Journal of Mathematical Physics, Analysis, Geometry
2022, Vol. 18, No. 1, pp. 75–104
doi: https://doi.org/10.15407/mag18.01.075

Controllability Problems for the Heat Equation in a Half-Plane Controlled by the Dirichlet Boundary Condition with a Point-Wise Control

Larissa Fardigola and Kateryna Khalina

In the paper, the problems of controllability and approximate controllability are studied for the control system $w_t = \Delta w$, $w(0, x_2, t) = u(t)\delta(x_2)$, $x_1 > 0$, $x_2 \in \mathbb{R}$, $t \in (0, T)$, where $u \in L^\infty(0, T)$ is a control. Both necessary and sufficient conditions for controllability and sufficient conditions for approximate controllability in a given time T under a control u bounded by a given constant are obtained in terms of solvability of a Markov power moment problem. Orthogonal bases are constructed in special spaces of Sobolev type. Using these bases, necessary and sufficient conditions for approximate controllability and numerical solutions to the approximate controllability problem are obtained. The results are illustrated by an example.

Key words: heat equation, controllability, approximate controllability, point-wise control, half-plane

Mathematical Subject Classification 2010: 93B05, 35K05, 35B30

1. Introduction

Controllability problems for the heat equation were studied both in bounded and unbounded domains. However, most of the papers studying these problems deal with domains bounded with respect to the spatial variables (see [2, 7, 8, 14–16, 19, 23, 25, 26, 28, 37] and the references therein). Although there are quite a few papers considering domains unbounded with respect to the spatial variables [3–6, 12, 13, 17, 20, 27, 29–32, 34, 35], we know only one paper (see [30]) where the boundary controllability of the heat equation was studied in a half-plain. The boundary controllability of the wave equation in a half-plane $x_1 > 0$, $x_2 \in \mathbb{R}$ with a point-wise control on the bound was studied in [9–11].

In the present paper we consider the heat equation in a half-plane

$$w_t = \Delta w, \qquad x_1 > 0,\ x_2 \in \mathbb{R},\ t \in (0, T), \tag{1.1}$$

controlled by the Dirichlet boundary condition

$$w\big(0, (\cdot)_{[2]}, t\big) = \delta_{[2]} u(t), \qquad x_2 \in \mathbb{R},\ t \in (0, T), \tag{1.2}$$

under the initial condition

$$w\big((\cdot)_{[1]},(\cdot)_{[2]},0\big)=w^0, \qquad\qquad x_1>0,\ x_2\in\mathbb{R}, \tag{1.3}$$

where $T>0$, $u\in L^\infty(0,T)$ is a control, $\delta_{[m]}$ is the Dirac distribution with respect to x_m, $m=1,2$, $\Delta=(\partial/\partial x_1)^2+(\partial/\partial x_2)^2$. The subscripts [1] and [2] associate with the variable numbers, e,g., $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x\in\mathbb{R}^2$. This problem is considered in spaces of Sobolev type (see details in Section 2).

In Section 2, some notations and definitions are given. In Section 3, the controllability problem is formulated for the control system (1.1)–(1.3), and preliminary results are gotten. In particular, properties of the solutions (Theorem 3.4) and properties of the reachability sets (Theorem 3.5) are studied for this system. In Section 4, for control system (1.1)–(1.3), the following assertions are obtained in a given time under the control bounded by a given constant: a necessary condition for 0-controllability (Theorem 4.1); necessary and sufficient conditions for controllability (Theorem 4.2); sufficient conditions for approximate controllability (Theorem 4.3). In Section 5, bases in special spaces of Sobolev type are constructed by using the generalized Laguerre polynomials. In Section 6, necessary and sufficient conditions for approximate controllability in a given time are obtained for system (1.1)–(1.3) (Theorems 6.1 and 6.2). Moreover, an algorithm is given for construction of controls solving the approximate controllability problem for this system. In Section 7, the results are illustrated by an example.

2. Notations

Let us introduce the spaces used in the paper. Let $n\in\mathbb{N}$. By $|\cdot|$, we denote the Euclidean norm in $\mathbb{R}^n$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions [33], $\mathcal{S}'(\mathbb{R}^n)$ be the dual space. Denote also $\mathcal{S}=\mathcal{S}(\mathbb{R})$. Let $\mathbb{R}_+=(0,+\infty)$. Let $\mathcal{D}(\mathbb{R}_+)$ be the space of infinitely differentiable functions on $\mathbb{R}$, which supports are bounded and they are contained in $\mathbb{R}_+$.

Let $D=\big(-i\partial/\partial x_1,\dots,-i\partial/\partial x_n\big)$, $D^\alpha=\big(-i(\partial/\partial x_1)^{\alpha_1},\dots,-i(\partial/\partial x_n)^{\alpha_n}\big)$, where $\alpha=(\alpha_1,\dots,\alpha_n)$ is multi-index.

For $s=\overline{0,3}$, consider

$$H^s(\mathbb{R}^n)=\big\{\varphi\in L^2(\mathbb{R}^n)\mid\forall\alpha\in\mathbb{N}_0^n\ \big(\alpha_1+\dots+\alpha_n\le s\Rightarrow D^\alpha\varphi\in L^2(\mathbb{R}^n)\big)\big\}$$

with the norm

$$\|\varphi\|^s=\left(\sum_{\alpha_1+\dots+\alpha_n\le s}\Big(\|D^\alpha\varphi\|_{L^2(\mathbb{R}^n)}\Big)^2\right)^{1/2},\quad \varphi\in H^s(\mathbb{R}^n),$$

and $H^{-s}(\mathbb{R}^n)=(H^s(\mathbb{R}^n))^*$ with the strong norm $\|\cdot\|^{-s}$ of the adjoint space. We also have $H^0(\mathbb{R}^n)=L^2(\mathbb{R}^n)=\big(H^0\big(\mathbb{R}^n)\big)^*=H^{-0}(\mathbb{R}^n)$.

For $m = \overline{-3,3}$, consider

$$H_m(\mathbb{R}^n) = \left\{\psi \in L^2_{\text{loc}}(\mathbb{R}^n) \mid \left(1+|\sigma|^2\right)^{m/2}\psi \in L^2(\mathbb{R}^n)\right\}$$

with the norm

$$\|\psi\|_m = \left\|\left(1+|\sigma|^2\right)^{m/2}\psi\right\|_{L^2(\mathbb{R}^n)}, \quad \psi \in H_m(\mathbb{R}^n).$$

Evidently, $H_{-m}(\mathbb{R}^n) = (H_m(\mathbb{R}^n))^*$.

Let $\langle f, \varphi\rangle$ be the the value of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ on a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

By $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $s = \overline{0,3}$, denote the Fourier transform operator with the domain $\mathcal{S}'(\mathbb{R}^n)$. This operator is an extension of the classical Fourier transform operator which is an isometric isomorphism of $L^2(\mathbb{R}^n)$. The extension is given by the formula

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}^{-1}\varphi\rangle, \quad f \in \mathcal{S}'(\mathbb{R}^n),\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The operator $\mathcal{F}$ is an isometric isomorphism of $H^m(\mathbb{R}^n)$ and $H_m(\mathbb{R}^n)$, $m = \overline{-3,3}$, [18, Chap. 1].

A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to be *odd with respect to* x_1, if $\left\langle f, \varphi\left((\cdot)_{[1]}, (\cdot)_{[2]}\right)\right\rangle = -\left\langle f, \varphi\left(-(\cdot)_{[1]}, (\cdot)_{[2]}\right)\right\rangle$, where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to be *even with respect to* x_1, if $\left\langle f, \varphi\left((\cdot)_{[1]}, (\cdot)_{[2]}\right)\right\rangle = \left\langle f, \varphi\left(-(\cdot)_{[1]}, (\cdot)_{[2]}\right)\right\rangle$, where $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Let $m = \overline{-3,3}$. By $\widetilde{H}^m\left(\mathbb{R}^2\right)$ (or $\widetilde{H}_m\left(\mathbb{R}^2\right)$), denote the subspace of all distributions in $H^m\left(\mathbb{R}^2\right)$ (or $H_m\left(\mathbb{R}^2\right)$, respectively) that are odd with respect to x_1. Evidently, $\widetilde{H}^m\left(\mathbb{R}^2\right)$ (or $\widetilde{H}_m\left(\mathbb{R}^2\right)$) is a closed subspace of $H^m\left(\mathbb{R}^2\right)$ (or $H_m\left(\mathbb{R}^2\right)$, respectively).

For $s = \overline{0,3}$, consider

$$H^s_{\circledcirc} = \Big\{\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \mid \Big(\forall \alpha \in \mathbb{N}_0^2\ \left(\alpha_1+\alpha_2 \le s \Rightarrow D^\alpha\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})\right)\Big) \wedge \Big(\forall k = \overline{0, s-1}\ D^{(k,0)}\varphi(0^+, (\cdot)_{[2]}) = 0\Big)\Big\}$$

with the norm

$$\|\varphi\|^s_{\circledcirc} = \left(\sum_{\alpha_1+\alpha_2 \le s}\left(\|D^\alpha\varphi\|_{L^2(\mathbb{R}_+\times\mathbb{R})}\right)^2\right)^{1/2}, \quad \varphi \in H^s_{\circledcirc},$$

and $H^{-s}_{\circledcirc} = \left(H^s_{\circledcirc}\right)^*$ with the strong norm $\|\cdot\|^{-s}_{\circledcirc}$ of the adjoint space. We have

$$H^0_{\circledcirc} = L^2(\mathbb{R}_+ \times \mathbb{R}).$$

We also need the following subspaces of the spaces $\widetilde{H}^s\left(\mathbb{R}^2\right)$ and $\widetilde{H}_s\left(\mathbb{R}^2\right)$.

$$\mathcal{H}^s = \left\{\nu \in \widetilde{H}^s\left(\mathbb{R}^2\right) \mid \exists \rho \in \mathcal{S}'\ \nu(x) = \frac{\partial}{\partial x_1}\rho(|x|)\right\}, \qquad s = \overline{-1,1},$$

$$\mathcal{H}_s = \left\{ g \in \widetilde{H}_s\left(\mathbb{R}^2\right) \mid \exists f \in \mathcal{S}'\ g(\sigma) = i\sigma_1 f(|\sigma|) \right\}, \qquad s = \overline{-1,1}.$$

Without less of generality, we can assume ρ and f are even in these definitions. Let $s = \overline{-1,1}$ and $L^2_{s,(3)}(\mathbb{R}_+)$ be the space of functions which are square-integrable with the weight $r^3(1+r^2)^s$ on $\mathbb{R}_+$. It is equipped with the norm

$$\begin{aligned} \|h\|_{L^2_{s,(3)}(\mathbb{R}_+)} &= \left(\int_0^\infty |h(r)|^2 r^3 \left(1+r^2\right)^s dr \right)^{1/2} \\ &= \left\| (\cdot)^{3/2} \left(1+(\cdot)^2\right)^{s/2} h \right\|_{L^2(\mathbb{R}_+)} \end{aligned} \tag{2.1}$$

and the inner product

$$\begin{aligned} \langle h, q \rangle_{L^2_{s,(3)}(\mathbb{R}_+)} &= \int_0^\infty h(r)\overline{q(r)} r^3 (1+r^2)^s\, dr \\ &= \left\langle (\cdot)^{3/2}\left(1+(\cdot)^2\right)^{s/2} h, (\cdot)^{3/2}\left(1+(\cdot)^2\right)^{s/2} q \right\rangle_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{2.2}$$

Let $g \in \mathcal{H}_s$. Then there exists $f \in \mathcal{S}'$ such that $g(\sigma) = i\sigma_1 f(|\sigma|)$. Using polar coordinates, we have

$$\|g\|_{-1} = \sqrt{\pi} \|f\|_{L^2_{-1,(3)}(\mathbb{R}_+)}. \tag{2.3}$$

Thus, $\mathcal{H}_s$ is complete. Obviously, $\mathcal{H}_s = \mathcal{F}\mathcal{H}^s$. Therefore, $\mathcal{H}^s$ is also complete.

We treat equality (1.2) as the value of the distribution w at $x_1 = 0$ (see Definition 2.1 below). To give this definition, we need some additional notations.

