)} \le \|v\|_{L^2(\Omega)}$.

As in [13], let φ be the solution of the problem

$$\Delta\varphi = \theta \quad \text{in} \quad \Omega, \qquad \varphi = 0 \quad \text{on } \partial\Omega.$$

Since $\theta \in C^1\left([0,+\infty); L^2(\Omega)\right)$, then $\varphi \in C^1\left([0,+\infty); H^2(\Omega) \cap H^1_0(\Omega)\right)$ and

$$\|\varphi(\cdot,t)\|_{H^2(\Omega)} \le c_0 \|\theta(\cdot,t)\|_{L^2(\Omega)}. \tag{2.6}$$

3. Decay of solutions

In this section, we focus our attention on the uniform decay of solutions to problem (1.1)–(1.7). For this purpose, we introduce the energy functional

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{(\mu+\lambda)}{2}\|\operatorname{div} u\|_2^2 + \frac{c}{2}\|\theta\|_2^2 + \frac{\tau_0}{2}\|q\|_2^2$$

$$+\frac{1}{2}\left(k_0-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t), \tag{3.1}$$

where

$$(g\circ v)(t)=\int_0^t g(t-s)\|v(t)-v(s)\|_2^2\,ds. \tag{3.2}$$

Adopting the proof of [20], we still have the following results.

Lemma 3.1. *For any $u\in C^1\left([0,+\infty);H^1(\Omega)\right)$, we have*

$$\int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u_t\,ds\,dx=-\frac{1}{2}\int_\Omega g(t)|\nabla u(t)|^2\,dx+\frac{1}{2}\left(g'\circ\nabla u\right)(t)$$
$$-\frac{1}{2}\frac{d}{dt}\left[(g\circ\nabla u)(t)-\int_\Omega\int_0^t g(s)\,ds\,|\nabla u(t)|^2\,dx\right]. \tag{3.3}$$

Lemma 3.2. *Let u be the solution of (1.1)–(1.7) under assumptions (G1)–(G4). Then the energy functional satisfies*

$$E'(t)=-\frac{1}{2}\int_\Omega g(t)|\nabla u(t)|^2\,dx+\frac{1}{2}\left(g'\circ\nabla u\right)(t)$$
$$-\alpha(t)\int_\Omega u_t w(u_t)\,dx-\int_\Omega|q|^2\,dx-\int_{\Gamma_1}u_t h(u_t)\,d\Gamma\le 0. \tag{3.4}$$

Proof. Multiplying (1.1) by u_t, (1.2) by θ, and (1.3) by q and integrating over Ω, using integration by parts and the boundary condition, hypotheses (G1)–(G4) and (3.3), we obtain ((3.4)). □

Now we will study the asymptotic behavior of the energy functional $E(t)$. First, we define some functionals and establish Lemma 3.3. Let

$$\mathcal{F}(t)=NE(t)+\Phi(t)+\Psi(t), \tag{3.5}$$

where

$$\Phi(t)=\int_\Omega u\left(u_t-\frac{\beta\tau_0}{k}q\right)\,dx, \tag{3.6}$$

$$\Psi(t)=-\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx-\tau_0\int_\Omega q\nabla\varphi dx, \tag{3.7}$$

and N is a positive constant to be specified later.

Lemma 3.3. *There exist two positive constants β_1 and β_2 such that the relation*

$$\beta_1 E(t)\le\mathcal{F}(t)\le\beta_2 E(t). \tag{3.8}$$

Proof. By Hölder's inequality, Young's inequality, Poincaré's inequality and (2.6), we deduce that

$$|\Phi(t)|\le\frac{1}{2}\|u_t\|_2^2+B^2\|\nabla u\|_2^2+\frac{\beta\tau_0}{2k}\|q\|_2^2, \tag{3.9}$$

$$
\begin{aligned}
|\Psi(t)| &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))\,ds\right)^2 dx \\
&\quad + \frac{\tau_0}{2}\|q\|_2^2 + \frac{1}{2}\|\nabla\varphi\|_2^2 \\
&\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\int_0^t g(s)\,ds \int_\Omega\int_0^t g(t-s)|u(t)-u(s)|^2\,ds\,dx \\
&\quad + \frac{\tau_0}{2}\|q\|_2^2 + \frac{B^2}{2}\|\theta\|_2^2 \\
&\leq \frac{1}{2}\|u_t\|_2^2 + \frac{(k_0-l)B^2}{2}(g\circ\nabla u)(t) + \frac{\tau_0}{2}\|q\|_2^2 + \frac{B^2}{2}\|\theta\|_2^2. \qquad (3.10)
\end{aligned}
$$