Let $g \in H_{-s}(\mathbb{R}^2)$, $\psi \in H_s(\mathbb{R}^2)$, $s = \overline{0,3}$. Since $1 + |\sigma|^2 \leq (1+\sigma_1^2)(1+\sigma_2^2)$, we have

$$\begin{aligned} \left(\|g\|_{-s}\right)^2 &= \iint_{\mathbb{R}^2} \left(1+|\sigma|^2\right)^{-s} |g(\sigma)|^2\, d\sigma \\ &\geq \int_{-\infty}^\infty \left(1+\sigma_1^2\right)^{-s} \int_{-\infty}^\infty \left(1+\sigma_2^2\right)^{-s} |g(\sigma)|^2\, d\sigma_2\, d\sigma_1 \\ &= \int_{-\infty}^\infty \left(1+\sigma_2^2\right)^{-s} \int_{-\infty}^\infty \left(1+\sigma_1^2\right)^{-s} |g(\sigma)|^2\, d\sigma_1\, d\sigma_2. \end{aligned}$$

Therefore, we have $g\big(\sigma_1, (\cdot)_{[2]}\big) \in H_{-s}(\mathbb{R})$ for almost all $\sigma_1 \in \mathbb{R}$, and we have $g\big((\cdot)_{[1]}, \sigma_2\big) \in H_{-s}(\mathbb{R})$ for almost all $\sigma_2 \in \mathbb{R}$. Moreover, denoting

$$\langle g, \psi \rangle_{[1]}(\sigma_2) = \int_{-\infty}^\infty g(\sigma_1, \sigma_2)\overline{\psi(\sigma_1, \sigma_2)}\, d\sigma_1, \qquad \sigma_2 \in \mathbb{R},$$

$$\langle g, \psi \rangle_{[2]}(\sigma_1) = \int_{-\infty}^\infty g(\sigma_1, \sigma_2)\overline{\psi(\sigma_1, \sigma_2)}\, d\sigma_2, \qquad \sigma_1 \in \mathbb{R},$$

we get

$$\langle g, \psi \rangle_{[1]} \in H_{-s}(\mathbb{R}) \quad \text{and} \quad \langle g, \psi \rangle_{[2]} \in H_{-s}(\mathbb{R}).$$

Let $f \in H^{-s}(\mathbb{R}^2)$, $\varphi \in H^s(\mathbb{R}^2)$, $s = \overline{0,3}$. Since the operator $\mathcal{F}$ is an isometric isomorphism of $H^m(\mathbb{R}^n)$ and $H_m(\mathbb{R}^n)$, $m = \overline{-3,3}$, [18, Chap. 1], denoting

$$\langle f, \varphi \rangle_{[1]} = \mathcal{F}^{-1}_{\sigma_2 \to x_2}\left(\langle \mathcal{F}_{x\to\sigma} f, \mathcal{F}\varphi \rangle_{[1]} \right) \quad \text{and} \quad \langle f, \varphi \rangle_{[2]} = \mathcal{F}^{-1}_{\sigma_1 \to x_1}\left(\langle \mathcal{F}_{x\to\sigma} f, \mathcal{F}\varphi \rangle_{[2]} \right),$$

we obtain

$$\langle f,\varphi\rangle_{[1]}\in H^{-s}(\mathbb{R})\quad\text{and}\quad\langle f,\varphi\rangle_{[2]}\in H^{-s}(\mathbb{R}).$$

According to the definition of the value of a distribution of one variable at a point [1, Chap. 1] and to the definition of the value of a distribution of several variables at a line [11], we give the following definition.

Definition 2.1. Let $s=\overline{1,3}$. We say that a distribution $f\in H^{-s}_{⓪}$ has the value $f_0\in H^{-s}(\mathbb{R})$ on the line $x_1=0$, i.e., $f\big(0^+,(\cdot)_{[2]}\big)=f_0\big((\cdot)_{[2]}\big)$, if for each $\varphi\in H^s(\mathbb{R})$ and $\psi\in\mathcal{D}(\mathbb{R}_+)$ we have

$$\Big\langle\big\langle f(\alpha(\cdot)_{[1]},(\cdot)_{[2]}),\varphi((\cdot)_{[2]})\big\rangle_{[2]},\psi((\cdot)_{[1]})\Big\rangle_{[1]}\to\Big\langle\langle f_0,\varphi\rangle_{[2]},\psi\Big\rangle_{[1]}\quad\text{as }\alpha\to0^+,$$

where $\langle h(\alpha(\cdot)),\psi\rangle=\Big\langle h((\cdot)),\frac{1}{\alpha}\psi\left(\frac{(\cdot)}{\alpha}\right)\Big\rangle$ for $h\in H^{-s}(\mathbb{R})$.

Remark 2.2. Let $\varphi\in H^s_{⓪}$, $s=\overline{0,3}$. Let $\widetilde{\varphi}$ be its odd extension with respect to x_1, i.e., $\widetilde{\varphi}(x_1,x_2)=\varphi(x_1,x_2)$ if $x_1\ge0$ and $\widetilde{\varphi}(x_1,x_2)=-\varphi(-x_1,x_2)$ otherwise. Then $\widetilde{\varphi}\in\widetilde{H}^s\left(\mathbb{R}^2\right)$, $s=\overline{0,3}$. The converse assertion is true for $s=0,1$, and it is not true for $s=2,3$. That is why the odd extension with respect to x_1 of a distribution $f\in H^{-s}_{⓪}$ may not belong to $\widetilde{H}^{-s}\left(\mathbb{R}^2\right)$, $s=2,3$. However, the following theorem holds.

Theorem 2.3. *Let $f\in H^{-1}_{⓪}$ and there exists $f\big(0^+,(\cdot)_{[2]}\big)\in H^{-1}(\mathbb{R})$. Then $f_{x_1x_1}\in H^{-3}_{⓪}$ can be extended to a distribution $F\in\widetilde{H}^{-3}\left(\mathbb{R}^2\right)$ such that F is odd with respect to x_1. This distribution is given by the formula*

$$F=\widetilde{f}_{x_1x_1}-2f(0^+,(\cdot)_{[2]})\delta'_{[1]},\tag{2.4}$$

where $\widetilde{f}$ is the odd extension of f with respect to x_1.

In the case $f\in H^{-1/2}_{⓪}$, corresponding theorem has been proved in [11]. The proof of Theorem 2.3 is analogous to the proof of the mentioned theorem.

3. Problem formulation and preliminary results

We consider control system (1.1)–(1.3) in $H^{-l}_{⓪}$, $l=\overline{1,3}$, i.e., $\left(\frac{d}{dt}\right)^s w:[0,T]\to H^{-1-2s}_{⓪}$, $s=0,1$, $w^0\in H^{-1}_{⓪}$.

Let $w^0,w(\cdot,t)\in H^{-1}_{⓪}$, $t\in[0,T]$. Let W^0 and $W(\cdot,t)$ be the odd extensions of w^0 and $w(\cdot,t)$ with respect to x_1, respectively, $t\in[0,T]$. If w is a solution to control system (1.1)–(1.3), then W is a solution to control system

$$W_t=\triangle W-2\delta'_{[1]}\delta_{[2]}u(t),\qquad t\in(0,T),\tag{3.1}$$

$$W\big((\cdot)_{[1]},(\cdot)_{[2]},0\big)=W^0\tag{3.2}$$

according to Theorem 2.3. Here $\left(\frac{d}{dt}\right)^s W:[0,T]\to\widetilde{H}^{-1-2s}\left(\mathbb{R}^2\right)$, $s=0,1$, $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$. The converse assertion is also true. Let $W^0,W(\cdot,t)\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$, $t\in$

$[0,T]$. Let w^0 and $w(\cdot,t)$ be the restrictions of W^0 and $W(\cdot,t)$ to $(0,+\infty)$ with respect to x_1, respectively, $t\in[0,T]$. If W is a solution to (3.1), (3.2), then w is a solution to (1.1)–(1.3) because

$$W\big(0^+,(\cdot)_{[2]},t\big)=\delta_{[2]}u(t)\quad\text{for almost all } t\in[0,T] \tag{3.3}$$

according to Lemma 3.6 (see below). Let $w^T\in H^{-1}_{⓪}$. One can see that $w\big((\cdot)_{[1]},(\cdot)_{[2]},T\big)=w^T$ iff $W\big((\cdot)_{[1]},(\cdot)_{[2]},T\big)=W^T$. Here W^T is the odd extension of w^T with respect to x_1 and $W^T\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$.

Thus, control systems (1.1)–(1.3) and (3.1), (3.2) are equivalent. Therefore, basing on this reason, we will further consider control system (3.1), (3.2) instead of original system (1.1)–(1.3).

Let $T>0$, $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$. By $\mathcal{R}_T\left(W^0\right)$, denote the set of all states $W^T\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ for which there exists a control $u\in L^\infty(0,T)$ such that there exists a unique solution W to system (3.1), (3.2) such that $W\big((\cdot)_{[1]},(\cdot)_{[2]},T\big)=W^T$.

Definition 3.1. A state $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ is said to be controllable to a target state $W^T\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ in a given time $T>0$ if $W^T\in\mathcal{R}_T\left(W^0\right)$.

In other words, a state $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ is said to be controllable to a target state $W^T\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ in a given time $T>0$ if there exists a control $u\in L^\infty(0,T)$ such that there exists a unique solution W to system (3.1), (3.2) and $W\big((\cdot)_{[1]},(\cdot)_{[2]},T\big)=W^T$.

Definition 3.2. A state $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ is said to be null-controllable in a given time $T>0$ if $0\in\mathcal{R}_T\left(W^0\right)$.

Definition 3.3. A state $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ is said to be approximately controllable to a target state $W^T\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ in a given time $T>0$ if $W^T\in\overline{\mathcal{R}_T\left(W^0\right)}$, where the closure is considered in the space $\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$.

In other words, a state $W^0\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ is approximately controllable to a target state $W^T\in\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ in a given time $T>0$ if for each $\varepsilon>0$, there exists $u_\varepsilon\in L^\infty(0,T)$ such that there exists a unique solution W_ε to system (3.1), (3.2) with $u=u_\varepsilon$ and $\left\|W_\varepsilon\big((\cdot)_{[1]},(\cdot)_{[2]},T\big)-W^T\right\|^{-1}<\varepsilon$.

Using the Poisson integral (see, e.g., [36]), we obtain the unique solution to system (3.1), (3.2)

$$W(x,t)=\mathcal{W}_0(x,t)+\mathcal{W}_u(x,t),\quad x\in\mathbb{R}^2,\ t\in[0,T], \tag{3.4}$$

where

$$\begin{aligned}
\mathcal{W}_0(x,t)&=\frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}*W^0(x), && x\in\mathbb{R}^2,\ t\in[0,T],\\
\mathcal{W}_u(x,t)&=\frac{x_1}{\pi}\int_0^t\frac{1}{4\xi^2}e^{-\frac{|x|^2}{4\xi}}u(t-\xi)\,d\xi, && x\in\mathbb{R}^2,\ t\in[0,T].
\end{aligned}$$

Theorem 3.4. *Let* $u \in L^\infty(0,T)$, $W^0 \in \widetilde{H}^{-1}(\mathbb{R}^2)$. *Then,*

(i) $\mathcal{W}_0(\cdot,t) \in \widetilde{H}^{-1}(\mathbb{R}^2)$, $t \in [0,T]$;

(ii) $\mathcal{W}_0(\cdot,t) \in C^\infty(\mathbb{R}^2)$, $t \in (0,T]$;

(iii) *if* $W^0 \in \mathcal{H}^{-1}$, *then* $\mathcal{W}_0(\cdot,t) \in \mathcal{H}^{-1}$, $t \in [0,T]$;

(iv) $\mathcal{W}_u(\cdot,t) \in \mathcal{H}^{-1}$ *and* $\|\mathcal{W}_u(\cdot,t)\|^{-1} \leq \sqrt{2/\pi}(t+1)\|u\|_{L^\infty(0,T)}$, $t \in [0,T]$.

Proof. Denote $V^0 = \mathcal{F}W^0$, $V(\cdot,t) = \mathcal{F}_{x\to\sigma}W(\cdot,t)$, $\mathcal{V}_0(\cdot,t) = \mathcal{F}_{x\to\sigma}\mathcal{W}_0(\cdot,t)$, $\mathcal{V}_u(\cdot,t) = \mathcal{F}_{x\to\sigma}\mathcal{W}_u(\cdot,t)$, $t \in [0,T]$. Then,

$$V(\sigma,t) = \mathcal{V}_0(\sigma,t) + \mathcal{V}_u(\sigma,t), \quad \sigma \in \mathbb{R}^2,\ t \in [0,T], \tag{3.5}$$

where

$$\mathcal{V}_0(\sigma,t) = e^{-t|\sigma|^2}V^0(\sigma), \qquad \sigma \in \mathbb{R}^2,\ t \in [0,T], \tag{3.6}$$

$$\mathcal{V}_u(\sigma,t) = -\frac{i\sigma_1}{\pi}\int_0^t e^{-\xi|\sigma|^2}u(t-\xi)\,d\xi, \qquad \sigma \in \mathbb{R}^2,\ t \in [0,T]. \tag{3.7}$$

Therefore,

$$\|\mathcal{W}_0(\cdot,t)\|^{-1} = \|\mathcal{V}_0(\cdot,t)\|_{-1} \leq \|V^0\|_{-1} = \|W^0\|^{-1}, \quad t \in [0,T], \tag{3.8}$$

i.e., (i) holds.

Let $\alpha = (\alpha_1,\alpha_2) \in \big(\mathbb{N}\cup\{0\}\big)^2$. We have

$$\begin{aligned}&|\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\mathcal{V}_0(\sigma,t)|^2\\ &\quad\leq \big(1+|\sigma|^2\big)^{1+\alpha_1+\alpha_2} e^{-2t|\sigma|^2}\big(1+|\sigma|^2\big)^{-1}|V^0(\sigma)|^2, \quad \sigma \in \mathbb{R}^2,\ t \in [0,T].\end{aligned}$$

For $m \in \mathbb{N}$ and $\beta > 0$ we have

$$\xi^m e^{-\beta\xi} \leq \left(\frac{m}{\beta e}\right)^m, \quad \xi \geq 0.$$

Therefore,

$$\begin{aligned}\|D^\alpha\mathcal{W}_0(\cdot,t)\|^0 &= \Big\|(\cdot)_{[1]}^{\alpha_1}(\cdot)_{[2]}^{\alpha_2}\mathcal{V}_0(\cdot,t)\Big\|_0 \leq e^t\left(\frac{1+\alpha_1+\alpha_2}{2te}\right)^{(1+\alpha_1+\alpha_2)/2}\|V^0\|_{-1}\\ &= e^t\left(\frac{1+\alpha_1+\alpha_2}{2te}\right)^{(1+\alpha_1+\alpha_2)/2}\|W^0\|^{-1}, \quad t \in (0,T],\end{aligned} \tag{3.9}$$

i.e., (ii) holds.

Assume that $W^0 \in \mathcal{H}^{-1}$. Then, $V^0 \in \mathcal{H}_{-1}$, therefore, $\mathcal{V}_0(\cdot,t) \in \mathcal{H}_{-1}$, hence, $\mathcal{W}_0(\cdot,t) \in \mathcal{H}^{-1}$, $t \in [0,T]$, i.e., (iii) holds.

Put

$$g(r,t) = \int_0^t e^{-\xi r^2}u(t-\xi)\,d\xi, \quad r \geq 0,\ t \in [0,T].$$

We have

$$\|\mathcal{W}_u(\cdot,t)\|^{-1} = \|\mathcal{V}_u(\cdot,t)\|_{-1} = \frac{1}{\pi}\left(\iint_{\mathbb{R}^2} |\sigma_1|^2 \left|g\left(|\sigma|,t\right)\right|^2 \left(1+|\sigma|^2\right)^{-1} d\sigma\right)^{1/2}$$
$$\leq \frac{1}{\sqrt{\pi}}\left(\int_0^\infty |g(r,t)|^2 r\,dr\right)^{1/2}, \quad t\in[0,T]. \tag{3.10}$$

Since

$$\frac{1-e^{-tr^2}}{r^2} \leq \frac{2(t+1)}{r^2+1}, \quad r>0,\ t>0,$$

we have

$$|g(r,t)| \leq \|u\|_{L^\infty(0,T)} \int_0^t e^{-\xi r^2}\,d\xi = \|u\|_{L^\infty(0,T)} \frac{1-e^{-tr^2}}{r^2}$$
$$\leq \|u\|_{L^\infty(0,T)} \frac{2(t+1)}{r^2+1}, \quad r>0,\ t>0.$$

With regard to (3.10), we obtain

$$\|\mathcal{W}_u(\cdot,t)\|^{-1} \leq \sqrt{\frac{2}{\pi}}(t+1)\|u\|_{L^\infty(0,T)} \left(\int_0^\infty \frac{2r\,dr}{(1+r^2)^2}\right)^{1/2}$$
$$= \sqrt{\frac{2}{\pi}}(t+1)\|u\|_{L^\infty(0,T)}, \quad t\in[0,T], \tag{3.11}$$

i.e., (iv) holds. □

According to (3.4), we have

$$\mathcal{R}_T(W^0) = \left\{W^T \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right) \mid \exists u\in L^\infty(0,T)\ W^T = \mathcal{W}_0(\cdot,T) + \mathcal{W}_u(\cdot,T)\right\}, \tag{3.12}$$

in particular,

$$\mathcal{R}_T(0) = \left\{W^T \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right) \mid \exists u\in L^\infty(0,T)\ W^T = \mathcal{W}_u(\cdot,T)\right\}. \tag{3.13}$$

Denote also

$$\mathcal{R}_T^L(0) = \Big\{W^T \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right) \mid \exists u\in L^\infty(0,T)$$
$$\left(\|u\|_{L^\infty(0,T)} \leq L \text{ and } W^T = \mathcal{W}_u(\cdot,T)\right)\Big\}. \tag{3.14}$$

Taking into account Theorem 3.4, we obtain the following theorem.

Theorem 3.5. *Let $T>0$. We have*

(i) $\mathcal{R}_T(0) = \bigcup_{L>0} \mathcal{R}_T^L(0) \subset \mathcal{H}^{-1}$;

(ii) $\mathcal{R}_T^L(0) \subset \mathcal{R}_T^{L'}(0)$, $0 < L < L'$;

(iii) $f \in \mathcal{R}_T^1(0) \Leftrightarrow Lf \in \mathcal{R}_T^L(0)$, $L > 0$;

(iv) $f \in \mathcal{R}_T^L(g) \Leftrightarrow f - \mathcal{W}_0(\cdot, T) \in \mathcal{R}_T^L(0)$, $L > 0$;

(v) $f \in \mathcal{R}_T(g) \Leftrightarrow f - \mathcal{W}_0(\cdot, T) \in \mathcal{R}_T(0)$.

Lemma 3.6. *Let $W^0 \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right)$, $t \in [0,T]$. Let W be a solution to (3.1), (3.2). Then (3.3) holds.*

Proof. According to Theorem 3.4(ii), $\mathcal{W}_0(\cdot, t)$ is continuous on $\mathbb{R}^2$ for each $t \in (0,T]$. Moreover, $\mathcal{W}_0(\cdot, t)$ is odd with respect to x_1, $t \in [0,T]$. Hence,

$$\mathcal{W}_0(0^+, (\cdot)_{[2]}, t) = 0, \quad t \in (0,T]. \tag{3.15}$$

Let us calculate $\mathcal{W}_u(0^+, (\cdot)_{[2]}, t)$, $t \in [0,T]$. We have

$$\mathcal{W}_u(x,t) = \frac{2}{\pi}\frac{x_1}{|x|^2}\int_{\frac{|x|}{2\sqrt{t}}}^{\infty} ye^{-y^2} u\left(t - \frac{|x|^2}{4y^2}\right) dy, \quad x \in \mathbb{R}^2,\ t \in [0,T].$$

Let $\varphi \in H^1(\mathbb{R})$, $\psi \in \mathcal{D}(\mathbb{R}_+)$. We have

$$\begin{aligned}&\left\langle \left\langle \mathcal{W}_u((\cdot)_{[1]}, (\cdot)_{[2]}), \varphi((\cdot)_{[2]})\right\rangle_{[2]}, \frac{1}{\alpha}\psi\left(\frac{(\cdot)_{[1]}}{\alpha}\right)\right\rangle_{[1]} \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\xi_1}{|\xi|^2}\int_{\frac{\alpha|\xi|}{2\sqrt{t}}}^{\infty} ye^{-y^2}u\left(t - \frac{\alpha^2|\xi|^2}{4y^2}\right) dy\, \varphi(\alpha\xi_2)\, d\xi_2\, \psi(\xi_1)\, d\xi_1, \\ &\qquad t \in [0,T].\end{aligned} \tag{3.16}$$

Since

$$\int_{\frac{\alpha|\xi|}{2\sqrt{t}}}^{\infty} ye^{-y^2}\left|u\left(t - \frac{\alpha^2|\xi|^2}{4y^2}\right)\right| dy \le \|u\|_{L^\infty(0,T)}\int_0^\infty ye^{-y^2}\, dy = \frac{1}{2}\|u\|_{L^\infty(0,T)}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\xi_1|}{|\xi|^2}|\varphi(\alpha\xi_2)|\, d\xi_2\, |\psi(\xi_1)|\, d\xi_1 &\le \sup_{\mu\in\mathbb{R}}|\varphi(\mu)|\int_0^\infty \xi_1|\psi(\xi_1)|\int_{-\infty}^{\infty}\frac{d\xi_2}{\xi_1^2+\xi_2^2}\, d\xi_1 \\ &= \pi\sup_{\mu\in\mathbb{R}}|\varphi(\mu)|\int_0^\infty |\psi(\xi_1)|\, d\xi_1 < \infty,\end{aligned}$$

we can apply Lebesgue's dominated convergence theorem to (3.16) as $\alpha \to 0^+$:

$$\begin{aligned}&\left\langle \left\langle \mathcal{W}_u((\cdot)_{[1]}, (\cdot)_{[2]}), \varphi((\cdot)_{[2]})\right\rangle_{[2]}, \frac{1}{\alpha}\psi\left(\frac{(\cdot)_{[1]}}{\alpha}\right)\right\rangle_{[1]} \\ &\quad\to \frac{2}{\pi}u(t)\varphi(0)\int_0^\infty\int_{-\infty}^{\infty}\frac{\xi_1}{|\xi|^2}\int_0^\infty ye^{-y^2}\, dy\, d\xi_2\, \psi(\xi_1)\, d\xi_1\end{aligned}$$

$$= \frac{1}{\pi} u(t)\varphi(0) \int_0^{\infty} \xi_1 \psi(\xi_1) \int_{-\infty}^{\infty} \frac{d\xi_2}{\xi_1^2 + \xi_2^2}\, d\xi_1 = u(t)\varphi(0) \int_{-\infty}^{\infty} \psi(\xi_1)\, d\xi_1$$
$$= \left\langle \left\langle u(t)\delta_{[2]}, \varphi \right\rangle_{[2]}, \psi \right\rangle_{[1]}, \quad t \in [0,T],$$

i.e.,

$$\mathcal{W}_u\big(0^+, (\cdot)_{[2]}, t\big) = u(t)\delta_{[2]}, \quad t \in [0,T]. \tag{3.17}$$

Taking into account (3.4), (3.15), and (3.17), we obtain (3.3). □

4. Controllability

The following theorem gives us the necessary condition for $f \in \mathcal{R}_T^L(0)$.

Theorem 4.1. *Let $T > 0$, $L > 0$. If $f \in \mathcal{R}_T^L(0)$, then $f \in \mathcal{H}^{-1}$ and*

$$\int_0^{\infty} \int_0^{\infty} e^{\frac{|x|^2}{4T^*}} |f(x)|\, dx_1\, dx_2 \le \frac{L}{\sqrt{\pi}} \sqrt{\frac{T^* T}{T^* - T}}, \quad T^* > T. \tag{4.1}$$

Proof. From Theorem 3.5 (i), it follows that $f \in \mathcal{H}^{-1}$. Taking into account (3.14), we obtain

$$\int_0^{\infty} \int_0^{\infty} e^{\frac{|x|^2}{4T^*}} |f(x)|\, dx_1\, dx_2 \le \frac{L}{\pi} \int_0^{\infty} \int_0^{\infty} e^{\frac{|x|^2}{4T^*}} x_1 \int_0^T e^{-\frac{|x|^2}{4\xi}} \frac{1}{4\xi^2} d\xi\, dx_1\, dx_2$$
$$= \frac{L}{\pi} \int_0^T \frac{1}{4\xi^2} \int_0^{\infty} \int_0^{\infty} x_1 e^{-|x|^2\left(\frac{1}{4\xi} - \frac{1}{4T^*}\right)}\, dx_1\, dx_2\, d\xi$$
$$= \frac{L}{4\pi} \int_0^T \frac{1}{\xi^2} \left(\int_0^{\infty} x_1 e^{-x_1^2\left(\frac{1}{4\xi} - \frac{1}{4T^*}\right)} dx_1 \int_0^{\infty} e^{-x_2^2\left(\frac{1}{4\xi} - \frac{1}{4T^*}\right)} dx_2 \right) d\xi. \tag{4.2}$$

It is easy to see that

$$\int_0^{\infty} x_1 e^{-x_1^2\left(\frac{1}{4\xi} - \frac{1}{4T^*}\right)} dx_1 = \frac{2T^*\xi}{T^* - \xi}, \qquad \int_0^{\infty} e^{-x_2^2\left(\frac{1}{4\xi} - \frac{1}{4T^*}\right)} dx_2 = \sqrt{\frac{\pi T^* \xi}{T^* - \xi}}.$$

Continuing (4.2), we obtain

$$\int_0^{\infty} \int_0^{\infty} e^{\frac{|x|^2}{4T^*}} |f(x)|\, dx_1\, dx_2 \le \frac{L}{4\pi} \int_0^T \frac{1}{\xi^2} \frac{2T^*\xi}{T^* - \xi} \sqrt{\frac{\pi T^* \xi}{T^* - \xi}}\, d\xi$$
$$= \frac{L (T^*)^{3/2}}{2\sqrt{\pi}} \int_0^T \frac{d\xi}{\sqrt{\xi}(T^* - \xi)^{3/2}}. \tag{4.3}$$

Replacing ξ by $T^* \sin^2 t$, we get

$$\int_0^T \frac{d\xi}{\sqrt{\xi}(T^* - \xi)^{3/2}} = \frac{2}{T^*} \int_0^{\arcsin\sqrt{\frac{T}{T^*}}} \frac{dt}{\cos^2 t}$$
$$= \frac{2}{T^*} \tan\left(\arcsin\sqrt{\frac{T}{T^*}} \right) = \frac{2}{T^*} \frac{\sqrt{T}}{\sqrt{T^* - T}}.$$

With regard to (4.3), we conclude that (4.1) is true. □

Theorem 4.2. *Let* $L > 0$, $T > 0$, $W^0 \in \widetilde{H}^{-1}(\mathbb{R}^2)$, $W^T \in \widetilde{H}^{-1}(\mathbb{R}^2)$, *and condition* (4.1) *holds for* $\widetilde{W} = W^T - \mathcal{W}_0(\cdot, T)$. *Let*

$$\omega_n = 2\frac{n!}{(2n+1)!}\int_0^\infty \int_0^\infty x_1^{2n+1}\widetilde{W}(x_1, x_2)\,dx_1\,dx_2, \qquad n = \overline{0, \infty}. \tag{4.4}$$

Then $W^T \in \mathcal{R}_T^L(W^0)$ *iff* $\widetilde{W} \in \mathcal{H}^{-1}$ *and there exists* $u \in L^\infty(0,T)$ *such that* $\|u\|_{L^\infty(0,T)} \leq L$ *and*

$$\int_0^T \xi^n u(T-\xi)\,d\xi = \omega_n, \qquad n = \overline{0, \infty}. \tag{4.5}$$

Proof. According to Theorem 3.5 (iv), $W^T \in \mathcal{R}_T^L(W^0)$ iff $\widetilde{W} \in \mathcal{R}_T^L(0)$. With regard to (3.14), we conclude that $\widetilde{W} \in \mathcal{R}_T^L(0)$ iff there exists $u \in L^\infty(0,T)$ such that $\|u\|_{L^\infty(0,T)} \leq L$ and $\widetilde{W} = \mathcal{W}_u(\cdot, T)$. Denote $\widetilde{V} = \mathcal{F}\widetilde{W}$.