Hence, taking (3.5), (3.9), and (3.10) into account, we have

$$
\begin{aligned}
\mathcal{F}(t) &= NE(t) + \Phi(t) + \Psi(t) \\
&\leq E(t) + \|u_t\|_2^2 + B^2\|\nabla u\|_2^2 + c_1(g\circ\nabla u)(t) + c_2\|q\|_2^2 + \frac{B^2}{2}\|\theta\|_2^2 \\
\mathcal{F}(t) &\geq E(t) - c_3\left(\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \|q\|_2^2 + \|\theta\|_2^2\right),
\end{aligned}
$$

where $c_1 = (k_0-l)B^2/2$, $c_2 = \frac{\beta\tau_0}{2k} + \frac{\tau_0}{2}$, and $c_3 = \max\left(1, B^2, c_1, c_2\right)$. Thus, there exist two positive constants β_1 and β_2 such that

$$
\beta_1 E(t) \leq \mathcal{F}(t) \leq \beta_2 E(t). \qquad \square \qquad (3.11)
$$

Lemma 3.4. *Assume that (G1)–(G4) hold. Then the functional*

$$
\Phi(t) = \int_\Omega u.\left(u_t - \frac{\beta\tau_0}{k}q\right)dx
$$

satisfies, along the solution of (1.1)–(1.7), the inequality

$$
\begin{aligned}
\Phi'(t) &\leq -\frac{l}{2}\int_\Omega |\nabla u|^2\,dx + 2\int_\Omega u_t^2\,dx + \frac{(k_0-l)}{2l}(g\circ\nabla u)(t) \\
&\quad - (\mu+\lambda)\int_\Omega |\operatorname{div} u|^2\,dx + \frac{(\gamma B)^2}{2l}\int_{\Gamma_1} u_t^2\,d\Gamma \\
&\quad - \alpha(t)\int_\Omega u w(u_t)\,dx + c\int_\Omega |q|^2\,dx. \qquad (3.12)
\end{aligned}
$$

Proof. By using the differential equation in (1.1)–(1.7), Green's identity, Poincaré's inequality and Young's inequality, we obtain

$$
\begin{aligned}
\Phi'(t) &= \int_\Omega |u_t|^2\,dx - k_0\int_\Omega |\nabla u|^2\,dx - (\mu+\lambda)\int_\Omega |\operatorname{div} u|^2\,dx \\
&\quad + \int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx - \int_{\Gamma_1} h(u_t)\,u\,d\Gamma \\
&\quad - \alpha(t)\int_\Omega u w(u_t)\,dx - \beta\int_\Omega u.\nabla\theta dx
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\beta\tau_0}{k}\int_\Omega u_t q\,dx+\beta\int_\Omega u\nabla\theta\,dx+\frac{\beta}{k}\int_\Omega uq\,dx \qquad (3.13)\\
&\le 2\int_\Omega |u_t|^2\,dx-\left(k_0-B_1^2\delta_1\right)\int_\Omega |\nabla u|^2\,dx-(\mu+\lambda)\int_\Omega |\operatorname{div} u|^2\,dx\\
&\quad+\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx-\int_{\Gamma_1} h(u_t)\,u\,d\Gamma\\
&\quad+c\int_\Omega |q|^2\,dx-\alpha(t)\int_\Omega uw(u_t)\,dx. \qquad (3.14)
\end{aligned}
$$

The fourth and the fifth terms on the right-hand side of (3.14) can be estimated as follows. From Hölder's inequality, Young's inequality, and $(G1)$, for $\eta>0$, we have

$$
\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,ds\,dx\\
&\quad\le \frac{k_0}{2}\int_\Omega |\nabla u|^2\,dx+\frac{1}{2k_0}\int_\Omega\left(\int_0^t g(t-s)\nabla u(s)\,ds\right)^2 dx\\
&\quad\le \frac{k_0}{2}\int_\Omega |\nabla u|^2\,dx\\
&\qquad+\frac{1}{2k_0}\int_\Omega\left(\int_0^t g(t-s)\left(\nabla u(s)-\nabla u(t)+\nabla u(t)\right)ds\right)^2 dx, \qquad (3.15)
\end{aligned}
$$

but

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-s)\left(\nabla u(s)-\nabla u(t)+\nabla u(t)\right)ds\right)^2 dx\\
&\quad\le \int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|\right)ds\right)^2 dx\\
&\quad\le \int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|\right)ds\right)^2 dx\\
&\qquad+\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(t)|\right)ds\right)^2 dx\\
&\qquad+2\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|\right)ds\right)\\
&\qquad\times\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(t)|\right)ds\right)dx\\
&\quad\le (1+\eta)\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(t)|\right)ds\right)^2 dx\\
&\qquad+\left(1+\frac{1}{\eta}\right)\int_\Omega\left(\int_0^t g(t-s)\left(|\nabla u(s)-\nabla u(t)|\right)ds\right)^2 dx. \qquad (3.16)
\end{aligned}
$$

Thus, by using the fact that

$$
\int_0^t g(s)\,ds\le\int_0^\infty g(s)\,ds=k_0-l, \qquad (3.17)
$$

we obtain

$$\int_\Omega \nabla u\,(t) \int_0^t g\,(t-s)\nabla u\,(s)\; ds\, dx$$
$$\le \left[\frac{k_0}{2} + \frac{1}{2k_0}\,(1+\eta)\,(k_0-l)^2\right]\int_\Omega |\nabla u|^2\, dx$$
$$+\frac{1}{2k_0}\left(1+\frac{1}{\eta}\right)(k_0-l)\,(g\circ\nabla u)\,(t)\,. \tag{3.18}$$

Employing Hölder's inequality, Young's inequality and (G_3), for $\delta_2>0$, we obtain

$$\int_{\Gamma_1} h\,(u_t)\,u\,d\Gamma \le \delta_2 B_2^2\,\|\nabla u\|_2^2 + \frac{\gamma^2}{4\delta_2}\int_{\Gamma_1} u_t^2\,d\Gamma. \tag{3.19}$$

A substitution of (3.18) and (3.19) into (3.14) yields, we arrive at

$$\begin{aligned}\Phi'\,(t) \le &-\left(\frac{k_0}{2} - \frac{1}{2k_0}\,(1+\eta)\,(k_0-l)^2 - B_1^2\,(\delta_1) - B_2^2\,(\delta_2)\right)\int_\Omega |\nabla u|^2\; dx\\ &+2\int_\Omega u_t^2\,dx + \frac{1}{2k_0}\left(1+\frac{1}{\eta}\right)(k_0-l)\,(g\circ\nabla u)\,(t) - (\mu+\lambda)\int_\Omega |\text{div}\, u|^2\; dx\\ &+\frac{\gamma^2}{4\delta_2}\int_{\Gamma_1} u_t^2\,d\Gamma - \alpha\,(t)\int_\Omega uw\,(u_t)\,dx + c\int_\Omega |q|^2\; dx.\end{aligned} \tag{3.20}$$

Letting $\eta = l/\,(k_0-l)$, $\delta_1 = l/2B_1^2$ and $\delta_2 = l/2B_2^2$, (3.12) is established. □

Lemma 3.5. *Let* $(u_0,u_1,\theta_0,q_0)\in\mathcal{H}$*and let* (G1)–(G3) *hold. Then the functional*

$$\Psi\,(t) = -\int_\Omega u_t\int_0^t g\,(t-s)\,(u\,(t)-u\,(s))\,dsdx - \tau_0\int_\Omega q.\nabla\varphi dx$$

satisfies, along the solution of (1.1)–(1.7) *the inequality*

$$\begin{aligned}\Psi'\,(t) \le &-\left(\int_0^t g\,(s)\,ds - \delta - c_4\right)\|u_t\|_2^2 + \delta c_5\,\|\nabla u\|_2^2 + c_6\,(g\circ\nabla u)\,(t)\\ &+(\mu+\lambda)^2\,\delta\int_\Omega (div\;u)^2\,dx + \delta\gamma^2\int_{\Gamma_1} u_t^2\,d\Gamma - \frac{g\,(0)\,B^2}{4\delta}\,(g'\circ\nabla u)\,(t)\\ &+\alpha^2\,(t)\,\delta\int_\Omega w^2\,(u_t)\,dx - \left(\frac{k}{2}-\beta^2\delta\right)\int_\Omega |\theta|^2\; dx + c_3\int_\Omega |q|^2\; dx,\end{aligned} \tag{3.21}$$

where c_3, c_4, c_5 *and* c_6 *are positive constants.*

Proof. Direct calculations give

$$\begin{aligned}\Psi'\,(t) = &-\int_\Omega u_{tt}\int_0^t g\,(t-s)\,(u\,(t)-u\,(s))\,ds\,dx - \left(\int_0^t g\,(s)\,ds\right)\int_\Omega u_t^2\,dx,\\ &-\int_\Omega u_t\int_0^t g'\,(t-s)\,(u\,(t)-u\,(s))\,ds\,dx\end{aligned}$$

$$- \tau_0 \int_\Omega q_t \nabla\varphi \, dx - \tau_0 \int_\Omega q \nabla\varphi_t \, dx. \tag{3.22}$$

Using (1.1)–(1.7), Green's identity and Young's inequality, we obtain

$$\begin{aligned}
\Psi'(t) &= k_0 \int_\Omega \nabla u(t) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \\
&+ (\mu + \lambda) \int_\Omega \operatorname{div} u \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \\
&- \int_\Omega \left(\int_0^t g(t-s) \nabla u(s)\, ds \right) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \\
&+ \alpha(t) \int_\Omega w(u_t) \left(\int_0^t g(t-s)(u(t) - u(s))\, ds \right) dx \\
&+ \int_{\Gamma_1} h(u_t) \int_0^t g(t-s)(u(t) - u(s))\, ds\, d\Gamma \\
&- \int_\Omega u_t \int_0^t g'(t-s)(u(t) - u(s))\, ds\, dx - \left(\int_0^t g(s)\, ds \right) \int_\Omega u_t^2\, dx \\
&- \beta \int_\Omega \theta \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds\, dx \\
&+ \int_\Omega (q + k\nabla\theta) . \nabla\varphi\, dx - \tau_0 \int_\Omega q . \nabla \left(\Delta^{-1} \theta_t \right) dx.
\end{aligned} \tag{3.23}$$