Let $u \in L^\infty(0,T)$ such that $\|u\|_{L^\infty(0,T)} \leq L$ and $\widetilde{W} = \mathcal{W}_u(\cdot, T)$. Hence,

$$\widetilde{V}(\sigma) = -\frac{i\sigma_1}{\pi}\int_0^T e^{-|\sigma|^2\xi}u(T-\xi)d\xi, \quad \sigma \in \mathbb{R}^2. \tag{4.6}$$

For

$$g(z) = -\frac{i}{\pi}\int_0^T e^{-z\xi}u(T-\xi)d\xi, \quad z \in \mathbb{C}, \tag{4.7}$$

we have

$$\widetilde{V} = \sigma_1 g\left(|\sigma|^2\right), \quad \sigma \in \mathbb{R}^2. \tag{4.8}$$

Due to the Paley–Wiener theorem, g is an entire function. Therefore,

$$g(z) = \sum_{n=0}^\infty \frac{g^{(n)}(0)}{n!}z^n, \quad z \in \mathbb{C}. \tag{4.9}$$

Differentiating (4.7), (4.8) and considering the derivatives at $z = 0$ and $\sigma = 0$, we get

$$g^{(n)}(0) = \frac{(-1)^{n+1}i}{\pi}\int_0^T \xi^n u(T-\xi)\,d\xi, \quad n = \overline{0, \infty}, \tag{4.10}$$

$$g^{(n)}(0) = \frac{k!(n-k)!}{(2k+1)!(2(n-k))!}\frac{\partial^{2n+1}\widetilde{V}(0,0)}{\partial\sigma_1^{2k+1}\partial\sigma_2^{2(n-k)}}, \quad k = \overline{0, n},\ n = \overline{0, \infty}. \tag{4.11}$$

Since

$$\begin{aligned}\frac{\partial^{2n+1}\widetilde{V}(0,0)}{\partial\sigma_1^{2k+1}\partial\sigma_2^{2(n-k)}} &= \frac{1}{2\pi}\iint_{\mathbb{R}^2}(-ix_1)^{2k+1}(-ix_2)^{2(n-k)}\widetilde{W}(x_1,x_2)\,dx_1\,dx_2 \\ &= 2\frac{(-1)^{n+1}i}{\pi}\int_0^\infty\int_0^\infty x_1^{2k+1}x_2^{2(n-k)}\widetilde{W}(x_1,x_2)\,dx_1\,dx_2, \\ &\qquad k = \overline{0, n},\ n = \overline{0, \infty},\end{aligned} \tag{4.12}$$

with regard to (4.4), (4.10) and (4.11), we conclude that (4.5) holds.

Let $\widetilde{W} \in \mathcal{H}^{-1}$ and there exists $u \in L^\infty(0,T)$ such that $\|u\|_{L^\infty(0,T)} \le L$ and (4.5) holds. Hence,

$$\widetilde{V}(\sigma) = \sigma_1 f\left(|\sigma|^2\right), \quad \sigma \in \mathbb{R}^2. \tag{4.13}$$

Taking into account (4.4) and (4.12), we get

$$\frac{(-1)^{n+1} i}{\pi} \int_0^T \xi^n u(T-\xi)\, d\xi = \frac{n!}{(2n+1)!} \frac{\partial^{2n+1} \widetilde{V}(0,0)}{\partial \sigma_1^{2n+1}}, \quad n = \overline{0,\infty}. \tag{4.14}$$

Multiplying this relation by $\sigma_1^{2n+1}/n!$, we obtain

$$\begin{aligned} -\frac{i\sigma_1}{\pi} \int_0^T e^{-\sigma_1^2 \xi} u(T-\xi) d\xi &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{\partial^{2n+1} \widetilde{V}(0,0)}{\partial \sigma_1^{2n+1}} \sigma_1^{2n+1} \\ &= \widetilde{V}(\sigma_1, 0), \quad \sigma_1 \in \mathbb{R}. \end{aligned} \tag{4.15}$$

Substituting $\sigma_2 = 0$ in (4.13), we have

$$\widetilde{V}(\sigma_1, 0) = \sigma_1 f\left(\sigma_1^2\right), \quad \sigma_1 \in \mathbb{R}.$$

With regard to (4.15), we get

$$-\frac{i\sigma_1}{\pi} \int_0^T e^{-\sigma_1^2 \xi} u(T-\xi) d\xi = \sigma_1 f\left(\sigma_1^2\right), \quad \sigma_1 \in \mathbb{R}.$$

Therefore,

$$f(z) = -\frac{i}{\pi} \int_0^T e^{-z\xi} u(T-\xi) d\xi, \quad z \in \mathbb{C}.$$

Taking into account (4.13), we have

$$\widetilde{V}(\sigma) = -\frac{i\sigma_1}{\pi} \int_0^T e^{-|\sigma|^2 \xi} u(T-\xi) d\xi, \quad \sigma \in \mathbb{R}^2.$$

Hence $\widetilde{W} = \mathcal{W}_u(\cdot, T)$. That completes the proof. □

Theorem 4.3. *Let $L > 0$, $T > 0$, $W^0 \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right)$, $W^T \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right)$, $\widetilde{W} = W^T - \mathcal{W}_0(\cdot, T) \in \mathcal{H}^{-1}$, and condition (4.1) holds for $\widetilde{W}$. Let $\{\omega_n\}_{n=0}^\infty$ be defined by (4.4). If for each $N \in \mathbb{N}$ there exists $u_N \in L^\infty(0,T)$ such that $\|u_N\|_{L^\infty(0,T)} \le L$ and*

$$\int_0^T \xi^n u_N(T-\xi) d\xi = \omega_n, \quad n = \overline{0,N}, \tag{4.16}$$

then $W^T \in \overline{\mathcal{R}_T^L\left(W^0\right)}$, where the closure is considered in $\widetilde{H}^{-1}\left(\mathbb{R}^2\right)$.

Proof. Let $N \in \mathbb{N}$, W_N be the solution to (3.1), (3.2) with $W^0 = 0$ and $u = u_N$. Then, $W_N = \mathcal{W}_{u_N}$. Denote $\widetilde{V} = \mathcal{F}\widetilde{W}$ and $V_N(\cdot, T) = \mathcal{F}_{x \to \sigma} W_N(\cdot, T)$. We have

$$V_N(\sigma) = \sigma_1 g_N\left(|\sigma|^2\right), \quad \sigma \in \mathbb{R}^2, \tag{4.17}$$

where

$$g_N(z) = -\frac{i}{\pi}\int_0^T e^{-z\xi} u_N(T-\xi)d\xi, \quad z \in \mathbb{C}. \tag{4.18}$$

According to the Paley–Wiener theorem, we conclude that g_N is an entire function. Since $\widetilde{W} \in \mathcal{H}^{-1}$, we get $\widetilde{V} \in \mathcal{H}_{-1}$ and

$$\widetilde{V}(\sigma) = \sigma_1 f\left(|\sigma|^2\right), \quad \sigma \in \mathbb{R}^2. \tag{4.19}$$

We have

$$\iint_{|\sigma|\geq a} \left(1+|\sigma|^2\right)^{-1} \left|\widetilde{V}(\sigma) - V_N(\sigma)\right|^2 d\sigma \to 0 \quad \text{as } a \to \infty. \tag{4.20}$$

Now, let us estimate the integral

$$\iint_{|\sigma|\leq a} \left(1+|\sigma|^2\right)^{-1} \left|\widetilde{V}(\sigma) - V_N(\sigma)\right|^2 d\sigma, \quad a > 0.$$

Let an arbitrary $a > 0$ be fixed. From (4.4), (4.10), (4.12), (4.16), and (4.18), it follows that

$$\frac{n!}{(2n+1)!}\frac{\partial^{2n+1}\widetilde{V}(0,0)}{\partial\sigma_1^{2n+1}} = \frac{(-1)^{n+1}i}{\pi}\omega_n, \qquad n = \overline{0,\infty}, \tag{4.21}$$

$$g_N^{(n)}(0) = \frac{(-1)^{n+1}i}{\pi}\omega_n, \qquad n = \overline{0,N}. \tag{4.22}$$

Taking into account (4.18), we obtain

$$g_N(\mu) = \sum_{n=0}^{N} \frac{g_N^{(n)}(0)}{n!}\mu^n + \frac{(-1)^N}{(N+1)!}\mu^{N+1}R_N^g(\mu), \quad \mu \in [0,a^2],$$

where

$$R_N^g(\mu) = \frac{i}{\pi}\int_0^T e^{-\xi\widetilde{\mu}}\xi^{N+1}u_N(T-\xi)\,d\xi, \quad \mu \in [0,a^2],$$

here $\widetilde{\mu} \in [0,\mu]$. Since

$$\left|\int_0^T e^{-\xi\widetilde{\mu}}\xi^{N+1}u_N(T-\xi)\,d\xi\right| \leq L\int_0^T \xi^{N+1}\,d\xi = L\frac{T^{N+2}}{N+2}, \quad \widetilde{\mu} \in [0,a^2],$$

we get

$$\left|\frac{(-1)^N}{(N+1)!}\mu^{N+1}R_N^g(\mu)\right| \leq \frac{LT^{N+2}a^{2(N+1)}}{\pi(N+2)!}, \quad \mu \in [0,a^2].$$

Therefore, with regard to (4.17) and (4.22), we get

$$V_N(\sigma) = \frac{i\sigma_1}{\pi}\sum_{n=0}^{N}\frac{(-1)^{n+1}}{n!}\omega_n|\sigma|^{2n} + \widetilde{R}_N^g(\sigma), \quad |\sigma| \leq a, \tag{4.23}$$

where

$$\widetilde{R}_N^g(\sigma) = \frac{(-1)^N}{(N+1)!} R_N^g\left(|\sigma|^2\right)|\sigma|^{2(N+1)}\sigma_1, \quad |\sigma| \le a,$$

and

$$\left|\widetilde{R}_N^g(\sigma)\right| \le \frac{LT^{N+2}a^{2N+3}}{\pi(N+2)!}, \quad |\sigma| \le a. \tag{4.24}$$

Taking into account (4.21), we obtain

$$\begin{aligned}\widetilde{V}(\sigma_1,0) &= \sum_{n=0}^{N} \frac{1}{(2n+1)!} \frac{\partial^{2n+1}\widetilde{V}(0,0)}{\partial\sigma_1^{2n+1}}\sigma_1^{2n+1} + \widetilde{R}_N^f(\sigma_1) \\ &= \frac{i\sigma_1}{\pi}\sum_{n=0}^{N}\frac{(-1)^{n+1}}{n!}\omega_n\sigma_1^{2n} + \widetilde{R}_N^f(\sigma_1), \quad \sigma_1 \in [0,a],\end{aligned} \tag{4.25}$$

where

$$\widetilde{R}_N^f(\sigma_1) = \frac{1}{(2N+3)!}\frac{\partial^{2N+3}\widetilde{V}(\widetilde{\sigma}_1,0)}{\partial\sigma_1^{2N+3}}\sigma_1^{2N+3}, \quad \sigma_1 \in [0,a], \tag{4.26}$$

here $\widetilde{\sigma}_1 \in [0,\sigma_1]$. With regard to (4.19), we get

$$\sigma_1 f(\sigma_1) = \widetilde{V}(\sigma_1,0), \quad \sigma_1 \in \mathbb{R}.$$

Therefore,

$$\widetilde{V}(\sigma) = \frac{i\sigma_1}{\pi}\sum_{n=0}^{N}\frac{(-1)^{n+1}}{n!}\omega_n|\sigma|^{2n} + \widetilde{R}_N^f(\sigma), \quad |\sigma| \le a, \tag{4.27}$$

where

$$\widetilde{R}_N^f(\sigma) = \frac{\sigma_1}{(2N+3)!}\frac{\partial^{2N+3}\widetilde{V}(\widetilde{\sigma}_1,0)}{\partial\sigma_1^{2N+3}}|\sigma|^{2(N+1)}, \quad |\sigma| \le a, \tag{4.28}$$

here $\widetilde{\sigma}_1 \in [0,\sigma_1]$. To estimate $\widetilde{R}_N^f$, let us first estimate $\partial^{2N+3}\widetilde{V}(\sigma_1,0)/\partial\sigma_1^{2N+3}$ for $\sigma_1 \in [0,a]$. Let $T^* > T$ be fixed. With regard to (4.1), we put

$$\Omega_{T^*} = \int_0^\infty\int_0^\infty e^{\frac{|x|^2}{4T^*}}\left|\widetilde{W}(x)\right| dx_1\,dx_2.$$

We have

$$\begin{aligned}\left|\frac{\partial^{2N+3}\widetilde{V}(\sigma_1,0)}{\partial\sigma_1^{2N+3}}\right| &\le \frac{1}{2\pi}\iint_{\mathbb{R}^2}\left|x_1^{2N+3}\widetilde{W}(x)\right| dx \\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty\left(x_1^{2N+3}e^{-\frac{|x|^2}{4T^*}}\right)\left(e^{\frac{|x|^2}{4T^*}}\left|\widetilde{W}(x)\right|\right)dx_1\,dx_2.\end{aligned}$$

Since

$$x_1^{2N+3}e^{-\frac{|x|^2}{4T^*}} \le \left(\frac{2T^*(2N+3)}{e}\right)^{N+3/2}, \quad x_1 \ge 0,\ x_2 \ge 0,$$

we obtain

$$\left|\frac{\partial^{2N+3}\widetilde{V}(\sigma_1,0)}{\partial\sigma_1^{2N+3}}\right| \le \Omega_{T^*}\frac{2}{\pi}\left(\frac{2T^*(2N+3)}{e}\right)^{N+3/2}, \quad \sigma_1\in\mathbb{R}. \tag{4.29}$$