Similarly to (3.13), we estimate the right-hand side of (3.23). Using Young's inequality, Hölder's inequality, (3.3), (G_1), (G_2), and (G_3), for $\delta > 0$, we have

$$\begin{aligned}
&\left| \int_\Omega k_0 \nabla u(t) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \right| \\
&\quad \le k_0^2 \delta \|\nabla u\|_2^2 + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right)^2 dx \\
&\quad \le k_0^2 \delta \|\nabla u\|_2^2 + \frac{1}{4\delta} \int_0^t g(s)\, ds \int_\Omega \int_0^t g(t-s) |\nabla u(t) - \nabla u(s)|^2\, ds\, dx \\
&\quad \le k_0^2 \delta \|\nabla u\|_2^2 + \frac{k_0 - l}{4\delta} (g \circ \nabla u)(t),
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&(\mu + \lambda) \int_\Omega \operatorname{div} u \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \\
&\quad \le (\mu + \lambda)^2 \delta \int_\Omega (\operatorname{div} u)^2 + \frac{k_0 - l}{4\delta} (g \circ \nabla u)(t),
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s) \nabla u(s)\, ds \right) \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right) dx \\
&\quad \le \delta \int_\Omega \left(\int_0^t g(t-s) \nabla u(s)\, ds \right)^2 dx \\
&\qquad + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right)^2 dx
\end{aligned}$$

$$\leq 2\delta\left(k_0-l\right)^2\|\nabla u\|_2^2+\left(2\delta+\frac{1}{4\delta}\right)\left(k_0-l\right)\left(g\circ\nabla u\right)(t). \tag{3.26}$$

For the fourth and fifth terms on the right-hand side of (3.23), we have

$$\begin{aligned}\alpha(t)\int_\Omega w(u_t)\left(\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\right)dx&\\ \leq \alpha^2(t)\,\delta\int_\Omega w^2(u_t)\,dx+\frac{(k_0-l)\,B^2}{4\delta}(g\circ\nabla u)(t)&\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\int_{\Gamma_1} h(u_t)\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\,d\Gamma&\\ \leq \delta\gamma^2\int_{\Gamma_1}u_t^2\,d\Gamma+\frac{(k_0-l)\,B^2}{4\delta}(g\circ\nabla u)(t).&\end{aligned} \tag{3.28}$$

As for the sixth and the seventh terms on the right-hand side of (3.23), using Hölder's inequality, Young's inequality, (G1) and (G2), we obtain

$$\int_\Omega u_t\int_0^t g'(t-s)\left(u(t)-u(s)\right)ds\,dx\leq\delta\|u_t\|_2^2-\frac{g(0)\,B^2}{4\delta}\left(g'\circ\nabla u\right)(t), \tag{3.29}$$

$$\begin{aligned}\beta\int_\Omega\theta\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx&\\ \leq\beta^2\delta\int_\Omega|\theta|^2\,dx+\frac{(k_0-l)}{4\delta}(g\circ\nabla u)(t),&\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}&\int_\Omega(q+k\nabla\theta)\,\nabla\varphi\,dx-\tau_0\int_\Omega q\nabla\left(\Delta^{-1}\theta_t\right)dx\\ &=-k\int_\Omega\theta^2dx+\int_\Omega q\nabla\varphi\,dx\\ &\quad-\tau_0\int_\Omega q\nabla\left(\Delta^{-1}\left(\frac{-k}{c}\operatorname{div}q-\frac{\beta}{c}\operatorname{div}u_t\right)\right)dx\\ &\leq-k\int_\Omega\theta^2dx+\frac{k}{2c_0}\int_\Omega|\nabla\varphi|^2\,dx+\frac{c_0}{2k}\int_\Omega|q|^2\,dx\\ &\quad+\frac{\tau_0}{2}\int_\Omega|q|^2\,dx+\frac{\tau_0}{2}\int_\Omega\left|\nabla\left(\Delta^{-1}\left(\operatorname{div}\left(\frac{-k}{c}q-\frac{\beta}{c}u_t\right)\right)\right)\right|^2dx.\end{aligned} \tag{3.31}$$

Since $\left(\frac{-k}{c}q-\frac{\beta}{c}u_t\right)\in\left(L^2(\Omega)\right)^n$, recalling Lemma 2.3, we get $\operatorname{div}\left(\frac{-k}{c}q-\frac{\beta}{c}u_t\right)\in H^{-1}(\Omega)$, then $\Delta^{-1}\left(\operatorname{div}\left(\frac{-k}{c}q-\frac{\beta}{c}u_t\right)\right)\in H^1(\Omega)$ with

$$\left\|\Delta^{-1}\left(\operatorname{div}\left(\frac{-k}{c}q-\frac{\beta}{c}u_t\right)\right)\right\|_{H^1(\Omega)}\leq c_0\left\|\operatorname{div}\left(\frac{-k}{c}q-\frac{\beta}{c}u_t\right)\right\|_{H^{-1}(\Omega)}$$

$$\leq c_0 \left\| \frac{k}{c} q + \frac{\beta}{c} u_t \right\|_{L^2(\Omega)} \leq c_1 \|q\|_{L^2(\Omega)} + c_2 \|u_t\|_{L^2(\Omega)} \tag{3.32}$$

for some positive constants c_1 and c_2. Thus, by using (2.6), we arrive at

$$\int_\Omega (q + k\nabla\theta) \nabla\varphi \, dx - \tau_0 \int_\Omega q\nabla \left(\Delta^{-1}\theta_t\right) dx$$
$$\leq -\frac{k}{2} \int_\Omega \theta^2 \, dx + c_3 \int_\Omega |q|^2 \, dx + c_4 \int_\Omega |u_t|^2 \, dx. \tag{3.33}$$