Using Stirling's formula:

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \le n! \le en^{n+\frac{1}{2}}e^{-n}, \qquad n\in\mathbb{N}, \tag{4.30}$$

we conclude that

$$(2N+3)! \ge \sqrt{2\pi(2N+3)}\left(\frac{2N+3}{e}\right)^{2N+3}.$$

With regard to (4.29), we get

$$\left|\frac{1}{(2N+3)!}\frac{\partial^{2N+3}\widetilde{V}(\sigma_1,0)}{\partial\sigma_1^{2N+3}}\right| \le \sqrt{\frac{2}{\pi^3}}\Omega_{T^*}\left(\frac{2T^*e}{2N+3}\right)^{N+3/2}, \quad \sigma_1\in\mathbb{R}. \tag{4.31}$$

According to (4.28), we obtain

$$\left|\widetilde{R}_N^f(\sigma)\right| \le \sqrt{\frac{2}{\pi^3}}\Omega_{T^*}\left(\frac{2T^*e}{2N+3}\right)^{N+3/2}a^{2N+3}, \quad |\sigma|\le a. \tag{4.32}$$

From (4.23) and (4.27), it follows that

$$\widetilde{V}(\sigma)-V_N(\sigma)=\widetilde{R}_N^g(\sigma)-\widetilde{R}_N^f(\sigma), \quad |\sigma|\le a.$$

Taking into account (4.24) and (4.32), we get

$$\iint_{|\sigma|\le a}\left(1+|\sigma|^2\right)^{-1}\left|\widetilde{V}(\sigma)-V_N(\sigma)\right|^2 d\sigma$$
$$\le\left(L\frac{T^{N+2}}{(N+2)!}+\sqrt{\frac{2}{\pi}}\Omega_{T^*}\left(\frac{2T^*e}{2N+3}\right)^{N+3/2}\right)^2\frac{a^{4N+8}}{\pi}\to 0 \quad \text{as } N\to\infty.$$

With regard to (4.20), we conclude that

$$\left\|\widetilde{W}-W_N\right\|^{-1}=\left\|\widetilde{V}-V_N\right\|_{-1}\to 0 \quad \text{as } N\to\infty,$$

therefore, $\widetilde{W}\in\overline{\mathcal{R}_T^L(0)}$, hence (see 3.5 (iv)), $W^T\in\overline{\mathcal{R}_T^L(W^0)}$, where the closure is considered in $\widetilde{H}^{-1}(\mathbb{R}^2)$. The theorem is proved. □

In Theorems 4.2 and 4.3 we deal with a control bounded by a given constant. In fact, in these theorems, we reduce the controllability problems to the Markov power moment problems. They may be solved by using the algorithms given in [21, 24]. Similar results were obtained for controllability problems for the heat equation on a half-axis [12, 13]. Theorem 4.2 gives us necessary and sufficient conditions for controllability of an initial state $W^0\in\widetilde{H}^{-1}(\mathbb{R}^2)$ to a target state $W^T\in\widetilde{H}^{-1}(\mathbb{R}^2)$ in a given time $T>0$ under the additional condition (4.1) on the function $W^T-\mathcal{W}_0(\cdot,T)$. Theorem 4.3 gives us the sufficient conditions for approximate controllability of an initial state $W^0\in\widetilde{H}^{-1}(\mathbb{R}^2)$ to a target state $W^T\in\widetilde{H}^{-1}(\mathbb{R}^2)$ in a given time $T>0$ under the same additional condition.

5. Bases in $\mathcal{H}^{-1}$ and $\mathcal{H}_{-1}$

In this section we introduce and study orthogonal bases in $\mathcal{H}^{-1}$ and $\mathcal{H}_{-1}$.

In the spaces $H_m(\mathbb{R}^2)$ and $H^m(\mathbb{R}^2)$, $m = \overline{-3,3}$, we consider the following inner products

$$\langle g,h\rangle_m = \left\langle \left(1+|\sigma|^2\right)^{\frac{m}{2}} g, \left(1+|\sigma|^2\right)^{\frac{m}{2}} h \right\rangle_0, \qquad f \in H_m(\mathbb{R}^2),\ g \in H_m(\mathbb{R}^2),$$
$$\langle \nu,\mu\rangle^m = \langle \mathcal{F}\nu, \mathcal{F}\mu\rangle_m, \qquad \nu \in H^m(\mathbb{R}^2),\ \mu \in H^m(\mathbb{R}^2),$$

where $\langle\cdot,\cdot\rangle_0$ is the inner product in $L^2(\mathbb{R}^2)$. Note that $\langle\cdot,\cdot\rangle^0 = \langle\cdot,\cdot\rangle_0$. For $f \in H_m(\mathbb{R}^2)$ and $g \in H_m(\mathbb{R}^2)$, if $g(\sigma) = i\sigma_1 f(|\sigma|)$ and $h(\sigma) = i\sigma_1 p(|\sigma|)$, $\sigma \in \mathbb{R}^2$, then

$$\langle g,h\rangle_m = \pi \langle f,p\rangle_{L^2_{m,(3)}(\mathbb{R}_+)}. \tag{5.1}$$

First, we prove the following lemma.

Lemma 5.1. *Let $T>0$,*

$$\varphi_m(\sigma) = i\sigma_1 |\sigma|^m e^{-T|\sigma|^2}, \qquad \sigma \in \mathbb{R}^2,\ m = \overline{0,\infty}. \tag{5.2}$$

Then the system $\{\varphi_m\}_{m=0}^\infty$ is complete in $\mathcal{H}_{-1}$.

Proof. Put

$$q(r) = \frac{r^{3/2}}{\sqrt{1+r^2}} e^{-Tr^2}, \qquad r>0.$$

Let us prove that the system $\{(\cdot)^m q\}_{m=0}^\infty$ is complete in $L^2(\mathbb{R}_+)$. Suppose the converse. Due to the Hahn–Banach theorem, there exists $h \in L^2(\mathbb{R}_+) \setminus \{0\}$ such that

$$0 = \langle (\cdot)^m q, h\rangle_{L^2(\mathbb{R}_+)} = \int_0^\infty r^m q(r)\overline{h(r)}\,dr, \qquad m = \overline{0,\infty}. \tag{5.3}$$

Extend the functions q and h to $\mathbb{R}$ by setting $q(r) = h(r) = 0$ on $(-\infty,0]$. Evidently, $q\overline{h} \in L^1(\mathbb{R})$ and its Fourier transform is given by

$$\left(\mathcal{F}\left(q\overline{h}\right)\right)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ir\lambda} q(r)\overline{h(r)}\,dr, \qquad \lambda \in \mathbb{C}.$$

Moreover, $\mathcal{F}\left(q\overline{h}\right)$ is an entire function. According to (5.3), we have

$$\left(\mathcal{F}\left(q\overline{h}\right)\right)^{(m)}(0) = 0, \qquad m = \overline{0,\infty}.$$

Therefore, $\mathcal{F}\left(q\overline{h}\right) = 0$. Hence, $q\overline{h} = 0$ in $L^1(\mathbb{R})$. Since $q \neq 0$, we see that $h=0$ in $L^2(\mathbb{R}_+)$. This contradiction proves that the system $\{(\cdot)^m q\}_{m=0}^\infty$ is complete in $L^2(\mathbb{R}_+)$.

Putting $f_m(r) = r^m e^{-Tr^2}$, $r>0$, we conclude that the system $\{f_m\}_{m=0}^\infty$ is complete in $L^2_{-1,(3)}(\mathbb{R}_+)$. Therefore, the system $\{\varphi_m\}_{m=0}^\infty$ is complete in $\mathcal{H}_{-1}$ according to (2.3) and (5.1). $\square$

Consider the generalized Laguerre polynomials [22, p. 775]:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\left(\frac{d}{dx}\right)^n\left(e^{-x}x^{n+\alpha}\right) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \quad n=\overline{0,\infty}. \tag{5.4}$$

Denote

$$\psi_n(x) = \frac{x_1}{\sqrt{2T}}e^{-\frac{|x|^2}{4T}}L_n^{(1)}\left(\frac{|x|^2}{2T}\right), \qquad x\in\mathbb{R}^2,\ n=\overline{0,\infty}. \tag{5.5}$$

With regard to [22, p. 775], we get

$$\langle\psi_n,\psi_m\rangle^0 = \iint_{\mathbb{R}^2}\psi_n(x)\psi_m(x)\,dx_1\,dx_2 = T\pi(n+1)\delta_{nm}, \qquad m,n=\overline{0,\infty}, \tag{5.6}$$

where δ_{mn} is the Kronecker delta.

Denote $\widehat{\psi}_n(\sigma) = \mathcal{F}\psi_n(\sigma)$, $n=\overline{0,\infty}$. One can easily obtain

$$\widehat{\psi}_n(\sigma) = (-1)^{n+1}i2T\psi_n(2T\sigma), \qquad n=\overline{0,\infty}. \tag{5.7}$$

Hence,

$$\langle\widehat{\psi}_n,\widehat{\psi}_m\rangle_0 = \iint_{\mathbb{R}^2}\widehat{\psi}_n(\sigma)\overline{\widehat{\psi}_m(\sigma)}\,d\sigma_1\,d\sigma_2 = (-1)^{n+m}T\pi(n+1)\delta_{nm}, \qquad m,n=\overline{0,\infty}. \tag{5.8}$$

Thus, $\{\psi_n\}_{n=0}^{\infty}$ and $\left\{\widehat{\psi}_n\right\}_{n=0}^{\infty}$ are orthogonal bases in $\mathcal{H}^0$ and $\mathcal{H}_0$, respectively.

Put

$$\psi_n^1(x) = 2T\sum_{p=0}^{n}\frac{(-1)^{p+1}L_p^{(1)}(-2T)}{p+1}\left(1+|D|^2\right)\psi_p(x), \quad x\in\mathbb{R}^2,\ n=\overline{0,\infty}, \tag{5.9}$$

and denote $\widehat{\psi}_n^1 = \mathcal{F}\psi_n^1$, $n=\overline{0,\infty}$. Obviously,

$$\widehat{\psi}_n^1(\sigma) = 2T\left(1+|\sigma|^2\right)\sum_{p=0}^{n}\frac{(-1)^{p+1}L_p^{(1)}(-2T)}{p+1}\widehat{\psi}_p(\sigma), \quad \sigma\in\mathbb{R}^2,\ n=\overline{0,\infty}. \tag{5.10}$$

Taking into account (5.5) and (5.7), for $n=\overline{0,\infty}$, we get

$$\widehat{\psi}_n^1(\sigma) = i(2T)^2\left(1+|\sigma|^2\right)\sum_{p=0}^{n}\frac{L_p^{(1)}(-2T)}{p+1}\psi_p(2T\sigma)$$
$$= i(2T)^2\left(1+|\sigma|^2\right)\sqrt{2T}\sigma_1e^{-T|\sigma|^2}\sum_{p=0}^{n}\frac{L_p^{(1)}(-2T)L_p^{(1)}(2T|\sigma|^2)}{p+1}, \quad \sigma\in\mathbb{R}^2. \tag{5.11}$$

Using the Christoffel–Darboux formula [22, pp. 785], from (5.11) we obtain

$$\widehat{\psi}_n^1(\sigma) = i(2T)^2\left(1+|\sigma|^2\right)\sqrt{2T}\sigma_1e^{-T|\sigma|^2}$$

$$\times \frac{L_n^{(1)}(2T|\sigma|^2)L_{n+1}^{(1)}(-2T) - L_{n+1}^{(1)}(2T|\sigma|^2)L_n^{(1)}(-2T)}{2T|\sigma|^2+2T}$$
$$= i2T\sqrt{2T}\sigma_1 e^{-T|\sigma|^2}\left(L_n^{(1)}(2T|\sigma|^2)L_{n+1}^{(1)}(-2T) - L_{n+1}^{(1)}(2T|\sigma|^2)L_n^{(1)}(-2T)\right)$$
$$= (-1)^{n+1}\left(L_{n+1}^{(1)}(-2T)\widehat{\psi}_n(\sigma) + L_n^{(1)}(-2T)\widehat{\psi}_{n+1}(\sigma)\right), \qquad \sigma\in\mathbb{R}^2. \quad (5.12)$$

Let $m \ge n$, $n=\overline{0,\infty}$. Taking into account (5.8), (5.10), and (5.12), we have

$$\left\langle \widehat{\psi}_m^1, \widehat{\psi}_n^1\right\rangle_{-1} = 2T(-1)^{m+1}\sum_{p=0}^{n}\frac{(-1)^{p+1}L_p^{(1)}(-2T)}{p+1}\left(L_{m+1}^{(1)}(-2T)\left\langle\widehat{\psi}_m,\widehat{\psi}_p\right\rangle_0\right.$$
$$\left. + L_m^{(1)}(-2T)\left\langle\widehat{\psi}_{m+1},\widehat{\psi}_p\right\rangle_0\right)$$
$$= 2T(-1)^{m+1}\frac{(-1)^{n+1}L_n^{(1)}(-2T)L_{m+1}^{(1)}(-2T)}{n+1}\left\langle\widehat{\psi}_m,\widehat{\psi}_n\right\rangle_0$$
$$= 2T^2\pi L_n^{(1)}(-2T)L_{m+1}^{(1)}(-2T)\delta_{nm}, \qquad m,n=\overline{0,\infty}. \quad (5.13)$$

Note that $L_n^{(1)}(-2T)>0$ for all $n=\overline{0,\infty}$.