After combining estimates (3.24)-(3.33) , (3.23) becomes

$$\begin{aligned}\Psi'(t) \leq & -\left(\int_0^t g(s)\, ds - \delta - c_4\right) \|u_t\|_2^2 + \delta c_5 \|\nabla u\|_2^2 + c_6 (g \circ \nabla u)(t) \\ & + (\mu+\lambda)^2 \delta \int_\Omega (\operatorname{div} u)^2 \, dx + \delta\gamma^2 \int_{\Gamma_1} u_t^2 \, d\Gamma - \frac{g(0) B^2}{4\delta} (g' \circ \nabla u)(t) \\ & + \alpha^2(t) \delta \int_\Omega w^2(u_t)\, dx + c_3 \int_\Omega |q|^2 \, dx - \left(\frac{k}{2} - \beta^2\delta\right) \int_\Omega |\theta|^2 \, dx,\end{aligned} \tag{3.34}$$

where

$$c_5 = k_0^2 + 2(k_0 - l)^2 > 0, \quad c_6 = (k_0 - l)\left(\frac{1}{\delta} + 2\delta + \frac{3B^2}{4\delta}\right) > 0.$$

Thus (3.21) is established. □

By using (2.2) , (3.4), (3.5), (3.12), and (3.21), we obtain

$$\begin{aligned}\mathcal{F}'(t) \leq & -(N - g_0 - c_4) \int_\Omega u_t^2 \, dx + \left((N+2) \int_\Omega u_t^2 \, dx - \alpha(t) \int_\Omega u \, w(u_t)\, dx\right) \\ & - N\alpha(t) \int_\Omega u_t w(u_t)\, dx - \left(\frac{l}{2} - \delta c_5\right) \int_\Omega |\nabla u|^2 \, dx \\ & + \left(\frac{(k_0 - l)}{2l} + c_6\right) (g \circ \nabla u)(t) \\ & - (\mu+\lambda)(1 - \delta(\mu+\lambda)) \int_\Omega |\operatorname{div} u|^2 \, dx + \alpha^2(t) \delta \int_\Omega w^2(u_t)\, dx \\ & - \left(\frac{k}{2} - \beta^2\delta\right) \int_\Omega |\theta|^2 \, dx - \left(\alpha N - \frac{(B_2\gamma)^2}{2l} - \delta\gamma^2\right) \int_{\Gamma_1} u_t^2 \, d\Gamma \\ & + \left(\frac{N}{2} - \frac{g(0) B^2}{4\delta}\right) (g' \circ \nabla u)(t) - (N - c - c_3) \int_\Omega |q|^2 \, dx.\end{aligned} \tag{3.35}$$

We have used the fact that for any $t_0 > 0$,

$$\int_0^t g(s)\, ds \geq \int_0^{t_0} g(s)\, ds = g_0, \quad t \geq t_0, \tag{3.36}$$

because g is positive and continuous with $g(0) > 0$. At this point, we choose N large and δ small enough such that

$$k_1 = N - g_0 - c_4 > 0, \tag{3.37}$$

$$k_2 = \frac{l}{2} - \delta c_5 > 0, \tag{3.38}$$

$$k_3 = 1 - \delta(\mu + \lambda) > 0, \tag{3.39}$$

$$k_4 = \frac{k}{2} - \beta^2 \delta > 0, \tag{3.40}$$

$$k_5 = \alpha N - \frac{(B_2\gamma)^2}{2l} - \delta\gamma^2 > 0, \tag{3.41}$$

$$k_6 = N - c - c_3 > 0. \tag{3.42}$$

Hence, for all $t_0 > 0$, we arrive at

$$\begin{aligned}\mathcal{F}'(t) \leq & -k_1 \|u_t\|_2^2 - k_2 \|\nabla u\|_2^2 + c_7 (g \circ \nabla u)(t) + \alpha^2(t)\,\delta \int_\Omega w^2(u_t)\, dx \\ & - k_3 \int_\Omega |\text{div}\, u|^2 dx + c_8 \int_\Omega \left(u_t^2 + |uw(u_t)|\right) dx - N\alpha(t) \int_\Omega u_t w(u_t)\, dx \\ & - k_4 \|\theta\|_2^2 - k_5 \int_{\Gamma_1} u_t^2\, d\Gamma + c_9 \left(g' \circ \nabla u\right)(t) - k_6 \|q\|_2^2, \end{aligned} \tag{3.43}$$

which yields

$$\begin{aligned}\mathcal{F}'(t) \leq & - c_{10} E(t) + c_{11} (g \circ \nabla u)(t) \\ & + c\left(\int_\Omega w^2(u_t)\, dx + \int_\Omega \left(u_t^2 + |uw(u_t)|\right) dx\right)\end{aligned} \tag{3.44}$$

for some positive constants $c_7 - c_{11}$.

Now, let us choose $0 < \varepsilon_1 \leq \varepsilon$ such that

$$sw(s) \leq \min\{\varepsilon,\ G(\varepsilon)\} \quad \text{for all} \quad |s| \leq \varepsilon_1. \tag{3.45}$$

Then it is easy to show that

$$\begin{cases} c_1' |s| \leq |w(s)| \leq c_2' \min\{|s|, |s|^p\} & \text{if } |s| \geq \varepsilon_3, \\ s^2 + w^2(s) \leq G^{-1}(sw(s)) & \text{if } |s| \leq \varepsilon_3. \end{cases} \tag{3.46}$$

Considering the following partition of Ω:

$$\Omega_1 = \{x \in \Omega :\ |u_t| \leq \varepsilon_1\}, \quad \Omega_2 = \{x \in \Omega :\ |u_t| > \varepsilon_1\},$$

from (3.46), we have

$$\int_{\Omega_2} w^2(u_t)\, dx \leq -c_{12} E'(t). \tag{3.47}$$

Using also Poincaré's inequality, (3.46), the Hölder's inequality and Young's inequality, we see that:

If $\min\{|u_t|, |u_t|^p\} = |u_t|$, then

$$\int_{\Omega_2} |uw(u_t)|\, dx \leq \int_{\Omega_2} |u||u_t|\, dx \leq Bc\|\nabla u\|_{L^2(\Omega)}\|u_t\|_{L^2(\Omega)} \leq c_{13}E(t). \tag{3.48}$$

If $\min\{|u_t|, |u_t|^p\} = |u_t|^p$, then

$$\begin{aligned}\int_{\Omega_2} |uw(u_t)|\, dx &\leq \left(\int_{\Omega_2} |u|^{q+1}\, dx\right)^{\frac{1}{p+1}} \left(\int_{\Omega_2} |w(u_t)|^{1+\frac{1}{p}}\, dx\right)^{\frac{p}{p+1}} \\ &\leq cB\left(\int_{\Omega} |\nabla u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_2} |w(u_t)|^{1+\frac{1}{q}}\, dx\right)^{\frac{p}{p+1}} \\ &\leq cB\left(\int_{\Omega} |\nabla u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_2} u_t w(u_t)\, dx\right)^{\frac{p}{p+1}}. \end{aligned} \tag{3.49}$$

Thus, from (3.48) and (3.49), we deduce that

$$\begin{aligned}&\int_{\Omega_2} \left(|u_t|^2 + |uw(u_t)|\right) dx \\ &\leq c\int_{\Omega_2} u_t w(u_t)\, dx + cB\left(\int_{\Omega} |\nabla u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_2} u_t w(u_t)\, dx\right)^{\frac{p}{p+1}} + c_{13}E(t) \\ &\leq -cE'(t) + cE(t)^{\frac{1}{2}}\left(-E'(t)\right)^{\frac{p}{p+1}} + c_{13}E(t),\end{aligned}$$

where c, $c_{13} > 0$.

Taking into account (3.47) and using Young's inequality and the boundedness of E, we arrive at

$$\int_{\Omega_2} w^2(u_t)\, dx + \int_{\Omega_2} \left(u_t^2 + |uw(u_t)|\right) dx \leq c\varepsilon E(t) - (c_\varepsilon + c_{12})E'(t). \tag{3.50}$$

Also

$$\begin{aligned}&\int_{\Omega_1} w^2(u_t)\, dx + \int_{\Omega_1} \left(u_t^2 + |uw(u_t)|\right) dx \\ &\leq \int_{\Omega_1} u_t^2 dx + \varepsilon\int_{\Omega_1} u^2\, dx + (C_\varepsilon + 1)\int_{\Omega_1} w^2(u_t)\, dx \\ &\leq \int_{\Omega_1} u_t^2 dx + c\varepsilon E(t) + (C_\varepsilon + 1)\int_{\Omega_1} w^2(u_t)\, dx.\end{aligned} \tag{3.51}$$

Hence, Lemma 3.3, (3.50), and (3.51) imply, for ε small enough, that the functional $\mathcal{L} = \mathcal{F} + C_\varepsilon E$ satisfies

$$\mathcal{L}'(t) \leq -dE(t) + c_{10}(g \circ \nabla u)(t) + c\int_{\Omega_1} \left(u_t^2 + w^2(u_t)\right) dx, \tag{3.52}$$

and

$$\mathcal{L}(t) \sim E(t). \tag{3.53}$$

We are ready to state and prove our main result.