According to Lemma 5.1, the system $\left\{\widehat{\psi}_n\right\}_{n=0}^{\infty}$ is complete in $\mathcal{H}_{-1}$. Taking into account (5.12), we conclude that the system $\left\{\widehat{\psi}_n^1\right\}_{n=0}^{\infty}$ is complete in $\mathcal{H}_{-1}$. Due to (5.13), we obtain the following theorem.

Theorem 5.2. *The systems $\left\{\widehat{\psi}_n^1\right\}_{n=0}^{\infty}$ and $\left\{\psi_n^1\right\}_{n=0}^{\infty}$ are orthogonal bases in $\mathcal{H}_{-1}$ and $\mathcal{H}^{-1}$, respectively.*

Denote

$$\varphi_{2n}^l(\sigma) = i\sigma_1|\sigma|^{2n}e^{-T|\sigma|^2}\left(\frac{e^{|\sigma|^2/l}-1}{|\sigma|^2/l}\right)^{n+1}, \qquad \sigma\in\mathbb{R}^2,\ l\in\mathbb{N},\ n=\overline{0,\infty},$$
$$u_l^n(\xi) = \begin{cases}(-1)^{n-j}\binom{n}{j}l^{n+1}, & \xi\in\left(\frac{j}{l},\frac{j+1}{l}\right),\ j=\overline{0,n}\\ 0, & \xi\notin\left[0,\frac{n+1}{l}\right]\end{cases}, \quad l\in\mathbb{N},\ n=\overline{0,\infty}. \quad (5.14)$$

Note that $u_l^n \to \delta^{(n)}$ as $l\to\infty$ in $H^{-1}(\mathbb{R})$ for each $n=\overline{0,\infty}$.

Lemma 5.3. *Let $l\in\mathbb{N}$, $n=\overline{0,\infty}$. Then*

$$\mathcal{F}W_{u_l^n}(\cdot,T) = -\frac{1}{\pi}\varphi_{2n}^l. \quad (5.15)$$

Proof. We have

$$\left(\mathcal{F}W_{u_l^n}(\cdot,T)\right)(\sigma) = -\frac{i}{\pi}\sigma_1\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}l^{n+1}\int_{j/l}^{(j+1)/l}e^{-(T-\xi)|\sigma|^2}d\xi$$
$$= -\frac{i}{\pi}\sigma_1 l^{n+1}\frac{e^{-T|\sigma|^2}}{|\sigma|^2}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\left(e^{(j+1)|\sigma|^2/l}-e^{j|\sigma|^2/l}\right)$$

$$= -\frac{i}{\pi}\sigma_1 l^{n+1}\frac{e^{-T|\sigma|^2}}{|\sigma|^2}\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^{n-j+1}e^{j|\sigma|^2/l}$$

$$= -\frac{i}{\pi}\sigma_1|\sigma|^{2n}e^{-T|\sigma|^2}\left(\frac{e^{|\sigma|^2/l}-1}{|\sigma|^2/l}\right)^{n+1} = -\frac{1}{\pi}\varphi_{2n}^l(\sigma), \qquad \sigma \in \mathbb{R}^2.$$

The lemma is proved. □

Lemma 5.4. *Let* $n=\overline{0,\infty}$ *and* $l>\frac{2(n+1)}{T}$. *Then*

$$\left\|\varphi_{2n}-\varphi_{2n}^l\right\|_{-1} \longrightarrow 0 \quad \text{as } l\to\infty, \quad n=\overline{0,\infty}. \tag{5.16}$$

Proof. We have

$$\varphi_{2n}(\sigma)=i\sigma_1 f_{2n}(|\sigma|), \quad \varphi_{2n}^l(\sigma)=i\sigma_1 f_{2n}^l(|\sigma|), \quad \sigma\in\mathbb{R}^2,\ l\in\mathbb{N},\ n=\overline{0,\infty},$$

where

$$f_{2n}(r)=r^{2n}e^{-Tr^2}, \quad f_{2n}^l(r)=r^{2n}e^{-Tr^2}\left(\frac{e^{r^2/l}-1}{r^2/l}\right)^{n+1}, \qquad r\ge 0,\ l\in\mathbb{N},\ n=\overline{0,\infty}.$$

Let $l>(2n+2)/T$. Then,

$$\left|f_{2n}^l(r)\right| \le r^{2n}e^{-\left(T-\frac{n+1}{l}\right)r^2} \le r^{2n}e^{-\frac{T}{2}r^2}, \quad r\ge 0,\ n=\overline{0,\infty}.$$

Since $r^2<1+r^2$, we obtain

$$\int_0^\infty \left|\frac{r^{2n+\frac{3}{2}}}{\sqrt{1+r^2}}e^{-\frac{T}{2}r^2}\right|^2 dr < \int_0^\infty r^{4n+1}e^{-Tr^2}dr = \frac{(2n)!}{2T^{2n+1}}, \qquad n=\overline{0,\infty}.$$

It is easy to see that

$$\frac{r^{\frac{3}{2}}f_{2n}^l(r)}{\sqrt{1+r^2}} \to \frac{r^{\frac{3}{2}}f_{2n}(r)}{\sqrt{1+r^2}} \quad \text{as } l\to\infty \ \text{ a.e. on } \mathbb{R}_+,\ n=\overline{0,\infty}.$$

According to Lebesgue's dominated convergence theorem (see also (2.2)), for each $n=\overline{0,\infty}$, we get

$$\left\|f_{2n}(r)-f_{2n}^l(r)\right\|_{L^2_{-1,(3)}(\mathbb{R}_+)} \to 0 \quad \text{as } l\to\infty.$$

Due to (2.3), we obtain (5.16). The lemma is proved. □

6. Approximate controllability

Theorem 6.1. *Let* $T > 0$. *Then* $\overline{\mathcal{R}_T(0)} = \mathcal{H}^{-1}$.

Proof. Let $f \in \mathcal{H}^{-1}$. Then $F = \mathcal{F}f \in \mathcal{H}_{-1}$. From Lemma 5.1, it follows that for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $\alpha_n^N \in \mathbb{R}$, $n = \overline{0, N}$, such that

$$\left\|F - F^N\right\|_{-1} < \frac{\varepsilon}{2}, \tag{6.1}$$

where

$$F^N = \sum_{n=0}^{N+1} \alpha_n^N \varphi_{2n}. \tag{6.2}$$

According to Lemma 5.4, for each $\varepsilon > 0$, there exist $l \in \mathbb{N}$ such that

$$\left\|\varphi_{2n} - \varphi_{2n}^l\right\|_{-1} < \varepsilon \left(2 \sum_{m=0}^{N+1} \left|\alpha_m^N\right|\right)^{-1}, \qquad n = \overline{0, N}.$$

Therefore,

$$\left\|F^N - F_l^N\right\|_{-1} < \varepsilon/2, \tag{6.3}$$

where

$$F_l^N = \sum_{n=0}^{N+1} \alpha_n^N \varphi_{2n}^l. \tag{6.4}$$

Combining (6.1) and (6.3), we obtain

$$\left\|F - F_l^N\right\|_{-1} \le \left\|F - F^N\right\|_{-1} + \left\|F^N - F_l^N\right\|_{-1} < \varepsilon.$$

Denote $f_l^N = \mathcal{F}^{-1} F_l^N$. By using (5.15) and (6.4), we get

$$f_l^N(x) = \mathcal{W}_{U_l^N}(x, T) = \frac{x_1}{\pi} \int_0^T e^{-\frac{|x|^2}{4\xi}} \frac{U_l^N(T - \xi)}{4\xi^2} \, d\xi, \qquad x \in \mathbb{R}^2,$$

where

$$U_l^N(\xi) = -\pi \sum_{n=0}^{N+1} \alpha_n^N u_l^n(\xi), \qquad \xi \ge 0.$$

Due to (3.13), we get $f_l^N \in \mathcal{R}_T(0)$. Thus, we have

$$\left\|f - f_l^N\right\|^{-1} = \left\|F - F_l^N\right\|_{-1} < \varepsilon.$$

Since we have considered an arbitrary $\varepsilon > 0$, we conclude that $f \in \overline{\mathcal{R}_T(0)}$. □

With regard to Theorems 3.5 (v) and 6.1, we obtain the following theorem

Theorem 6.2. *Each state* $W^0 \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ *is approximately controllable to a state* $W^T \in \widetilde{H}^{-1}\left(\mathbb{R}^2\right)$ *in a given time* $T > 0$ *iff* $W^T - \mathcal{W}_0(\cdot, T) \in \mathcal{H}^{-1}$.

For the state W^0, let us construct controls approximately targeting the state W^T.

Denote $\widetilde{W} = W^T - \mathcal{W}_0(\cdot, T)$, $\widetilde{V} = \mathcal{F}\widetilde{W}$. Then $\widetilde{W} \in \mathcal{H}^{-1}$, $\widetilde{V} \in \mathcal{H}_{-1}$, and, with regard to Theorem 5.2, we have

$$\widetilde{V}(\sigma) = \sum_{n=0}^{\infty} \omega_n \widehat{\psi}_n^1(\sigma), \qquad \sigma \in \mathbb{R}^2,$$

where $\omega_n = \left\langle \widetilde{V}, \widehat{\psi}_n^1 \right\rangle_{-1} \left(\left\| \widehat{\psi}_n^1 \right\|_{-1} \right)^{-2}$, $n = \overline{0, \infty}$.

According to (5.13), for each $\varepsilon_1 > 0$, there exists $N = N(\varepsilon_1) \in \mathbb{N}$ such that

$$\begin{aligned} S_N &= \left\| \sum_{n=N+1}^{\infty} \omega_n \widehat{\psi}_n^1 \right\|_{-1} \\ &= T\sqrt{2\pi} \left(\sum_{n=N+1}^{\infty} |\omega_n|^2 L_n^{(1)}(-2T) L_{n+1}^{(1)}(-2T) \right)^{1/2} < \varepsilon_1. \end{aligned} \tag{6.5}$$

Put

$$\widetilde{V}^N(\sigma) = \sum_{n=0}^{N} \omega_n \widehat{\psi}_n^1(\sigma), \qquad \sigma \in \mathbb{R}^2.$$

Hence,

$$\left\| \widetilde{V} - \widetilde{V}^N \right\|_{-1} = S_N < \varepsilon_1. \tag{6.6}$$

Put

$$\Omega_p^N = \sum_{n=p}^{N} \omega_n, \quad p = \overline{0, N}. \tag{6.7}$$

Due to (5.11), (5.4), and (5.2), we get

$$\begin{aligned} \widetilde{V}^N(\sigma) &= \sum_{n=0}^{N} \omega_n i(2T)^2 \left(1 + |\sigma|^2\right) \sqrt{2T} \sigma_1 e^{-T|\sigma|^2} \sum_{p=0}^{n} \frac{L_p^{(1)}(-2T) L_p^{(1)}(2T|\sigma|^2)}{p+1} \\ &= i(2T)^{5/2} \left(1 + |\sigma|^2\right) \sigma_1 e^{-T|\sigma|^2} \\ &\qquad \times \sum_{p=0}^{N} \Omega_p^N \frac{L_p^{(1)}(-2T)}{p+1} \sum_{k=0}^{p} (-1)^k \binom{p+1}{p-k} \frac{\left(2T|\sigma|^2\right)^k}{k!} \\ &= (2T)^{5/2} \sum_{p=0}^{N} \Omega_p^N \frac{L_p^{(1)}(-2T)}{p+1} \sum_{k=0}^{p} (-1)^k \binom{p+1}{p-k} \frac{(2T)^k}{k!} \varphi_{2k}(\sigma) \\ &+ (2T)^{5/2} \sum_{p=0}^{N} \Omega_p^N \frac{L_p^{(1)}(-2T)}{p+1} \sum_{k=0}^{p} (-1)^k \binom{p+1}{p-k} \frac{(2T)^k}{k!} \varphi_{2(k+1)}(\sigma) \\ &= (2T)^{5/2} \sum_{k=0}^{N} \frac{(-2T)^k}{k!} \sum_{p=k}^{N} \Omega_p^N \frac{L_p^{(1)}(-2T)}{p+1} \binom{p+1}{p-k} \left(\varphi_{2k}(\sigma) + \varphi_{2(k+1)}(\sigma)\right) \end{aligned}$$

$$= (2T)^{5/2} \sum_{k=0}^{N} \frac{(-2T)^k}{(k+1)!} h_k^N \left(\varphi_{2k}(\sigma) + \varphi_{2(k+1)}(\sigma) \right), \qquad \sigma \in \mathbb{R}^2, \tag{6.8}$$

where

$$h_k^N = \sum_{p=k}^{N} \binom{p}{k} \Omega_p^N L_p^{(1)}(-2T), \qquad k = \overline{0, N}. \tag{6.9}$$

Summing the coefficients at φ_{2k}, $k = \overline{0, N}$, in (6.8), we obtain

$$\widetilde{V}^N(\sigma) = \sum_{k=0}^{N+1} \alpha_k^N \varphi_{2k}(\sigma), \qquad \sigma \in \mathbb{R}^2, \tag{6.10}$$

where

$$\begin{aligned} \alpha_0^N &= (2T)^{5/2} h_0^N, \\ \alpha_k^N &= (2T)^{5/2} \left(\frac{(-2T)^{k-1}}{k!} h_{k-1}^N + \frac{(-2T)^k}{(k+1)!} h_k^N \right), \quad k = \overline{1, N}, \\ \alpha_{N+1}^N &= (2T)^{5/2} \frac{(-2T)^N}{(N+1)!} h_N^N. \end{aligned} \tag{6.11}$$