Theorem 3.6. *Assume that (G1)–(G4) hold. Then there exist positive constants c_1, c_2, c_3 and ε_0 such that the solution of (1.1)–(1.7) satisfies*

$$E(t) \le \varepsilon_1 G_1^{-1}\left(k_1 \int_0^t \alpha(s)\,ds + c_2\right), \quad t \ge 0, \tag{3.54}$$

where

$$G_1(t) = \int_t^1 \frac{1}{G_2(s)}ds \quad \text{and} \quad G_2(t) = tG'(\varepsilon_0 t).$$

Here G_1 is strictly decreasing and convex on $(0,1]$ with $\lim_{t\to 0} G_1(t) = +\infty$.

Proof. It follows from (3.52), (2.2) and (3.4) that

$$\begin{aligned}\alpha(t)\mathcal{L}'(t) &\le -d\alpha(t)E(t) + c_{10}\alpha(t)(g\circ\nabla u)(t) + c\alpha(t)\int_{\Omega_1}\left(u_t^2 + w^2(u_t)\right)dx\\ &\le -d\alpha(t)E(t) - c_{14}\left(g'\circ\nabla u\right)(t) + c\alpha(t)\int_{\Omega_1}\left(u_t^2 + w^2(u_t)\right)dx\\ &\le -d\alpha(t)E(t) - mE'(t) + c\alpha(t)\int_{\Omega_1}\left(u_t^2 + w^2(u_t)\right)dx, \quad t\ge 0,\end{aligned} \tag{3.55}$$

where m and c_{14} are positive constants.

We then introduce $\mathcal{H}(t) = \alpha(t)\mathcal{L}(t) + mE(t)$, (clearly that $\mathcal{H} \sim E$), using (G2), so that (3.55) becomes

$$\mathcal{H}'(t) \le -d\alpha(t)E(t) + c\alpha(t)\int_{\Omega_1}\left(u_t^2 + w^2(u_t)\right)dx. \tag{3.56}$$

Then we have two estimates:

Case 1. G is linear on $[0,\varepsilon)$: We deduce that

$$\mathcal{H}'(t) \le -d\alpha(t)E(t) + c\alpha(t)\int_{\Omega_1} u_t w(u_t)\,dx = -d\alpha(t)E(t) - cE'(t),$$

which gives

$$(\mathcal{H}+cE)'(t) \le -d\alpha(t)E(t).$$

Hence, using the fact that $\mathcal{H}+cE \sim E$, we easily obtain

$$E(t) \le c' e^{-c''\int_0^t \alpha(s)ds} = c' G_1^{-1}\left(c''\int_0^t \alpha(s)\,ds\right).$$

Case 2. G is nonlinear on $[0,\varepsilon)$: To estimate the last integral in (3.56), we use, for $I(t)$ defined by

$$I(t) = \frac{1}{|\Omega_1|}\int_{\Omega_1} u_t w(u_t)\,dx,$$

Jensen's inequality to get

$$G^{-1}(I(t)) \ge c\int_{\Omega_1} G^{-1}(u_t w(u_t))\,dx. \tag{3.57}$$

Thus, using (2.5), (3.57), we get

$$\alpha(t)\int_{\Omega_1}\left(|u_t|^2+w(u_t)^2\right)dx \le \alpha(t)\int_{\Omega_1} G^{-1}(u_t w(u_t))\,dx \le c\alpha(t)G^{-1}(I(t)).$$

Therefore, (3.56) becomes

$$R_0'(t) \le -d\alpha(t)E(t) + c\alpha(t)G^{-1}(I(t)), \tag{3.58}$$

where $R_0=\alpha\mathcal{L}+E$, and $R_0\sim E$ because of (3.53).

Now, for $\varepsilon_0<\varepsilon$ and $c_0>0$, using (3.58) and the fact that $E'\le 0$, $G'>0$, and $G''>0$ on $(0,\varepsilon]$, we find that the functional R_1, defined by

$$R_1(t)=G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)R_0(t)+c_0E(t),$$

satisfies, for some $a_1,a_2>0$,

$$a_1R_1(t)\le E(t)\le a_2R_1(t), \tag{3.59}$$

and

$$\begin{aligned} R_1'(t) &= \varepsilon_0\frac{E'(t)}{E(0)}G''\left(\varepsilon_0\frac{E(t)}{E(0)}\right)R_0(t)+G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)R_0'(t)+c_0E'(t)\\ &\le -d\alpha(t)E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+c\alpha(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G^{-1}(I(t))\\ &\qquad + c_0E'(t). \end{aligned} \tag{3.60}$$

Let G^* be the convex conjugate of G in the sense of Young (see [2, p. 61–64]). Then

$$G^*(s)=s(G')^{-1}(s)-G[(G')^{-1}(s)] \quad \text{if } s\in(0,G'(\varepsilon)], \tag{3.61}$$

and G^* satisfies the following generalized Young's inequality:

$$AB\le G^*(A)+G(B),\quad \text{if } A\in(0,G'(\varepsilon)],\ B\in(0,\varepsilon]. \tag{3.62}$$

With $A=G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$, and $B=G^{-1}(I(t))$, using (3.4), (3.45), (3.60)–(3.62), we arrive at

$$\begin{aligned} R_1'(t) &\le -d\alpha(t)E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+c\alpha(t)G^*\left(G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\right)\\ &\qquad + c\alpha(t)I(t)+c_0E'(t)\\ &\le -d\alpha(t)E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)+c\varepsilon_0\alpha(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\ &\qquad - cE'(t)+c_0E'(t). \end{aligned}$$

Consequently, with suitable choices of ε_0 and c_0, we obtain

$$R_1'(t)\le -k\alpha(t)\left(\frac{E(t)}{E(0)}\right)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)=-k\alpha(t)G_2\left(\frac{E(t)}{E(0)}\right), \tag{3.63}$$

where $G_2(t) = tG'(\varepsilon_0 t)$. Since

$$G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 t G''(\varepsilon_0 t),$$

then, using the strict convexity of G on $(0, \varepsilon]$, we find that $G_2'(t), G_2(t) > 0$ on $(0, 1]$. Thus, with $R(t) = \frac{a_1 R_1(t)}{E(0)}$, and using (3.59) and (3.63), we have

$$R(t) \sim E(t) \tag{3.64}$$

and, for some $k_1 > 0$,

$$R'(t) \leq -k_1 \alpha(t) G_2(R(t)).$$

Then a simple integration gives, for some $k_2 > 0$,

$$R(t) \leq G_1^{-1}(k_1 \int_0^t \alpha(s) ds + k_2), \tag{3.65}$$

where $G_1(t) = \int_t^1 \frac{1}{G_2(s)} ds$. Here we used the properties of G_2 and the fact that G_1 is strictly decreasing on $(0, 1]$. Using (3.64)–(3.65), we obtain (3.54). □

3.1. Remark. If w satisfies

$$w_0(|s|) \leq |w(s)| \leq w_0^{-1}(|s|) \quad \text{for all } |s| \leq \varepsilon \tag{3.66}$$

and

$$c_1 |s| \leq |w(s)| \leq c_2 |s| \quad \text{for all } |s| \geq \varepsilon$$

for some strictly increasing function $w_0 \in C^1([0, +\infty))$, with $w_0(0) = 0$, and positive constants c_1, c_1, ε and the function G, defined by $G(s) = \sqrt{\frac{s}{2}} w_0 \left(\sqrt{\frac{s}{2}}\right)$, is strictly convex C^2 function on $(0, \varepsilon]$ when w_0 is nonlinear, then (G2) is satisfied. This kind of hypothesis, where (G2) is weaker, was considered by Liu and Zuazua [11], and Alabau-Boussouira [1].

3.2. Examples. We give same examples which were considered by Messaoudi and Mustafa [14] to illustrate the energy decay rates given by Theorem 3.6. Here we assume that w satisfies (3.66) near the origin with the following various examples for w_0:

(**1**) If $w_0(s) = cs^p$ and $p \geq 1$, then $G(s) = cs^{\frac{p+1}{2}}$ satisfies (G2). By using Theorem 3.6, we easily obtain

$$E(t) \leq c e^{-c' \int_0^t \alpha(s) ds} \quad \text{if } p = 1,$$

$$E(t) \leq c \left(c' \int_0^t \alpha(s) ds + c'' \right)^{-\frac{2}{p-1}} \quad \text{if } p > 1.$$

(**2**) If $w_0(s) = e^{-1/s}$, then (G2) is satisfied for $G(s) = \sqrt{\frac{s}{2}} e^{-\sqrt{2}/\sqrt{s}}$ near zero. Therefore, we get

$$E(t) \leq c \left(\ln \left(c' \int_0^t \alpha(s) ds + c'' \right) \right)^{-2}.$$

(**3**) If $w_0(s) = \frac{1}{s}e^{-1/s^2}$, then (G2) is satisfied for $G(s) = e^{-2/s}$ near zero. Then we obtain

$$E(t) \leq c\left(\ln\left(c'\int_0^t \alpha(s)ds + c''\right)\right)^{-1}.$$

(**4**) If $w_0(s) = \frac{1}{s}e^{-(\ln s)^2}$, then (G2) is satisfied for $G(s) = e^{-\frac{1}{4}(\ln \frac{s}{2})^2}$ near zero. Thus, we have the following energy decay rate

$$E(t) \leq ce^{-2\left(\ln(c'\int_0^t \alpha(s)ds + c'')\right)^{\frac{1}{2}}}.$$

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Результат щодо загального згасання для слабко демпфованої термов'язкопружної системи з другим звуком

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У статті розглядається n-вимірна термов'язкопружна система з другим звуком зі слабким демпфуванням тертя. Використовуючи деякі властивості опуклих функцій, встановлено явний та загальний результат згасання. Наш результат одержано без будь яких обмежувальних припущень щодо зростання демпфування тертя.

Ключові слова: загальне згасання, слабке демпфування тертя, термов'язкопружна система з другим звуком, опуклість