Let $l > \frac{N+2}{T}$. Put

$$\widetilde{V}_l^N(\sigma) = \sum_{k=0}^{N+1} \alpha_k^N \varphi_{2k}^l(\sigma), \qquad \sigma \in \mathbb{R}^2. \tag{6.12}$$

Evidently,

$$\left\| \widetilde{V}^N - \widetilde{V}_l^N \right\|_{-1} \le \sum_{k=0}^{N+1} |\alpha_k^N| \left\| \varphi_{2k} - \varphi_{2k}^l \right\|_{-1}.$$

Let us estimate $\left\| \varphi_{2k} - \varphi_{2k}^l \right\|_{-1}$ under the condition $l > \frac{k+1}{T}$, $k = \overline{0, N+1}$. Due to (2.3), we have

$$\left\| \varphi_{2k} - \varphi_{2k}^l \right\|_{-1} = \sqrt{\pi} \left\| f_{2k} - f_{2k}^l \right\|_{L^2_{-1,(3)}(\mathbb{R}_+)}, \qquad l > \frac{k+1}{T}, \ k = \overline{0, N+1},$$

where

$$f_{2k}(r) = r^{2k} e^{-Tr^2}, \quad f_{2k}^l(r) = r^{2k} e^{-Tr^2} \left(\frac{e^{r^2/l} - 1}{r^2/l} \right)^{k+1},$$
$$r \ge 0, \ l > \frac{k+1}{T}, \ k = \overline{0, N+1}.$$

Taking into account the following obvious inequalities

$$|y^m - 1| \le m y^{m-1} |y - 1|, \qquad y > 0, \ m > 0,$$

$$\left|\frac{e^z-1}{z}\right| \le e^z, \qquad \left|\frac{e^z-1}{z}-1\right| \le \frac{z}{2}e^z, \qquad\qquad z>0,$$

one can easily obtain

$$\left|\left(\frac{e^z-1}{z}\right)^m-1\right| \le m\left|\frac{e^z-1}{z}\right|^{m-1}\left|\frac{e^z-1}{z}-1\right| \le \frac{m}{2}ze^{mz}, \qquad z>0,\ m>0.$$

Due to (2.1) and (2.3), we get

$$\begin{aligned}\left\|\varphi_{2k}-\varphi_{2k}^l\right\|_{-1} &= \sqrt{\pi}\left\|f_{2k}-f_{2k}^l\right\|_{L^2_{-1,(3)}(\mathbb{R}_+)}\\ &= \left(\int_0^\infty \frac{r^{4k+3}e^{-2Tr^2}}{1+r^2}\left(\left(\frac{e^{r^2/l}-1}{r^2/l}\right)^{k+1}-1\right)^2 dr\right)^{1/2}\\ &\le \frac{\sqrt{\pi}(k+1)}{2l}\left(\int_0^\infty \frac{r^{4k+7}}{1+r^2}e^{-2\left(T-\frac{k+1}{l}\right)r^2}dr\right)^{1/2}\\ &\le \frac{k+1}{l}\frac{\sqrt{\pi}}{2}\left(\int_0^\infty r^{4k+5}e^{-2\left(T-\frac{k+1}{l}\right)r^2}dr\right)^{1/2}, \qquad k=\overline{0,N+1}.\end{aligned}$$

Replacing $2\left(T-\frac{k+1}{l}\right)r^2$ by z, we obtain

$$\begin{aligned}\left\|\varphi_{2k}-\varphi_{2k}^l\right\|_{-1} &\le \frac{k+1}{l}\frac{\sqrt{\pi}}{2\sqrt{2}}\left(\int_0^\infty \frac{z^{2k+2}e^{-z}}{\left(2\left(T-\frac{k+1}{l}\right)\right)^{2k+3}}dz\right)^{1/2}\\ &= \frac{k+1}{l}\frac{\sqrt{\pi}}{2^{k+3}}\frac{\sqrt{(2k+2)!}}{\left(T-\frac{k+1}{l}\right)^{k+3/2}}, \qquad k=\overline{0,N+1}.\end{aligned}$$

Thus, for $N=N(\varepsilon_1)$ and for each $\varepsilon_2>0$, there exists $l>\frac{N+2}{T}$ such that

$$\left\|\widetilde{V}^N-\widetilde{V}_l^N\right\|_{-1} \le \frac{\sqrt{\pi}}{8l}\sum_{k=0}^{N+1}\left|\alpha_k^N\right|\frac{k+1}{2^k}\frac{\sqrt{(2k+2)!}}{\left(T-\frac{k+1}{l}\right)^{k+3/2}} \le \varepsilon_2. \tag{6.13}$$

Denote $\widetilde{W}_l^N = \mathcal{F}^{-1}\widetilde{V}_l^N$. Due to (6.6), and (6.13), we have

$$\left\|\widetilde{W}-\widetilde{W}_l^N\right\|^{-1} = \left\|\widetilde{V}-\widetilde{V}_l^N\right\|_{-1} \le \varepsilon_1+\varepsilon_2.$$

Put

$$u_{N,l}(\xi) = -\pi\sum_{n=0}^{N+1}\alpha_n^N u_l^n(\xi), \tag{6.14}$$

where u_l^n is defined by (5.14), α_n^N is defined by (6.11). Using (5.15) and (6.12), we obtain $\widetilde{W}_l^N = \mathcal{W}_{u_{N,l}}(\cdot,T)$. Due to (3.13), we get $\widetilde{W}_l^N \in \mathcal{R}_T(0)$. Since we have considered arbitrary $\varepsilon_1>0$ and $\varepsilon_2>0$, we conclude that $\widetilde{W} \in \overline{\mathcal{R}_T(0)}$. Hence, according to Theorem 3.5 (v), $W^T \in \overline{\mathcal{R}_T(W^0)}$. In other words, the state W^0 is approximately controllable to the state W^T in the time $T > 0$ by the controls (6.14). We have

$$\left\|W^T - W_l^N\right\|^{-1} \le \varepsilon_1 + \varepsilon_2, \tag{6.15}$$

where $W_l^N = \mathcal{W}_0 + \widetilde{W}_l^N = \mathcal{W}_0 + \mathcal{W}_{u_{N,l}}$.

7. Examples

Example 7.1. Let $T = 1/2$,

$$w^0(x) = \frac{x_1}{T^2} e^{-\frac{|x|^2}{4T}}, \quad w^T(x) = \frac{x_1}{8T^2} e^{-\frac{|x|^2}{8T}}, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R}.$$

Consider $W^0 = w^0$ and $W^T = w^T$ in system (3.1), (3.2). Let us investigate whether the state W^0 is approximately controllable to a target state W^T in the time T.

We have $W^0 \in \mathcal{H}^{-1}$ and $W^T \in \mathcal{H}^{-1}$. Denote $\widetilde{W} = W^T - \mathcal{W}_0$. One can easily obtain $\widetilde{W}(x) = -\frac{x_1}{8T^2} e^{-\frac{|x|^2}{8T}}$, $x \in \mathbb{R}^2$. Therefore, $\widetilde{V} = \mathcal{F}\widetilde{W} = 2i\sigma_1 e^{-2T|\sigma|^2}$. Since $\widetilde{V} \in \mathcal{H}_{-1}$, then

$$\widetilde{V}(\sigma) = \sum_{n=0}^{\infty} \omega_n \widehat{\psi}_n^1(\sigma), \qquad \sigma \in \mathbb{R}^2,$$

where $\omega_n = \left\langle \widetilde{V}, \widehat{\psi}_n^1 \right\rangle_{-1} \left(2T^2 \pi L_n^{(1)}(-2T) L_{n+1}^{(1)}(-2T)\right)^{-1}$, $n = \overline{0, \infty}$.

Let $n = \overline{0, \infty}$. Using (5.11) and (5.4), we have

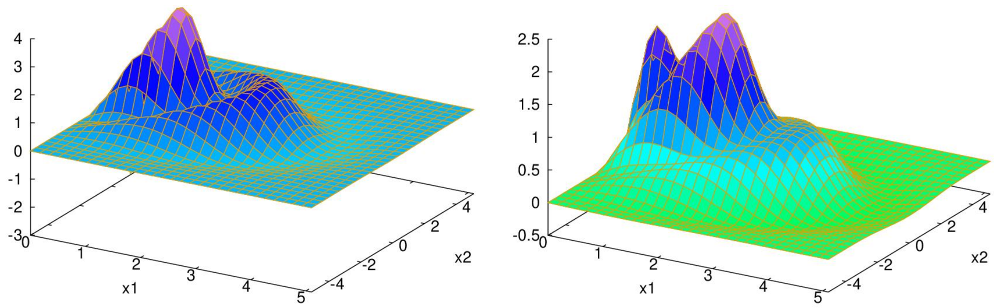
$$\begin{aligned}
&\left\langle \widetilde{V}, \widehat{\psi}_n^1 \right\rangle_{-1} \\
&= \iint_{\mathbb{R}^2} 2i\sigma_1 e^{-2T|\sigma|^2} i(2T)^{5/2} \sigma_1 e^{-T|\sigma|^2} \sum_{p=0}^{n} \frac{L_p^{(1)}(-2T) L_p^{(1)}(2T|\sigma|^2)}{p+1} d\sigma_1 d\sigma_2 \\
&= -2(2T)^{5/2} \sum_{p=0}^{n} \frac{L_p^{(1)}(-2T)}{p+1} \iint_{\mathbb{R}^2} \sigma_1^2 e^{-3T|\sigma|^2} \sum_{k=0}^{p} (-1)^k \binom{p+1}{p-k} \frac{(2T|\sigma|^2)^k}{k!} d\sigma_1 d\sigma_2 \\
&= -2(2T)^{5/2} \sum_{p=0}^{n} L_p^{(1)}(-2T) \sum_{k=0}^{p} \frac{(-2T)^k}{(k+1)!} \binom{p}{k} \iint_{\mathbb{R}^2} \sigma_1^2 |\sigma|^{2k} e^{-3T|\sigma|^2} d\sigma_1 d\sigma_2.
\end{aligned}$$

Using polar coordinates, we obtain from here

$$\begin{aligned}
\left\langle \widetilde{V}, \widehat{\psi}_n^1 \right\rangle_{-1} &= -2(2T)^{\frac{5}{2}} \sum_{p=0}^{n} L_p^{(1)}(-2T) \sum_{k=0}^{p} \frac{(-2T)^k}{(k+1)!} \binom{p}{k} \frac{\pi}{2} \frac{(k+1)!}{(3T)^{k+2}} \\
&= -\frac{(2T)^{\frac{5}{2}} \pi}{(3T)^2} \sum_{p=0}^{n} L_p^{(1)}(-2T) \sum_{k=0}^{p} \binom{p}{k} \left(-\frac{2}{3}\right)^k = -\frac{(2T)^{\frac{5}{2}} \pi}{T^2} \sum_{p=0}^{n} \frac{L_p^{(1)}(-2T)}{3^{p+2}}.
\end{aligned}$$

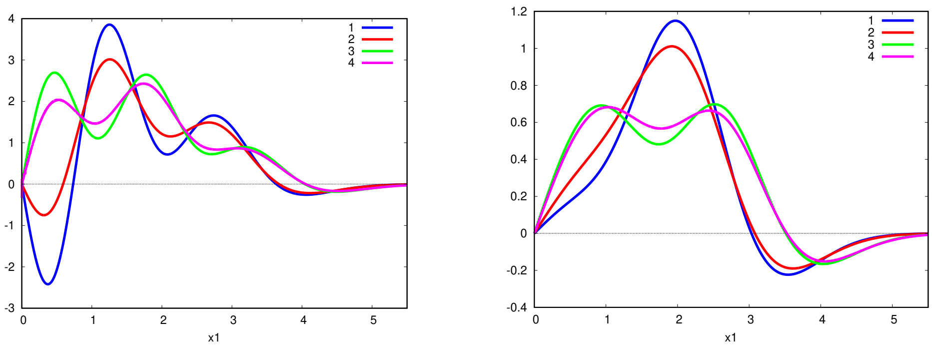
Hence,

$$\omega_n = -\left(\frac{2}{T}\right)^{\frac{3}{2}} \frac{1}{L_n^{(1)}(-2T) L_{n+1}^{(1)}(-2T)} \sum_{p=0}^{n} \frac{L_p^{(1)}(-2T)}{3^{p+2}}, \qquad n = \overline{0, \infty}.$$



(a) $W_l^N(\cdot,T) - W^T$, $N = 3$, $l = 50$. (b) $W_l^N(\cdot,T) - W^T$, $N = 4$, $l = 200$.

Fig. 7.1: The influence of the controls $u_{N,l}$ on the difference $W_l^N(\cdot,T) - W^T$.



(a) $W_l^N\big((\cdot)_{[1]},0,T\big) - W^T\big((\cdot)_{[1]},0\big)$ for 1) $N = 3$, $l = 50$; 2) $N = 3$, $l = 100$; 3) $N = 4$, $l = 100$; 4) $N = 4$, $l = 200$.

(b) $W_l^N\big((\cdot)_{[1]},2,T\big) - W^T\big((\cdot)_{[1]},2\big)$ for 1) $N = 3$, $l = 50$; 2) $N = 3$, $l = 100$; 3) $N = 4$, $l = 100$; 4) $N = 4$, $l = 200$.

Fig. 7.2: The influence of the controls $u_{N,l}$ on the difference $W_l^N(\cdot,T) - W^T$ (vertical sections for $x_2 = 0$ and $x_2 = 2$).

Let $N \in \mathbb{N}$. Put

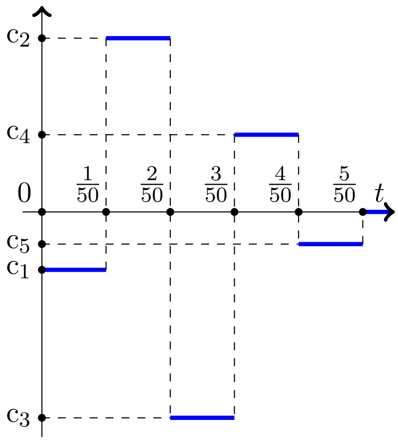
$$u_{N,l}(\xi) = -\pi \sum_{n=0}^{N+1} \alpha_n^N u_l^n(\xi), \tag{7.1}$$

where u_l^n is defined by (5.14), α_n^N is defined by (6.11), and

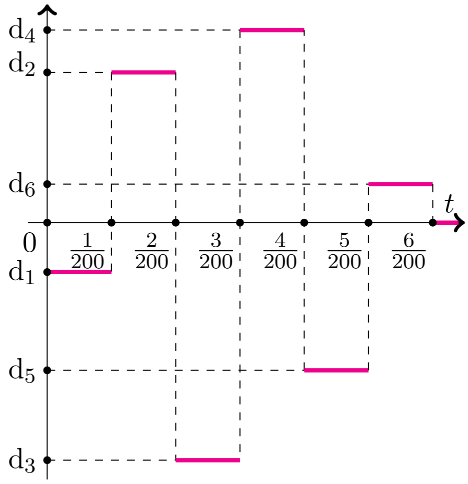
$$\begin{aligned} W_l^N(x) &= \mathcal{W}_0(x,T) + \mathcal{W}_{u_{N,l}}(x,T) \\ &= \frac{x_1}{4T^2} e^{-\frac{|x|^2}{8T}} + \frac{x_1}{\pi} \int_0^T \frac{e^{-\frac{|x|^2}{4\xi}}}{4\xi^2} u_{N,l}(T-\xi)\, d\xi, \quad x \in \mathbb{R}^2. \end{aligned}$$

Let us estimate S_N. Taking into account the following formula [22, p. 784]:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = (1-z)^{-\alpha-1} e^{\frac{xz}{z-1}},$$



(a) $N = 3$, $l = 50$,
$c_1 \approx -31414950.29038$,
$c_2 \approx 115438209.43400$,
$c_3 \approx -158753249.50461$,
$c_4 \approx 96837151.30739$,
$c_5 \approx -22106385.63675$.



(b) $N = 4$, $l = 200$,
$d_1 \approx -698686882813.66$,
$d_2 \approx 3395749137654.21$,
$d_3 \approx -6600596465358.46$,
$d_4 \approx 6414088809555.09$,
$d_5 \approx -3115950570459.56$,
$d_6 \approx 605395974997.15$.

Fig. 7.3: The controls $u_{N,l}$ defined by (5.14).

and monotonicity of the sequence of partial sums, we conclude

$$\sum_{p=0}^{n} \frac{L_p^{(1)}(-2T)}{3^p} \le \frac{9}{4} e^T.$$

From (5.4) it follows that $L_{n+1}^{(1)}(-2T) > L_n^{(1)}(-2T) \ge n+1$. Therefore, according to (6.5), we get

$$S_N = T\sqrt{2\pi}\left(\frac{2}{T}\right)^{3/2}\left(\sum_{n=N+1}^{\infty}\frac{1}{L_n^{(1)}(-2T)L_{n+1}^{(1)}(-2T)}\left(\sum_{p=0}^{n}\frac{L_p^{(1)}(-2T)}{3^{p+2}}\right)^2\right)^{1/2}$$
$$\le \frac{T\sqrt{2\pi}}{9}\left(\frac{2}{T}\right)^{3/2}\left(\sum_{n=N+1}^{\infty}\frac{1}{(n+1)^2}\left(\frac{9}{4}e^T\right)^2\right)^{1/2} = e^T\sqrt{\frac{\pi}{T}\sum_{n=N+1}^{\infty}\frac{1}{(n+1)^2}}$$
$$\le e^T\sqrt{\frac{\pi}{T(N+2)}} = \varepsilon_1.$$

Thus, due to (6.15) for $l \ge \frac{N+2}{T} = 2(N+2)$, we get

$$\left\|W^T - W_l^N\right\|^{-1} \le \varepsilon_1 + \varepsilon_2,$$

where

$$W_l^N(x) = x_1 e^{-\frac{|x|^2}{4}} + \frac{x_1}{\pi}\int_0^{1/2}\frac{e^{-\frac{|x|^2}{4\xi}}}{4\xi^2} u_{N,l}(1/2-\xi)\,d\xi, \qquad x \in \mathbb{R}^2,$$

$$\varepsilon_1 = e^T \sqrt{\frac{\pi}{T(N+2)}} = \sqrt{\frac{2\pi e}{N+2}},$$

$$\varepsilon_2 = \frac{\sqrt{\pi}}{8l} \sum_{k=0}^{N+1} \left|\alpha_k^N\right| \frac{k+1}{2^k} \frac{\sqrt{(2k+2)!}}{\left(T - \frac{k+1}{l}\right)^{k+3/2}}$$

$$= \frac{\sqrt{\pi}}{8l} \sum_{k=0}^{N+1} \left|\alpha_k^N\right| \frac{k+1}{2^k} \frac{\sqrt{(2k+2)!}}{\left(\frac{1}{2} - \frac{k+1}{l}\right)^{k+3/2}}.$$

The controls $u_{N,l}$, $l = \overline{2(N+2), \infty}$, $N = \overline{1, \infty}$, defined by (7.1), solve the approximate controllability problem for the given system.

The influence of the control $u_{N,l}$ on the difference $W_l^N(\cdot, T) - W^T$ is shown in Figs. 7.1 and 7.2. The controls $u_{N,l}$ are given in Fig. 7.3 for the cases of $N = 3$, $l = 50$ and $N = 4$, $l = 200$. The shape of the control in the cases of $N = 3$, $l = 200$ and $N = 4$, $l = 150$ are similar to these cases, respectively.

References

[1] P. Antosik, J. Mikusinski, and R. Sikorski, *Theory of Distributions. The Sequential Approach,* Elsevier, Amsterdam, 1973.

[2] J. Apraiz, *Observability Inequalities for Parabolic Equations over Measurable Sets and Some Applications Related to the Bang-Bang Property for Control Problems,* Appl. Math. Nonlin. Sci. **2** (2017), No. 2, 543–558.

[3] V. Barbu, *Exact null internal controllability for the heat equation on unbounded convex domains,* ESAIM Control Optim. Calc. Var. **20** (2014), No. 1, 222–235.

[4] Y. Duan, L.Wang, and Can Zhang, *Observability Inequalities for the Heat Equation with Bounded Potentials on the Whole Space,* SIAM J. Control Optim. **58** (2020), No. 4, 1939–1960.

[5] V.R. Cabanillas, S.B. De Menezes, and E. Zuazua, *Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms,* J. Optim. Theory Appl. **110** (2001), 245–264.

[6] P. Cannarsa, P. Martinez, and J. Vancostenoble, *Null controllability of the heat equation in unbounded domains by a finite measure control region,* ESAIM Control Optim. Calc. Var. **10** (2004), 381–408.

[7] O. Y. Emanuilov, *Boundary controllability of parabolic equations,* Russian Math. Surveys **48** (1993), No. 3, 192–194.

[8] O. Y. Emanuilov, *Controllability of parabolic equations,* Sb. Math. **186** (1995), No. 6, 879–900.

[9] L.V. Fardigola, *On controllability problems for the wave equation on a half-plane,* Zh. Mat. Fiz. Anal. Geom. **1** (2005), 93–115.

[10] L.V. Fardigola, *Modified Sobolev spaces in controllability problems for the wave equation on a half-plane,* Zh. Mat. Fiz. Anal. Geom. **11** (2015), 18–44.

[11] L.V. Fardigola, *Transformation Operators and Influence Operators in Control Problems,* Thesis (Dr. Hab.), Kharkiv, 2016 (Ukrainian).

[12] L. Fardigola and K. Khalina, *Reachability and Controllability Problems for the Heat Equation on a Half-Axis,* Zh. Mat. Fiz. Anal. Geom. **15** (2019), 57–78.

[13] L. Fardigola and K. Khalina, *Controllability Problems for the Heat Equation on a Half-Axis with a Bounded Control in the Neumann Boundary Condition,* Math. Control Relat. Fields **1** (2021), 211–236.

[14] D. L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, Stud. Appl. Math. **52** (1973), 189–211.

[15] H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension,* Arch. Rational Mech. Anal. **43**, No. 4 (1971), 272–292.

[16] H. O. Fattorini and D. L. Russell, *Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations,* Quart. Appl. Math. **32**, (1974/75), 45–69.

[17] H. O. Fattorini, *Boundary control of temperature distributions in a parallelepipedon,* SIAM J. Control **13**, (1975), 1–13.

[18] S.G. Gindikin and L.R. Volevich, *Distributions and Convolution Equations,* Gordon and Breach Sci. Publ., Philadelphia, 1992.

[19] W. Gong and B. Li, *Improved error estimates for semidiscrete finite element solutions of parabolic Dirichlet boundary control problems,* IMA J. Numer. Anal. **40** (2020), No. 4, 2898–2939.

[20] M. González-Burgos and L. de Teresa, *Some results on controllability for linear and nonlinear heat equations in unbounded domains,* Adv. Differential Equations **12**, (2007), No. 11, 1201–1240.

[21] L. Gosse and O. Runberg, Resolution of the finite Markov moment problem, C. R. Acad. Sci. Paris, Ser. I, **341** (2005), 775–789.

[22] Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables, *Eds. M. Abramowitz and I.A. Stegun,* National Bureau of Standards, Applied Mathematics Series, **55**, Washington, DC, 1972.

[23] O. Yu. Imanuvilov and M. Yamamoto, *Carleman Inequalities for Parabolic Equations in Sobolev Spaces of Negative Order and Exact Controllability for Semilinear Parabolic Equations,* Publ. RIMS, Kyoto Univ. **39** (2003), 227–274.

[24] V.I. Korobov and G.M. Sklyar, *Time optimality and the power moment problem,* Sb. Math. **134** (1987), No. 2, 186–206.

[25] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur,* Comm. Partial Differential Equations **20** (1995), No. 1–2, 335–356.

[26] J. Lohéac, E. Trélat, and E. Zuazua, *Minimal controllability time for the heat equation under unilateral state or control constraints*, Math. Models Methods Appl. Sci. **27** (2017), No. 9, 1587–1644.

[27] S.B. De Menezes and V.R. Cabanillas, *Null controllability for the semilinear heat equation in unbounded domains,* Pesquimat **4**, (2001), 35–54.

[28] S. Micu, I. Roventa, and M. Tucsnak, *Time optimal boundary controls for the heat equation,* J. Funct. Anal. **263** (2012), 25–49.

[29] S. Micu and E. Zuazua, *On the lack of null controllability of the heat equation on the half-line,* Trans. Amer. Math. Soc. **353** (2001), No. 4, 1635–1659.

[30] S. Micu and E. Zuazua, *On the lack of null controllability of the heat equation on the half-space,* Port. Math. (N.S.) **58** (2001), No. 1, 1–24.

[31] L. Miller, *On the null-controllability of the heat equation in unbounded domains,* Bull. Sci. Math. **129** (2005), 175–185.

[32] L. L. D. Njoukoue and G. Deugoue, *Stackelberg control in an unbounded domain for a parabolic equation,* J. Nonl. Evol. Equ. Appl. **5** (2021), 95–118.

[33] L. Schwartz, *Théorie des distributions, I, II,* Hermann, Paris, 1950–1951.

[34] L. de Teresa, *Approximate controllability of a semilinear heat equation in* $\mathbb{R}^N$, SIAM J. Control Optim. **36**, (1998), No. 6, 2128–2147.

[35] L. de Teresa and E. Zuazua, *Approximate controllability of a semilinear heat equation in unbounded domains,* Nonlinear Anal. **37** (1999), No. 8, 1059–1090.

[36] V.S. Vladimirov, *Equations of Mathematical Physics,* Imported Pubn., 1985.

[37] E.H. Zerrik and A. El Kabouss, *Bilinear Boundary Control Problem of an Output of Parabolic Systems,* In: Recent Advances in Modeling, Analysis and Systems Control: Theoretical Aspects and Applications (Eds. E. Zerrik, S. Melliani, O. Castillo). Studies in Systems, Decision and Control, **243**, Springer, Cham., 2020.

Received December 18, 2021.

Larissa Fardigola,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: fardigola@ilt.kharkov.ua

Kateryna Khalina,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: khalina@ilt.kharkov.ua

Проблеми керованості для рівняння теплопровідності на півплощині кероване крайовими умовами Діріхле за допомогою точкового керування

Larissa Fardigola and Kateryna Khalina

У роботі вивчено проблеми керованості та наближеної керованості для керованої системи $w_t = \Delta w$, $w(0, x_2, t) = u(t)\delta(x_2)$, $x_1 > 0$, $x_2 \in \mathbb{R}$, $t \in (0, T)$, де $u \in L^\infty(0, T)$ є керуванням. У термінах розв'язності степеневої проблеми моментів Маркова одержано необхідні і достатні умови керованості, а також достатні умови наближеної керованості за заданий час T, коли керування u є обмеженим заданою сталою. Побудовано ортогональні базиси в спеціальних просторах соболєвського типу.

Застосовуючи ці базиси, одержано необхідні і достатні умови наближеної керованості, ф також чисельне розв'язання проблеми наближеної керованості. Результати проілюстровано прикладом.

Ключові слова: рівняння теплопровідності, керованість, наближена керованість, точкове керування, півплощина